Modern Physics Letters A
Vol. 24, No. 21 (2009) 1677–1688

EXACT ONE-PERIODIC AND TWO-PERIODIC WAVE SOLUTIONS TO HIROTA BILINEAR EQUATIONS IN (2 + 1) DIMENSIONS

WEN-XIU MA

Department of Mathematics, Zhejiang Normal University, Jinhua 312004, P. R. China
Department of Mathematics and Statistics, University of South Florida,
Tampa, FL 33620-5700, USA
mawx@cas.usf.edu

RUGUANG ZHOU

School of Mathematical Science, Xuzhou Normal University, Xuzhou 221116, P. R. China
rgzhou@public.xz.js.cn

LIANG GAO

Department of Applied Mathematics, Northwestern Polytechnical University,
Xi'an 710072, P. R. China
Department of Mathematics and Statistics, University of South Florida,
Tampa, FL 33620-5700, USA
gaoliang@mail.nwpu.edu.cn

Received 27 May 2008
Revised 13 November 2008

Riemann theta functions are used to construct one-periodic and two-periodic wave solutions to a class of (2 + 1)-dimensional Hirota bilinear equations. The basis for the involved solution analysis is the Hirota bilinear formulation, and the particular dependence of the equations on independent variables guarantees the existence of one-periodic and two-periodic wave solutions involving an arbitrary purely imaginary Riemann matrix. The resulting theory is applied to two nonlinear equations possessing Hirota bilinear forms: $u_t + u_{xxy} - 3uu_y - 3u_xv = 0$ and $u_t + u_{xxxxy} - (5u_{xx}v + 10u_{xy}u - 15u^2v)_x = 0$ where $v_x = u_y$, thereby yielding their one-periodic and two-periodic wave solutions describing one-dimensional propagation of waves.

Keywords: Hirota bilinear equations; Riemann theta functions; one-periodic and two-periodic wave solutions.

PACS Nos.: 02.30.Gp, 02.30.Ik, 02.30.Jr

1. Introduction

It is always important to search for exact solutions to nonlinear differential equations. Different approaches, particularly in soliton theory, provide many tools for constructing explicit and exact solutions. Various kinds of exact solutions such as solitons, positons, complexitons, solitonoffs and dromions have been presented for nonlinear integrable equations.[1–9] Successful methods include the inverse scattering transform,[1] the Darboux transformation,[2] Hirota direct method,[3] and algebro-geometrical approach.[4]

The algebro-geometrical approach presents quasi-periodic or algebro-geometric solutions to many soliton equations, which contain the KdV equation, the sine-Gordon equation and the nonlinear Schrödinger equation. In recent years, such an approach have been applied to many $(2+1)$-dimensional nonlinear integrable equations.[10–13] Nonlinearization of Lax pairs[14–17] plays a crucial role in connecting the resulting algebro-geometric solutions with Liouville integrable Hamiltonian systems. The approach, however, needs Lax pair representations and involves complicated calculus on Riemann surfaces.

On the other hand, the Hirota direct method provides a powerful way to derive soliton solutions to nonlinear integrable equations and its basis is the Hirota bilinear formulation.[3] Once the corresponding bilinear forms are obtained, multi-soliton solutions and rational solutions to nonlinear differential equations can be computed in quite a systematic way,[3] even through Wronskian, Casoratian or Pfaffian determinants.[18–24] It is based on Hirota bilinear forms that Nakamura presented an approach to multi-periodic wave solutions of nonlinear integrable equations,[25,26] using directly Riemann theta functions. Such a method of solution does not need any Lax pairs and their induced Riemann surfaces for the considered equations. The presented multi-periodic solutions can be reduced to soliton solutions under asymptotic limits.[27,28] The advantage of the method is that it only relies on the existence of Hirota bilinear forms. Moreover, all parameters appearing in Riemann matrices are completely arbitrary, whereas algebro-geometric solutions involve specific Riemann constants, which are usually difficult to compute.

In this paper, motivated by Nakamura's idea,[25,26] we would like to use Riemann theta functions to generate one-periodic and two-periodic wave solutions to a particular class of $(2+1)$-dimensional Hirota bilinear equations, and the corresponding solution analysis will be made to guarantee the existence of one-periodic and two-periodic wave solutions to the selected class of $(2+1)$-dimensional nonlinear equations. As illustrative examples of the resulting theory, we will discuss two nonlinear equations in $(2+1)$ dimensions possessing Hirota bilinear forms:

$$u_t + u_{xxy} - 3uu_y - 3u_x v = 0$$

and

$$u_t + u_{xxxxy} - (5u_{xx}v + 10u_{xy}u - 15u^2 v)_x = 0\,,$$

where $v_x = u_y$, and their one-periodic and two-periodic wave solutions involving an arbitrary purely imaginary Riemann matrix will be explicitly presented.

