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Darboux transformations of integrable couplings and applications

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A formulation of Darboux transformations is proposed for integrable couplings, based on non-semisimple matrix Lie algebras. Applications to a kind of integrable couplings of the AKNS equations are made, along with an explicit formula for the associated Bäcklund transformation. Exact one-soliton-like solutions are computed for the integrable couplings of the second- and third-order AKNS equations, and a type of reduction is created to generate integrable couplings and their one-soliton-like solutions for the NLS and MKdV equations.

Keywords: Matrix spectral problem; integrable coupling; Darboux transformation; non-semisimple Lie algebra; soliton.

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1. Introduction

Integrable systems are usually generated from matrix spectral problems or Lax pairs associated with matrix loop algebras (see, e.g., [1–5]). Among celebrated examples, with dependent variables less than three, are the Korteweg–de Vries (KdV) hierarchy [6], the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [7], the Dirac hierarchy [8], the Kaup–Newell hierarchy [9], the coupled AKNS–Kaup–Newell hierarchy [10] and the Wadati–Konno–Ichikawa hierarchy [11].

Simple matrix loop algebras generate typical integrable systems (see, e.g., [7, 9] and [12, 13] for examples associated with $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$, respectively). Semisimple matrix loop algebras engender separated integrable systems, i.e. collections of typical integrable systems, each of which corresponds to a simple matrix loop algebra. Non-semisimple matrix loop algebras yield integrable couplings [14–17]. Integrable systems often possess bi-Hamiltonian structures [18], which generate hereditary recursion operators [19] and guarantee the Liouville integrability [20]. The associated Hamiltonian structures can be furnished by the trace identity [21] if the underlying matrix loop algebras are semisimple, and by the variational identity [22, 23] otherwise.

Darboux transformations are a direct and powerful approach to integrable systems, generating new solutions from known ones [24, 25], and they can be applied to diverse analytical solution situations [26–28], including solitons (see, e.g., [29, 30]) and rogue waves (see, e.g., [31, 32]), like Hirota's bilinear method [33]. The matrix Lie algebras which underlie various existing Darboux transformations are all semisimple [24, 25]. Integrable couplings are triangular multiple component integrable systems associated with non-semisimple matrix Lie algebras [34]. Therefore, it is natural to ask what kind of Darboux transformations there exists for integrable couplings. How can we construct Darboux transformations associated with non-semisimple matrix Lie algebras? This will create new studies to supplement the mathematical literature on Darboux transformations.

Let us recall the zero curvature equation formulation and the corresponding Darboux transformations for integrable systems [24, 25]. An integrable system of partial differential equations:

$$u_t = K(u) = K(x,t,u,u_x,\ldots), \tag{1.1}$$

is said to possess a zero curvature equation representation, if it is generated from a zero curvature equation:

$$U_t - V_x + [U,V] = 0, \tag{1.2}$$

where the two square matrices, U and V, called a Lax pair, belong to a matrix loop algebra [4, 14]. The above zero curvature equation is the compatibility condition of the spectral problems

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \phi_t = V\phi = V(u,\lambda)\phi, \tag{1.3}$$

where λ is the spectral parameter and ϕ is the vector eigenfunction. One of important tasks in the field of integrable systems is to construct, from the zero curvature equation formulation, Darboux transformations of the underlying spectral problems.

A transformation of $\phi' = D\phi$ and $u' = u'(u)$, with $D = D(u, \lambda)$ being a square matrix, is called a Darboux transformation of the spectral problems (1.3), if ϕ' satisfies the same type spectral problems:

$$\phi'_x = U'\phi' = U'(u', \lambda)\phi', \quad \phi'_t = V'\phi' = V'(u', \lambda)\phi', \tag{1.4}$$

where U' and V' should possess the same form as U and V, respectively. The matrix D is called a Darboux matrix of the spectral problems (1.3). The new Lax pair can be easily worked out:

$$U' = DUD^{-1} + D_x D^{-1}, \quad V' = DVD^{-1} + D_t D^{-1}. \tag{1.5}$$

Assume that U and V in the underlying Lax pair are $N \times N$ matrices. A Darboux matrix of first-order in λ can be taken as

$$D(\lambda) = \lambda I + S, \tag{1.6}$$

where I is the identity matrix of order N and S is an $N \times N$ matrix independent of λ. We introduce N distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ and their corresponding eigenfunctions:

$$\phi_x^{(s)} = U(u, \lambda_s)\phi^{(s)}, \quad \phi_t^{(s)} = V(u, \lambda_s)\phi^{(s)}, \quad 1 \leq s \leq N, \tag{1.7}$$

where u is a given solution to (1.1). Then a class of Darboux matrices can be generated [25, 26] from

$$S = H\Lambda H^{-1}, \tag{1.8}$$

where

$$H = (\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(N)}), \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N). \tag{1.9}$$

An integrable coupling of an integrable system (1.1) is a triangular integrable system of the following form [35, 36]:

$$\begin{cases} u_t = K(u), \\ v_t = T(u, v), \end{cases} \tag{1.10}$$

which is compactly written as

$$\bar{u}_t = \bar{K}(\bar{u}) \tag{1.11}$$

with the enlarged dependent variable $\bar{u} = (u^T, v^T)^T$. If T is nonlinear with respect to the second sub-vector v of dependent variables, the integrable coupling (1.10) is called nonlinear. An example of integrable couplings is the first-order perturbation system [35]:

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \end{cases} \tag{1.12}$$

where K' is the Gateaux derivative of K.

General Lie algebras possess semi-direct sum decompositions [37]:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \quad \mathfrak{g}\text{-semisimple}, \quad \mathfrak{g}_c\text{-solvable}, \tag{1.13}$$

and so do non-semisimple Lie algebras. Therefore, semi-direct sums of Lie algebras lay a foundation for constructing integrable couplings [14, 15]. The notion of semi-direct sums

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c \tag{1.14}$$

precisely means that the two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{g}_c$ satisfy

$$[\mathfrak{g}, \mathfrak{g}_c] \subseteq \mathfrak{g}_c, \tag{1.15}$$

where $[\mathfrak{g}, \mathfrak{g}_c] = \{[A, B] \,|\, A \in \mathfrak{g}, B \in \mathfrak{g}_c\}$, with $[\cdot, \cdot]$ denoting the Lie bracket of $\bar{\mathfrak{g}}$. Obviously, $\mathfrak{g}_c$ is an ideal Lie sub-algebra of $\bar{\mathfrak{g}}$. The subscript c indicates a contribution to the construction of coupling systems. It is recognized [14, 15] that integrable couplings are integrable systems associated with semi-direct sums of Lie algebras and thus non-semisimple Lie algebras, and Lax pairs for integrable couplings of integrable systems generated from (1.2) must be of the form:

$$\bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U(u, \lambda) & U_1(u, v, \lambda) \\ 0 & U(u, \lambda) \end{bmatrix}, \quad \bar{V}(\bar{u}, \lambda) = \begin{bmatrix} V(u, \lambda) & V_1(u, v, \lambda) \\ 0 & V(u, \lambda) \end{bmatrix}, \tag{1.16}$$

where $\bar{U}$ and $\bar{V}$ are elements in a non-semisimple matrix Lie algebra.

In this paper, we would like to propose a formulation of Darboux transformations for integrable couplings, which gives a positive answer to the previous two questions. Darboux transformations of integrable couplings will be explicitly presented, on the basis of matrix structures of non-semisimple matrix Lie algebras. Applications will be made to a kind of integrable couplings of the AKNS equations, and reductions to integrable couplings of the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (MKdV) equation will be created, along with exact one-soliton-like solutions to all the obtained integrable couplings. The resulting theory opens a new research area in the field of integrable systems. We will end the paper with a few concluding remarks.