2. Existence of One-Periodic and Two-Periodic Wave Solutions

Let us consider an evolution equation in $(2+1)$ dimensions:

$$u_t = K(u, u_x, u_y, \ldots)\,, \tag{2.1}$$

where $t \in \mathbb{R}$ is the time variable and $x, y \in \mathbb{R}$ are the space variables. We assume that under a transformation

$$u = u_0 - 2(\ln f(x,y,t))_{xx}\,, \tag{2.2}$$

where u_0 is a special solution to (2.1), the evolution equation (2.1) can be transformed into a Hirota bilinear equation

$$F(D_x, D_y, D_t) f \cdot f = 0\,, \tag{2.3}$$

where F is a polynomial in the three variables. Here and below, the Hirota bilinear differential operators[3] are defined by

$$\begin{aligned} &D_x^p D_y^q D_t^r f(x,y,t) \cdot g(x,y,t) \\ &\quad = (\partial_x - \partial_{x'})^p (\partial_y - \partial_{y'})^q (\partial_t - \partial_{t'})^r f(x,y,t) g(x',y',t')|_{x'=x, y'=y, t'=t}\,, \end{aligned} \tag{2.4}$$

where p, q, r are non-negative integers. We will focus on a particular class of Hirota bilinear equations in $(2+1)$ dimensions:

$$F(D_x, D_y, D_t) f \cdot f = (D_t P(D_x) + D_y Q(D_x) + R(D_x)) f \cdot f = 0\,, \tag{2.5}$$

where P and Q are nonzero odd polynomials and R is a nonzero even polynomial, namely, P, Q and R are nonzero polynomials and satisfy

$$P(-z) = -P(z)\,, \qquad Q(-z) = -Q(z)\,, \qquad R(-z) = R(z)\,. \tag{2.6}$$

When the Hirota operators act on exponential functions, the following derivative formula holds:

$$D_x^p D_y^q D_t^r e^{\eta_1} \cdot e^{\eta_2} = (k_1 - k_2)^p (l_1 - l_2)^q (\omega_1 - \omega_2)^r e^{\eta_1 + \eta_2}\,, \tag{2.7}$$

where $\eta_j = k_j x + l_j y + \omega_j t + \eta_{j0}$, $j = 1, 2$, with $k_j, l_j, \omega_j, \eta_{j0}$ being constants. More generally, we have

$$G(D_x, D_y, D_t) e^{\eta_1} \cdot e^{\eta_2} = G(k_1 - k_2, l_1 - l_2, \omega_1 - \omega_2) e^{\eta_1 + \eta_2}\,, \tag{2.8}$$

where G is a polynomial in the three variables. This derivative formula will be a crucial key to our success in generating one-periodic and two-periodic wave solutions.

We would like to consider the multi-dimensional special Riemann theta function solution[29]:

$$f = f(x,y,t) = \sum_{n \in \mathbb{Z}^N} e^{2\pi i \langle \eta, n \rangle + \pi i \langle \tau n, n \rangle}\,, \tag{2.9}$$

where $\langle \cdot\,, \cdot \rangle$ is the standard inner product of $\mathbb{R}^N$, $n = (n_1, \ldots, n_N)^{\mathrm{T}}$, $\eta = (\eta_1, \ldots, \eta_N)^{\mathrm{T}}$ with $\eta_j = k_j x + l_j y + \omega_j t + \eta_{j0}$, and $\tau = (\tau_{pq})_{N \times N}$ is a symmetric matrix whose imaginary part is positive definite (i.e. $\mathrm{Im}\,\tau > 0$). Based on (2.8), we can compute in general $G(D_x, D_t, \ldots) f \cdot f$ for such a Riemann theta function f,[30] but we will make direct computations to provide a complete solution process and capture more of special solution structures.

2.1. *One-periodic wave solutions*

Let us first consider the case of $N = 1$. Then the Riemann theta function in (2.9) becomes

$$f = f(x, y, t) = \sum_{n=-\infty}^{\infty} e^{2\pi i n\eta + \pi i n^2 \tau}, \tag{2.10}$$

where $\operatorname{Im}\tau > 0$ and $\eta = kx + ly + \omega t + \eta_0$ with k, l, ω, η_0 being real constants.

Based on the derivative formula (2.8), we can compute that

$$\begin{aligned}
&F(D_x, D_y, D_t) f \cdot f \\
&\quad = F(D_x, D_y, D_t) \sum_{n=-\infty}^{\infty} e^{2\pi i n\eta + \pi i n^2\tau} \cdot \sum_{m=-\infty}^{\infty} e^{2\pi i m\eta + \pi i m^2\tau} \\
&\quad = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F(D_x, D_y, D_t) e^{2\pi i n\eta + \pi i n^2\tau} \cdot e^{2\pi i m\eta + \pi i m^2\tau} \\
&\quad = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F(2\pi i(n-m)k, 2\pi i(n-m)l, 2\pi i(n-m)\omega) e^{2\pi i(n+m)\eta + \pi i(n^2+m^2)\tau} \\
&\quad = \sum_{m'=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} F(2\pi i(2n-m')k, 2\pi i(2n-m')l, 2\pi i(2n-m')\omega) \right. \\
&\qquad \left. \times e^{\pi i[(n^2+(n-m')^2]\tau} \right\} e^{2\pi i m'\eta} \\
&\quad = \sum_{m'=-\infty}^{\infty} \tilde{F}(m') e^{2\pi i m'\eta},
\end{aligned}$$

where the new summation $m' = m + n$ has been introduced and $\tilde{F}(m')$ is defined by

$$\tilde{F}(m') = \sum_{n=-\infty}^{\infty} F(2\pi i(2n-m')k, 2\pi i(2n-m')l, 2\pi i(2n-m')\omega) e^{\pi i[n^2+(n-m')^2]\tau}. \tag{2.11}$$

Shifting index n by introducing $n' = n - 1$, we have

$$\begin{aligned}
\tilde{F}(m') &= \sum_{n=-\infty}^{\infty} F(2\pi i(2n-m')k, 2\pi i(2n-m')l, 2\pi i(2n-m')\omega) e^{\pi i[n^2+(n-m')^2]\tau} \\
&= \sum_{n'=-\infty}^{\infty} F(2\pi i[2n' - (m'-2)]k, 2\pi i[2n' - (m'-2)]l, 2\pi i[2n' - (m'-2)]\omega) \\
&\quad \times e^{\pi i\{n'^2 + [n' - (m'-2)]^2\}\tau} e^{2\pi i(m'-1)\tau} \\
&= \tilde{F}(m'-2) e^{2\pi i(m'-1)\tau}, \qquad m' \in \mathbb{Z}.
\end{aligned}$$

It then follows that if $\tilde{F}(0) = \tilde{F}(1) = 0$, then $\tilde{F}(m') = 0$ for all $m' \in \mathbb{Z}$.

Noticing the specific form of Eq. (2.5), one-periodic wave solutions can be obtained, if we require

$$\begin{cases} \tilde{F}(0) = \displaystyle\sum_{n=-\infty}^{\infty} [4n\pi i\omega P(4n\pi ik) + 4n\pi ilQ(4n\pi ik) + R(4n\pi ik)]e^{2n^2\pi i\tau} = 0\,, \\ \tilde{F}(1) = \displaystyle\sum_{n=-\infty}^{\infty} [2(2n-1)\pi i\omega P(2(2n-1)\pi ik) + 2(2n-1)\pi ilQ(2(2n-1)\pi ik) \\ \qquad\qquad + R(2(2n-1)\pi ik)]e^{(2n^2-2n+1)\pi i\tau} = 0\,. \end{cases} \tag{2.12}$$

Upon introducing

$$\begin{cases} a_{11}(k) = \displaystyle\sum_{n=-\infty}^{\infty} 4n\pi iP(4n\pi ik)e^{2n^2\pi i\tau}\,, \\ a_{12}(k) = \displaystyle\sum_{n=-\infty}^{\infty} 4n\pi iQ(4n\pi ik)e^{2n^2\pi i\tau}\,, \\ a_{21}(k) = \displaystyle\sum_{n=-\infty}^{\infty} 2(2n-1)\pi iP(2(2n-1)\pi ik)e^{(2n^2-2n+1)\pi i\tau}\,, \\ a_{22}(k) = \displaystyle\sum_{n=-\infty}^{\infty} 2(2n-1)\pi iQ(2(2n-1)\pi ik)e^{(2n^2-2n+1)\pi i\tau}\,, \end{cases} \tag{2.13}$$