2. Formulation of Darboux Transformations of Integrable Couplings

Assume that a soliton hierarchy of integrable couplings:

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}) = (K_m^T(u), T_m^T(u, v))^T, \quad \bar{u} = (u^T, v^T)^T, \; m \geq 0, \tag{2.1}$$

i.e.

$$u_{t_m} = K_m(u), \quad v_{t_m} = T_m(u, v), \quad m \geq 0, \tag{2.2}$$

is associated with a hierarchy of enlarged spectral problems:

$$\bar{\phi}_x = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \quad \bar{\phi}_{t_m} = \bar{V}^{[m]}(\bar{u}, \lambda)\bar{\phi}, \quad m \geq 0, \tag{2.3}$$

where the enlarged Lax pairs are block upper triangular matrices:

$$\bar{U}(\bar{u},\lambda)=\begin{bmatrix} U(u,\lambda) & U_1(\bar{u},\lambda) \\ 0 & U(u,\lambda), \end{bmatrix}, \quad \bar{V}^{[m]}(\bar{u},\lambda)=\begin{bmatrix} V^{[m]}(u,\lambda) & V_1^{[m]}(\bar{u},\lambda) \\ 0 & V^{[m]}(u,\lambda) \end{bmatrix}, \quad m\geq 0. \tag{2.4}$$

We will focus on the matrix blocks defined by

$$\begin{cases} U(u,\lambda)=J\lambda+P, \quad U_1(\bar{u},\lambda)=J_1\lambda+P_1, \\ V^{[m]}(u,\lambda)=\displaystyle\sum_{j=0}^{m} V_j(u)\lambda^{m-j}, \ V_1^{[m]}(\bar{u},\lambda)=\displaystyle\sum_{j=0}^{m} V_{1j}(\bar{u})\lambda^{m-j}, \quad m\geq 0, \end{cases} \tag{2.5}$$

where J and J_1 are two given constant diagonal matrices of order N, and P and P_1 are two $N\times N$ matrices, consisting of dependent variables, whose diagonal elements are all zero. Upon introducing

$$\bar{J}=\begin{bmatrix} J & J_1 \\ 0 & J \end{bmatrix}, \quad \bar{P}=\begin{bmatrix} P & P_1 \\ 0 & P \end{bmatrix}, \quad \bar{V}_j=\begin{bmatrix} V_j & V_{1j} \\ 0 & V_j \end{bmatrix}, \quad 0\leq j\leq m, \tag{2.6}$$

then the enlarged spectral problems (2.3) can be rewritten as

$$\bar{\phi}_x=\bar{U}\bar{\phi}=(\lambda\bar{J}+\bar{P})\bar{\phi}, \quad \bar{\phi}_{t_m}=\bar{V}^{[m]}\bar{\phi}=\sum_{j=0}^{m}\bar{V}_j\lambda^{m-j}\bar{\phi}, \ m\geq 0. \tag{2.7}$$

Let $\bar{D}=\bar{D}(x,t,\lambda)$ be a $2N\times 2N$ matrix. If $\bar{\phi}'=\bar{D}\bar{\phi}$ satisfies the same type spectral problems as (2.7):

$$\bar{\phi}'_x=\bar{U}'\bar{\phi}'=(\lambda\bar{J}+\bar{P}')\bar{\phi}', \quad \bar{\phi}'_{t_m}=\bar{V}^{[m]\prime}\bar{\phi}'=\sum_{j=0}^{m}V^{[m]}{}'_j\lambda^{m-j}\bar{\phi}', \quad m\geq 0, \tag{2.8}$$

where $\bar{P}'$ has the same form as $\bar{P}$:

$$\bar{P}'=\begin{bmatrix} P' & P_1' \\ 0 & P' \end{bmatrix}, \tag{2.9}$$

P' and P_1' being $N\times N$ matrices with zero diagonal elements, then the transformation

$$(\bar{\phi},\bar{P})\to(\bar{\phi}',\bar{P}') \tag{2.10}$$

presents a Darboux transformation of the enlarged spectral problems (2.7) and $\bar{D}$ is a Darboux matrix of (2.7).

We are interested in a class of Darboux matrices of first-order in λ:

$$\bar{D}=\lambda\bar{I}-\bar{S}, \quad \bar{I}=\mathrm{diag}(I,I), \tag{2.11}$$

where I is the identity matrix of order N as before, and $\bar{S}$ is a $2N\times 2N$ matrix to be determined. The first equation in (2.8) reads

$$(\lambda\bar{J}+\bar{P}')(\lambda\bar{I}-\bar{S})\bar{\phi}=((\lambda\bar{I}-\bar{S})\bar{\phi})_x=(\lambda\bar{I}-\bar{S})(\lambda\bar{J}+\bar{P})\bar{\phi}-\bar{S}_x\bar{\phi}.$$

The coefficients of the first and second powers of λ require

$$\bar{P}' = \bar{P} + [\bar{J}, \bar{S}], \tag{2.12}$$

and

$$\bar{S}_x = \bar{P}'\bar{S} - \bar{S}\bar{P} = \bar{P}\bar{S} - \bar{S}\bar{P} + \bar{J}\bar{S}'^2 - \bar{S}\bar{J}\bar{S}, \tag{2.13}$$

which is equivalent to

$$\bar{S}_x + [\bar{S}, \bar{J}\bar{S} + \bar{P}] = 0. \tag{2.14}$$

The second equation in (2.8) reads

$$\sum_{j=0}^{m} \bar{V}_i'\lambda^{m-j}(\lambda\bar{I} - \bar{S})\bar{\phi} = ((\lambda\bar{I} - \bar{S})\bar{\phi})_{t_m} = (\lambda\bar{I} - \bar{S})\sum_{j=0}^{m} \bar{V}_j\lambda^{m-j}\bar{\phi} - \bar{S}_{t_m}\bar{\phi}.$$

Comparing coefficients of powers of λ leads to

$$\bar{V}_0' = \bar{V}_0, \quad \bar{V}_{j+1}' = \bar{V}_{j+1} + \bar{V}_j'\bar{S} - \bar{S}\bar{V}_j, \quad 0 \le j \le m-1,$$

and

$$\bar{S}_{t_m} = \bar{V}_m'\bar{S} - \bar{S}\bar{V}_m.$$

These two equations equivalently requires

$$\bar{V}_0' = \bar{V}_0, \quad \bar{V}_j' = \bar{V}_j + \sum_{k=1}^{j}[\bar{V}_{j-k}, \bar{S}]\bar{S}^{k-1}, \quad 1 \le j \le m, \tag{2.15}$$

and

$$\bar{S}_{t_m} + \left[\bar{S}, \sum_{j=0}^{m} \bar{V}_j\bar{S}^{m-j}\right] = 0. \tag{2.16}$$

To sum up, we have the following theorem on Darboux matrices of the enlarged spectral problems (2.7).

Theorem 2.1. *$\bar{D} = \lambda\bar{I} - \bar{S}$ is a Darboux matrix of the enlarged spectral problems* (2.7) *if and only if $\bar{S}$ satisfies*

$$\bar{S}_x + [\bar{S}, \bar{J}\bar{S} + \bar{P}] = 0 \tag{2.17}$$

and

$$\bar{S}_{t_m} + \left[\bar{S}, \sum_{j=0}^{m} \bar{V}_j\bar{S}^{m-j}\right] = 0. \tag{2.18}$$