and

$$\begin{cases} b_1(k) = -\displaystyle\sum_{n=-\infty}^{\infty} R(4n\pi ik)e^{2n^2\pi i\tau}\,, \\ b_2(k) = -\displaystyle\sum_{n=-\infty}^{\infty} R(2(2n-1)\pi ik)e^{(2n^2-2n+1)\pi i\tau}\,, \end{cases} \tag{2.14}$$

the linear system (2.12) of ω and l can be compactly written as

$$a_{11}(k)\omega + a_{12}(k)l = b_1(k)\,, \qquad a_{21}(k)\omega + a_{22}(k)l = b_2(k)\,. \tag{2.15}$$

We will see that there are a lot of choices for the angular wave number k. In order to generate real solutions (ω, l) to the system (2.15), we assume that

$$\operatorname{Re}\tau = 0\,. \tag{2.16}$$

The determinant of the coefficient matrix $A(k) = (a_{rs}(k))_{2\times 2}$ is a polynomial in k, and so, if $\det(A(k)) \not\equiv 0$ (this condition will be satisfied in our concrete examples), then

$$A_0 := \{k \in \mathbb{R} \,|\, \det(A(k)) = 0\} \tag{2.17}$$

is either an empty set or a finite set. This guarantees the existence of real solutions (ω, l) to the system (2.15) at least for $k \notin A_0$. About nonzero solutions, we can have the following analysis.

If $\deg(R) = 0$, i.e. $R = c$, where c is a nonzero real constant, then it follows from (2.14) that $b(k)$ does not depend on k and

$$b(k) = (b_1(k), b_2(k))^{\mathrm{T}} \neq 0\,,$$

and so, there is the unique nonzero solution of (ω, l) to the system (2.15) for $k \notin A_0$.

If $\deg(R) \geq 2$, then

$$B_0 := \{k \in \mathbb{R} \,|\, (b_1(k))^2 + b_2(k))^2 = 0\} \tag{2.18}$$

is either an empty set or a finite set, since each of $b_1(k)$ and $b_2(k)$ is a polynomial in k of degree $\deg(R)$. Therefore, there is the unique nonzero solution of (ω, l) to the system (2.15) for $k \notin A_0 \cup B_0$.

2.2. *Two-periodic wave solutions*

Let us now consider the case of $N = 2$ and the corresponding two-periodic wave solutions. Similarly, based on the derivative formula (2.8) and introducing $m' = n + m$, we can have

$$\begin{aligned}
F(D_x, D_y, D_t)f \cdot f &= \sum_{m,n\in\mathbb{Z}^2} F(D_x, D_y, D_t)e^{2\pi i\langle\eta,n\rangle+\pi i\langle\tau n,n\rangle} \cdot e^{2\pi i\langle\eta,m\rangle+\pi i\langle\tau m,m\rangle} \\
&= \sum_{m,n\in\mathbb{Z}^2} F(2\pi i\langle n-m, k\rangle, 2\pi i\langle n-m, l\rangle, 2\pi i\langle n-m,\omega\rangle) \\
&\quad \times e^{2\pi i\langle\eta,n+m\rangle+\pi i(\langle\tau m,m\rangle+\langle\tau n,n\rangle)} \\
&= \sum_{m'\in\mathbb{Z}^2}\sum_{n\in\mathbb{Z}^2} F(2\pi i\langle 2n-m', k\rangle, 2\pi i\langle 2n-m', l\rangle, 2\pi i\langle 2n-m',\omega\rangle) \\
&\quad \times e^{\pi i(\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle)}e^{2\pi i\langle\eta,m'\rangle} \\
&= \sum_{m'\in\mathbb{Z}^2} \tilde{F}(m_1', m_2')e^{2\pi i\langle\eta,m'\rangle}\,,
\end{aligned}$$

where $\tilde{F}(m_1', m_2') = \tilde{F}(m')$ is defined by

$$\begin{aligned}
&\tilde{F}(m_1', m_2') \\
&\quad = \sum_{n\in\mathbb{Z}^2} F(2\pi i\langle 2n-m', k\rangle, 2\pi i\langle 2n-m', l\rangle, 2\pi i\langle 2n-m',\omega\rangle)e^{\pi i(\langle\tau(n-m'),n-m'\rangle+\langle\tau n,n\rangle)}\,.
\end{aligned} \tag{2.19}$$

Shifting index n as $n' = n - e_r$ with $r = 1$ or $r = 2$, where $e_1 = (1,0)^{\mathrm{T}}$ and $e_2 = (0,1)^{\mathrm{T}}$, we can compute that

$$\begin{aligned}
\tilde{F}(m_1', m_2') = \tilde{F}(m') &= \tilde{F}(m' - 2e_r)e^{2\pi i(\langle\tau(m'-2e_r),e_r\rangle+\langle\tau e_r,e_r\rangle)} \\
&= \begin{cases} \tilde{F}(m_1'-2, m_2')e^{2\pi i(m_1'-1)\tau_{11}+2\pi i m_2'\tau_{12}}\,, & r = 1\,, \\ \tilde{F}(m_1', m_2'-2)e^{2\pi i(m_2'-1)\tau_{22}+2\pi i m_1'\tau_{12}}\,, & r = 2\,, \end{cases}
\end{aligned}$$

where $\tau = (\tau_{pq})_{2\times 2}$. It now follows that if

$$\tilde{F}(0,0) = \tilde{F}(0,1) = \tilde{F}(1,0) = \tilde{F}(1,1) = 0\,, \tag{2.20}$$

then $\tilde{F}(m_1', m_2') = 0$ for all $m_1', m_2' \in \mathbb{Z}$.

For our selected equation (2.5), we have

$$\begin{aligned}\tilde{F}(m_1,m_2) = \sum_{n\in\mathbb{Z}^2} &[2\pi i\langle 2n-m,\omega\rangle P(2\pi i\langle 2n-m,k\rangle) \\ &+ 2\pi i\langle 2n-m,l\rangle Q(2\pi i\langle 2n-m,k\rangle) \\ &+ R(2\pi i\langle 2n-m,k\rangle)]e^{\pi i(\langle \tau(n-m),n-m\rangle + \langle \tau n,n\rangle)}\,,\end{aligned} \tag{2.21}$$

where we set

$$\begin{gathered} m = (m_1,m_2)^{\mathrm{T}}\,, \qquad n = (n_1,n_2)^{\mathrm{T}}\,, \qquad k = (k_1,k_2)^{\mathrm{T}}\,, \\ l = (l_1,l_2)^{\mathrm{T}}\,, \qquad \omega = (\omega_1,\omega_2)^{\mathrm{T}}\,. \end{gathered} \tag{2.22}$$

For simplicity, define

$$\theta_r(n) = e^{\pi i(\langle \tau(n-m^{(r)}),n-m^{(r)}\rangle + \langle \tau n,n\rangle)}\,, \qquad 1\le r\le 4\,, \tag{2.23}$$

where $m^{(r)} = (m_1^{(r)}, m_2^{(r)})^{\mathrm{T}}$, $1 \le r \le 4$, are given by

$$m^{(1)} = (0,0)^{\mathrm{T}}\,, \qquad m^{(2)} = (0,1)^{\mathrm{T}}\,, \qquad m^{(3)} = (1,0)^{\mathrm{T}}\,, \qquad m^{(4)} = (1,1)^{\mathrm{T}}.$$

Then, upon introducing

$$\begin{cases} a_{rs}(k) = \displaystyle\sum_{n_1,n_2=-\infty}^{\infty} 2\pi i(2n_s - m_s^{(r)})P(2\pi i\langle 2n-m^{(r)},k\rangle)\theta_r(n)\,, \\ a_{r,s+2}(k) = \displaystyle\sum_{n_1,n_2=-\infty}^{\infty} 2\pi i(2n_s - m_s^{(r)})Q(2\pi i\langle 2n-m^{(r)},k\rangle)\theta_r(n)\,, \end{cases} \tag{2.24}$$

where $1\le r\le 4$ and $1 \le s \le 2$, and

$$b_r(k) = -\sum_{n_1,n_2=-\infty}^{\infty} R(2\pi i\langle 2n-m^{(r)},k\rangle)\theta_r(n)\,, \qquad 1\le r\le 4\,, \tag{2.25}$$

the linear system (2.20) of (ω, l) can be compactly written as

$$A(k)\begin{bmatrix}\omega_1\\ \omega_2\\ l_1\\ l_2\end{bmatrix} = b(k) = \begin{bmatrix} b_1(k)\\ b_2(k)\\ b_3(k)\\ b_4(k)\end{bmatrix}\,, \tag{2.26}$$

where $A(k) = (a_{rs}(k))_{4\times 4}$. If τ is purely imaginary, i.e. it satisfies (2.16), then $A(k)$ and $b(k)$ are real, due to our assumption on the polynomials P, Q, R. Note that if

$\det(A(k)) \not\equiv 0$ (this condition will be satisfied in our concrete examples), then

$$A_0 := \{k \in \mathbb{R}^2 \,|\, \det(A(k)) = 0\} \tag{2.27}$$

is either an empty set or a finite set.