Moreover, the corresponding Darboux transformation generates the Bäcklund transformation of (2.1):

$$\bar{P}' = \bar{P} + [\bar{J}, \bar{S}]. \tag{2.19}$$

To present a Darboux transformation for an integrable coupling (2.1), we introduce N enlarged eigenfunctions associated with N eigenvalues: $\lambda_s, 1 \leq s \leq N$:

$$\bar{\phi}_x^{(s)} = \bar{U}(\bar{u}, \lambda_s)\bar{\phi}^{(s)}, \quad \bar{\phi}_{t_m}^{(s)} = \bar{V}^{[m]}(\bar{u}, \lambda_s)\bar{\phi}^{(s)}, \quad 1 \leq s \leq N, \tag{2.20}$$

where $\bar{u}$ is a given solution to (2.1). We denote

$$\bar{\phi}^{(s)} = (\phi_1^{(s)T}, \phi^{(s)T})^T, \quad 1 \leq s \leq N, \tag{2.21}$$

where $\phi^{(s)}$ and $\phi_1^{(s)}$ are N-dimensional column vector functions, and then formulate a block upper triangular matrix

$$\bar{H} = \begin{bmatrix} H & H_1 \\ 0 & H \end{bmatrix}, \quad H = [\phi^{(1)}, \ldots, \phi^{(N)}], \quad H_1 = [\phi_1^{(1)}, \ldots, \phi_1^{(N)}]. \tag{2.22}$$

Obviously, we have

$$H_x = JH\Lambda + PH, \quad H_{1x} = JH_1\Lambda + PH_1 + J_1H\Lambda + P_1H, \tag{2.23}$$

and

$$H_t = \sum_{j=0}^{m} V_jH\Lambda^{m-j}, \quad H_{1t} = \sum_{j=0}^{m} V_jH_1\Lambda^{m-j} + \sum_{j=0}^{m} V_{1j}H\Lambda^{m-j}, \tag{2.24}$$

where Λ is defined as in (1.9).

Theorem 2.2. *Let $\bar{H}$ be defined by* (2.22) *and $\bar{\Lambda} = \mathrm{diag}(\Lambda, \Lambda)$. Then $\bar{H}$ is invertible if and only if H is invertible. When H is invertible, $\bar{S} = \bar{H}\bar{\Lambda}\bar{H}^{-1}$ can be represented as*

$$\bar{S} = \begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \quad S = H\Lambda H^{-1}, \quad S_1 = -H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1}, \tag{2.25}$$

and $\bar{D} = \lambda\bar{I} - \bar{S}$ is a Darboux matrix of the enlarged spectral problems (2.7), *which leads to the Bäcklund transformation for the integrable coupling* (2.1):

$$P' = P + [J, S], \quad P_1' = P_1 + [J, S_1] + [J_1, S]. \tag{2.26}$$

Proof. Noting that $\bar{H}$ defined by (2.22) has a block upper triangular form and the diagonal blocks are the same as H, we know that $\bar{H}$ is invertible if and only if H is invertible.

Assume now that H is invertible. It is easy to see that $\bar{H}$ can be inverted blockwise as follows:

$$\bar{H}^{-1} = \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{-1} \\ 0 & H^{-1} \end{bmatrix},$$

and so, we can work out the formula for $\bar{S}$:

$$\begin{aligned}\bar{S} &= \bar{H}\bar{\Lambda}\bar{H}^{-1}\\ &= \begin{bmatrix} H & H_1 \\ 0 & H \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{1} \\ 0 & H^{-1} \end{bmatrix}\\ &= \begin{bmatrix} H\Lambda & H_1\Lambda \\ 0 & H\Lambda \end{bmatrix} \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{1} \\ 0 & H^{-1} \end{bmatrix}\\ &= \begin{bmatrix} H\Lambda H^{-1} & -H\Lambda H^{-1}H_1H^{1} + H_1\Lambda H^{-1} \\ 0 & H\Lambda H^{-1} \end{bmatrix},\end{aligned}$$

which exactly tells the expressions for S and S_1 in (2.25).

We need to show that $\bar{S}$ satisfies the two conditions in (2.17) and (2.18), i.e.

$$\bar{S}_x + [\bar{S}, \bar{J}\bar{S} + \bar{P}] = 0, \quad \bar{S}_{t_m} + \left[\bar{S}, \sum_{j=0}^{m} \bar{V}_j \bar{S}^{m-j}\right] = 0, \tag{2.27}$$

to guarantee that $\bar{D} = \lambda\bar{I} - \bar{S}$ is a Darboux matrix of the enlarged spectral problems (2.7).