Now if $\deg(R) = 0$, i.e. $R = c$, where c is a nonzero real constant, then it follows from (2.25) that $b(k)$ does not depend on k and $b(k) \neq 0$, and so, there is the unique nonzero solution of $(\omega_1, \omega_2, l_1, l_2)$ to the system (2.26) for $k \notin A_0$.

If $\deg(R) \geq 2$, then

$$B_0 := \left\{ k \in \mathbb{R}^2 \,\middle|\, \sum_{r=0}^{4} (b_r(k))^2 = 0 \right\} \tag{2.28}$$

is either an empty set or a finite set, since each of $b_r(k)$, $1 \leq r \leq 4$, is a polynomial in k_1 and k_2 of degree $\deg R$. Therefore, there is the unique nonzero solution of $(\omega_1, \omega_2, l_1, l_2)$ to the system (2.26) for $k \notin A_0 \cup B_0$.

3. Two Illustrative Examples

Let us illustrate our idea of generating one-periodic and two-periodic wave solutions through two particular Hirota bilinear equations. The first example is

$$u_t + u_{xxy} - 3uu_y - 3u_x v = 0\,, \qquad v_x = u_y\,, \tag{3.1}$$

in the physical field. This nonlinear equation is related to the breaking soliton equation[31]:

$$u_t + u_{xxy} - 4uu_y - 2u_x \partial_x^{-1} u_y = 0\,,$$

and it can be transformed into

$$(D_t D_x + D_y D_x^3 + c) f \cdot f = 0\,, \tag{3.2}$$

where c can be an arbitrary function of y and t, under the transformation

$$u = -2(\ln f)_{xx}\,, \qquad v = -2(\ln f)_{xy}\,. \tag{3.3}$$

Actually, we have

$$u_t + u_{xxy} - 3uu_y - 3u_x v = -\left(\frac{(D_t D_x + D_y D_x^3) f \cdot f}{f^2} \right)_x .$$

The second example is

$$u_t + u_{xxxxy} - (5u_{xx}v + 10u_{xy}u - 15u^2 v)_x = 0\,, \qquad v_x = u_y\,, \tag{3.4}$$

in the physical field. This nonlinear equation can be transformed into

$$(D_t D_x + D_y D_x^5 + c) f \cdot f = 0\,, \tag{3.5}$$

where c can be an arbitrary function of y and t, under the same transformation (3.3). Similarly, we have

$$u_t + u_{xxxxy} - (5u_{xx}v + 10u_{xy}u - 15u^2 v)_x = -\left(\frac{(D_t D_x + D_y D_x^5) f \cdot f}{f^2} \right)_x .$$

The involved arbitrary function c of y and t shows the diversity of solutions to $(2+1)$-dimensional differential equations.

To generate one-periodic and two-periodic wave solutions by the solution method in the last section, we need to assume that the above function c is constant, based on which the angular wave number l (or the angular wave numbers l_1 and l_2) and the frequency ω (or the frequencies ω_1 and ω_2) are constant and thus the derivative formula (2.8) will hold. Obviously, we have

$$P(z) = z\,, \qquad Q(z) = z^3\,, \qquad R(z) = c\,, \tag{3.6}$$

for Eq. (3.1) and

$$P(z) = z\,, \qquad Q(z) = z^5\,, \qquad R(z) = c\,, \tag{3.7}$$

for Eq. (3.4). The polynomials P and Q defined above are odd and the polynomials R defined above are even, and so, the property (2.6) is satisfied. The determinants of the corresponding coefficient matrices of the linear systems (2.15) and (2.26) are not identically equal to zero, namely,

$$\det(A(k)) \neq 0 \qquad \text{and} \qquad \det(A(k_1, k_2)) \neq 0$$

in the two examples. For instance, in the case of one-periodic wave solutions, we have