Let us firstly show that the $(1,2)$th block in the x part of the conditions for $\bar{S}$ in (2.27) is equal to zero. On one hand, based on (2.25), we can compute that

$$\begin{aligned}S_{1x} &= -H_x\Lambda H^{-1}H_1H^{-1} + H\Lambda H^{-1}H_xH^{-1}H_1H^{-1} - H\Lambda H^{-1}H_{1x}H^{-1}\\ &\quad + H\Lambda H^{-1}H_1H^{-1}H_xH^{-1} + H_{1x}\Lambda H^{-1} - H_1\Lambda H^{-1}H_xH^{-1}\\ &= -(JH\Lambda + PH)\Lambda H^{-1}H_1H^{-1} + H\Lambda H^{-1}(JH\Lambda + \underline{PH})H^{-1}H_1H^{-1}\\ &\quad - H\Lambda H^{-1}(JH_1\Lambda + \underline{PH_1} + J_1H\Lambda + P_1H)H^{-1}\\ &\quad + H\Lambda H^{-1}H_1H^{-1}(JH\Lambda + PH)H^{-1}\\ &\quad + (JH_1\Lambda + PH_1 + J_1H\Lambda + P_1H)\Lambda H^{-1} - H_1\Lambda H^{-1}(JH\Lambda + PH)H^{-1}\\ &= -JH\Lambda^2H^{-1}H_1H^{-1} - PH\Lambda H^{-1}H_1H^{-1} + H\Lambda H^{-1}JH\Lambda H^{-1}H_1H^{-1}\\ &\quad - H\Lambda H^{-1}JH_1\Lambda H^{-1} - H\Lambda H^{-1}J_1H\Lambda H^{-1} - H\Lambda H^{-1}P_1\\ &\quad + H\Lambda H^{-1}H_1H^{-1}JH\Lambda H^{-1} + H\Lambda H^{-1}H_1H^{-1}P\\ &\quad + JH_1\Lambda^2H^{-1} + PH_1\Lambda H^{-1} + J_1H\Lambda^2H^{-1} + P_1H\Lambda H^{-1}\\ &\quad - H_1\Lambda H^{-1}JH\Lambda H^{-1} - H_1\Lambda H^{-1}P,\end{aligned}$$

where (2.23) was used and the two terms involving the underlined expressions cancel each other out. On the other hand, we have

$$[\bar{S}, \bar{J}\bar{S}+\bar{P}] = \left[\begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \begin{bmatrix} J & J_1 \\ 0 & J \end{bmatrix}\begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix} + \begin{bmatrix} P & P_1 \\ 0 & P \end{bmatrix}\right]$$
$$= \begin{bmatrix} [S, JS+P] & \begin{matrix}(SJS_1 + SJ_1S + S_1JS + SP_1 + S_1P \\ -JSS_1 - JS_1S - J_1S^2 - PS_1 - P_1S)\end{matrix} \\ 0 & [S, JS+P] \end{bmatrix}.$$

Therefore, again based on (2.25), the $(1,2)$th block of $[\bar{S}, \bar{J}\bar{S}+\bar{P}]$ can be computed as follows:

$$\begin{aligned}
[\bar{S}, \bar{J}\bar{S}+\bar{P}]_{12} &= SJS_1 + SJ_1S + S_1JS + SP_1 + S_1P \\
&\quad - JSS_1 - JS_1S - J_1S^2 - PS_1 - P_1S \\
&= H\Lambda H^{-1}J(-H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1}) \\
&\quad + H\Lambda H^{-1}J_1H\Lambda H^{-1} + (-H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1})JH\Lambda H^{-1} \\
&\quad + H\Lambda H^{-1}P_1 + (-H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1})P \\
&\quad - JH\Lambda H^{-1}(-H\Lambda H^{-1}H_1H^{-1} + \underline{H_1\Lambda H^{-1}}) \\
&\quad - J(\underline{-H\Lambda H^{-1}H_1H^{-1}} + H_1\Lambda H^{-1})H\Lambda H^{-1} - J_1H\Lambda^2H^{-1} \\
&\quad - P(-H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1}) - P_1H\Lambda H^{-1} \\
&= -H\Lambda H^{-1}JH\Lambda H^{-1}H_1H^{-1} + H\Lambda H^{-1}JH_1\Lambda H^{-1} \\
&\quad + H\Lambda H^{-1}J_1H\Lambda H^{-1} - H\Lambda H^{-1}H_1H^{-1}JH\Lambda H^{-1} \\
&\quad + H_1\Lambda H^{-1}JH\Lambda H^{-1} + H\Lambda H^{-1}P_1 - H\Lambda H^{-1}H_1H^{-1}P \\
&\quad + H_1\Lambda H^{-1}P + JH\Lambda^2H^{-1}H_1H^{-1} - JH_1\Lambda^2H^{-1} - J_1H\Lambda^2H^{-1} \\
&\quad + PH\Lambda H^{-1}H_1H^{-1} - PH_1\Lambda H^{-1} - P_1H\Lambda H^{-1},
\end{aligned}$$

where the two terms involving the underlined expressions cancel each other out. Now a careful but direct comparison between the above two equalities tells that the $(1,2)$th block in the x part of the conditions in (2.27) is equal to zero.

Let us secondly verify that the $(1,2)$th block in the t_m part of the conditions for $\bar{S}$ in (2.27) is equal to zero, too. We begin to observe

$$\bar{S}^{m-j} = \begin{bmatrix} S^{m-j} & \sum_{k=1}^{m-j} T_k \\ 0 & S^{m-j} \end{bmatrix},$$

where

$$T_k = \underbrace{S,\ldots,S}_{k-1}\, S_1 \underbrace{S,\ldots,S}_{m-j-k}, \quad 1 \le k \le m-j.$$

It directly follows that

$$[\bar{S}, \bar{V}_j \bar{S}^{m-j}] = \left[\begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \begin{bmatrix} V_j & V_{1j} \\ 0 & V_j \end{bmatrix} \begin{bmatrix} S^{m-j} & \sum_{k=1}^{m-j} T_k \\ 0 & S^{m-j} \end{bmatrix} \right]$$

$$= \begin{bmatrix} [S, V_j S^{m-j}] & \begin{bmatrix} SV_j \sum_{k=1}^{m-j} T_k + SV_{1j}S^{m-j} + S_1 V_j S^{m-j} \\ -V_j S^{m-j} S_1 - V_j \left(\sum_{k=1}^{m-j} T_k \right) S - V_{1j} S^{m-j+1} \end{bmatrix} \\ 0 & [S, V_j S^{m-j}] \end{bmatrix},$$

$$0 \leq j \leq m.$$

Note that applying (2.25), we have

$$\begin{aligned} \sum_{k=1}^{m-j} T_k &= \sum_{k=1}^{m-j} \underbrace{S, \ldots, S}_{k-1} S_1 \underbrace{S, \ldots, S}_{m-j-k} \\ &= \sum_{k=1}^{m-j} H\Lambda^{k-1}H^{-1}(-H\Lambda H^{-1}H_1 H^{-1} + H_1 \Lambda H^{-1}) H\Lambda^{m-j-k}H^{-1} \\ &= -\sum_{k=1}^{m-j} H\Lambda^k H^{-1} H_1 \Lambda^{m-j-k} H^{-1} + \sum_{k=1}^{m-j} H\Lambda^{k-1}H^{-1}H_1\Lambda^{m-j-k+1}H^{-1} \\ &= -H\Lambda^{m-j}H^{-1}H_1H^{-1} + H_1\Lambda^{m-j}H^{-1}. \end{aligned}$$