$$\det(A(k)) = ak^4 \quad \text{or} \quad \det(A(k)) = bk^6\,, \tag{3.8}$$

where

$$\begin{aligned} a = {} & -256\pi^6 \sum_{n=-\infty}^{\infty} n^2 e^{2n^2\pi i\tau} \sum_{n=-\infty}^{\infty} (2n-1)^4 e^{(2n^2-2n+1)\pi i\tau} \\ & + 1024\pi^6 \sum_{n=-\infty}^{\infty} n^4 e^{2n^2\pi i\tau} \sum_{n=-\infty}^{\infty} (2n-1)^2 e^{(2n^2-2n+1)\pi i\tau}\,, \\ b = {} & 1024\pi^8 \sum_{n=-\infty}^{\infty} n^2 e^{2n^2\pi i\tau} \sum_{n=-\infty}^{\infty} (2n-1)^6 e^{(2n^2-2n+1)\pi i\tau} \\ & - 16384\pi^8 \sum_{n=-\infty}^{\infty} n^6 e^{2n^2\pi i\tau} \sum_{n=-\infty}^{\infty} (2n-1)^2 e^{(2n^2-2n+1)\pi i\tau}\,. \end{aligned}$$

A direct computation by Maple 11 with Digits = 30 shows that

$$a|_{\tau=0.1i} \approx 4563.212514\,, \qquad a|_{\tau=0.2i} \approx 140396.7042\,, \qquad a|_{\tau=0.5i} \approx 25831.08621\,,$$

$$b|_{\tau=0.1i} \approx 11012599.24\,, \qquad b|_{\tau=0.2i} \approx 28544399.95\,, \qquad b|_{\tau=0.5i} \approx -4884657.870\,,$$

which are all nonzero. Generally, our general analysis made before is valid for the two equations (3.2) and (3.5), and so, one-periodic and two-periodic wave solutions to the two $(2+1)$-dimensional Hirota bilinear equations (3.1) and (3.4) can be computed explicitly.

4. Conclusion and Remarks

The Riemann theta functions have been used to generate one-periodic and two-periodic wave solutions of a particular class of $(2+1)$-dimensional Hirota bilinear equations, and the corresponding solution analysis has been made to guarantee the existence of such multi-periodic wave solutions. Two illustrative examples:

$$u_t + u_{xxy} - 3uu_y - 3u_x v = 0$$

and

$$u_t + u_{xxxxy} - (5u_{xx}v + 10u_{xy}u - 15u^2 v)_x = 0\,,$$

where $v_x = u_y$, have been discussed in details, along with their one-periodic and two-periodic wave solutions involving an arbitrary purely imaginary Riemann matrix.

Our solution analysis provides a way to construct one-periodic and two-periodic wave solutions to $(2+1)$-dimensional nonlinear differential equations. It allows different angular wave numbers k (or k_1 and k_2), but the angular wave number l (or the angular wave numbers l_1 and l_2) and the frequency ω (or the frequencies ω_1 and ω_2) are determined in terms of the angular wave number k (or the angular wave numbers k_1 and k_2) and hence the obtained solutions describe one-dimensional propagation of waves.

We also remark that the proposed approach can be applied to other nonlinear differential equations. For example, the following combined equation with the Sawada–Kotera vector field:

$$u_t + u_{xxy} - 3uu_y - 3u_x v + u_{xxxxx} - 15(uu_{xx} - u^3)_x = 0\,, \qquad v_x = u_y\,,$$

can be analyzed similarly. Under the transformation (3.3), this equation can be put into the following bilinear equation:

$$(D_t D_x + D_y D_x^3 + D_x^6 + c) f \cdot f = 0\,,$$

where c is an arbitrary function of y and t. The corresponding polynomials P, Q, R read

$$P(z) = z\,, \qquad Q(z) = z^3\,, \qquad R(z) = z^6 + c\,, \tag{4.1}$$

where c is assumed to be constant. Therefore, the same analysis on one-periodic and two-periodic wave solutions will work for this equation as well. On the other hand, soliton solutions to Eqs. (3.1) and (3.4) can be computed by using Hirota's direct method. For example, one soliton solutions to Eqs. (3.1) and (3.4) are determined by

$$f = 1 + e^{\pm k^3 t + kx \mp ky} \qquad \text{and} \qquad f = 1 + e^{\pm k^5 t + kx \mp ky}, \ k\text{-arbitrary const.},$$

respectively. This can also be verified by using (2.8). It should be, however, interesting to establish any relations between soliton solutions and multi-periodic wave solutions.

It is our hope that our analysis on one-periodic and two-periodic wave solutions made for the particularly selected class of Hirota bilinear equations could help to better understand the diversity and integrability of nonlinear differential equations.

Acknowledgments

The work was supported in part by the Established Researcher Grant of the University of South Florida, the CAS faculty development grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, Wang Kuancheng foundation, the National Natural Science Foundation of China (Grant Nos. 10471120, 10332030, 10472091, 10502042), the NCET-04-0518, FANEDD (No. 200013), the Excellent Young Teachers Program of MOEPRC, the Project-sponsored by SRF for ROCS, SEM, the "333 project" of Jiangsu Province, and the Doctorate Foundation of Northwestern Polytechnical University (Grant No. CX200616).

References

1. M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge Univ. Press, 1991).
2. V. B. Matveev and M. A. Salle, *Darboux Transformation and Solitons* (Springer, 1991).
3. R. Hirota, *Direct Methods in Soliton Theory* (Springer, 2004).
4. E. Belokolos, A. Bobenko, V. Enol'skij, A. Its and V. Matveev, *Algebro-Geometrical Approach to Nonlinear Integrable Equations* (Springer, 1994).
5. M. Boiti, J. Jp. Leon, L. Martina and F. Pempinelli, *Phys. Lett. A* **132**, 432 (1988).
6. J. Hietarinta, *Phys. Lett. A* **149**, 113 (1990).
7. S. Y. Lou, X. Y. Tang, X. M. Qian, C. L. Chen, J. Lin and S. L. Zhang, *Mod. Phys. Lett. B* **16**, 1075 (2002).
8. W. X. Ma and Y. C. You, *Trans. Amer. Math. Soc.* **357**, 1753 (2005).
9. L. Gao, W. Xu, Y. N. Tang and G. F. Meng, *Phys. Lett. A* **366**, 411 (2007).
10. C. W. Cao, Y. T. Wu and X. G. Geng, *J. Math. Phys.* **40**, 3948 (1999).
11. C. W. Cao, X. G. Geng and H. Y. Wang, *J. Math. Phys.* **43**, 621 (2002).
12. R. G. Zhou, *Nuovo Cimento B* **117**, 925 (2002).
13. X. G. Geng and H. H. Dai, *Phys. A* **319**, 270 (2003).
14. C. W. Cao, *Sci. China Ser. A* **33**, 528 (1990).
15. W. X. Ma and W. Strampp, *Phys. Lett. A* **185**, 277 (1994).
16. W. X. Ma and X. G. Geng, in *Bäcklund and Darboux Transformations – The Geometry of Solitons* (Halifax, NS, 1999), pp. 313–323, CRM Proc. Lecture Notes, Vol. 29 (Amer. Math. Soc., 2001).
17. W. X. Ma and Y. B. Zeng, *ANZIAM J.* **44**, 129 (2002).
18. R. Hirota and Y. Ohta, *J. Phys. Soc. Jpn.* **60**, 798 (1991).
19. W. X. Ma, *Phys. Lett. A* **301**, 35 (2002).
20. W. X. Ma and K. Maruno, *Phys. A* **343**, 219 (2004).
21. J. X. Zhao, C. X. Li and X. B. Hu, *J. Phys. Soc. Jpn.* **73**, 1159 (2004).
22. X. B. Hu, C. X. Li, J. J. C. Nimmo and G. F. Yu, *J. Phys A*: *Math. Gen.* **38**, 195 (2005).
23. C. X. Li, W. X. Ma, X. J. Liu and Y. B. Zeng, *Inv. Probl.* **23**, 279 (2007).

24. W. X. Ma, J. S. He and C. X. Li, A second Wronskian formulation of the Boussinesq equation, *Nonlinear Anal. Theor. Meth. Appl.*, Available online 5 October 2008.
25. A. Nakamura, *J. Phys. Soc. Jpn.* **47**, 1701 (1979).
26. A. Nakamura, *J. Phys. Soc. Jpn.* **48**, 1365 (1980).
27. Y. Matsuno, *Bilinear Transformation Method* (Academic Press, 1984).
28. Y. Zhang, L. Y. Ye, Y. N. Lv and H. Q. Zhao, *J. Phys. A*: *Math. Theor.* **40**, 5539 (2007).
29. H. E. Rauch and H. M. Farkas, *Theta Functions with Applications to Riemann Surfaces* (The Williams & Wilkins Co., 1974).
30. R. Hirota and M. Ito, *J. Phys. Soc. Jpn.* **50**, 338 (1981).
31. F. Calogero and A. Degasperis, *Nuovo Cimento B* **31**, 201 (1977).