Therefore, we can then compute that

$$\begin{aligned} S_{1t} = {} & -H_t \Lambda H^{-1} H_1 H^{-1} + H\Lambda H^{-1} H_t H^{-1} H_1 H^{-1} - H\Lambda H^{-1} H_{1t} H^{-1} \\ & + H\Lambda H^{-1} H_1 H^{-1} H_t H^{-1} + H_{1t} \Lambda H^{-1} - H_1 \Lambda H^{-1} H_t H^{-1} \\ = {} & \sum_{j=0}^{m} [-V_j H \Lambda^{m-j+1} H^{-1} H_1 H^{-1} + H\Lambda H^{-1} V_j H \Lambda^{m-j} H^{-1} H_1 H^{-1} \\ & - H\Lambda H^{-1}(V_j H_1 \Lambda^{m-j} + V_{1j} H \Lambda^{m-j}) H^{-1} + H\Lambda H^{-1} H_1 H^{-1} V_j H \Lambda^{m-j} H^{-1} \\ & + (V_j H_1 \Lambda^{m-j} + V_{1j} H \Lambda^{m-j}) \Lambda H^{-1} - H_1 \Lambda H^{-1} V_j H \Lambda^{m-j} H^{-1}] \\ = {} & \sum_{j=0}^{m} (-V_j H \Lambda^{m-j+1} H^{-1} H_1 H^{-1} + H\Lambda H^{-1} V_j H \Lambda^{m-j} H^{-1} H_1 H^{-1} \\ & - H\Lambda H^{-1} V_j H_1 \Lambda^{m-j} H^{-1} - H\Lambda H^{-1} V_{1j} H \Lambda^{m-j} H^{-1} \\ & + H\Lambda H^{-1} H_1 H^{-1} V_j H \Lambda^{m-j} H^{-1} + V_j H_1 \Lambda^{m-j+1} H^{-1} \\ & + V_{1j} H \Lambda^{m-j+1} H^{-1} - H_1 \Lambda H^{-1} V_j H \Lambda^{m-j} H^{-1}), \end{aligned}$$

where (2.24) was used, and again by use of (2.25), the $(1,2)$th block of $[\bar{S}, \sum_{j=1}^{m} \bar{V}_j \bar{S}^{m-j}]$ is

$$\left[\bar{S}, \sum_{j=1}^{m} \bar{V}_j \bar{S}^{m-j}\right]_{12}$$

$$= \sum_{j=0}^{m} \left[SV_j \sum_{k=1}^{m-j} T_k + SV_{1j}S^{m-j} + S_1 V_j S^{m-j} - V_j S^{m-j} S_1 - V_j \left(\sum_{k=1}^{m-j} T_k \right) S - V_{1j} S^{m-j+1} \right]$$

$$= \sum_{j=0}^{m} [H\Lambda H^{-1} V_j (-H\Lambda^{m-j} H^{-1} H_1 H^{-1} + H_1 \Lambda^{m-j} H^{-1}) + H\Lambda H^{-1} V_{1j} H\Lambda^{m-j} H^{-1} + (-H\Lambda H^{-1} H_1 H^{-1} + H_1 \Lambda H^{-1}) V_j H\Lambda^{m-j} H^{-1} - V_j H\Lambda^{m-j} H^{-1} (-H\Lambda H^{-1} H_1 H^{-1} + \underline{H_1 \Lambda H^{-1}}) - V_j (\underline{-H\Lambda^{m-j} H^{-1} H_1 H^{-1}} + H_1 \Lambda^{m-j} H^{-1}) H\Lambda H^{-1} - V_{1j} H\Lambda^{m-j+1} H^{-1}]$$

$$= \sum_{j=0}^{m} (-H\Lambda H^{-1} V_j H\Lambda^{m-j} H^{-1} H_1 H^{-1} + H\Lambda H^{-1} V_j H_1 \Lambda^{m-j} H^{-1} + H\Lambda H^{-1} V_{1j} H\Lambda^{m-j} H^{-1} - H\Lambda H^{-1} H_1 H^{-1} V_j H\Lambda^{m-j} H^{-1} + H_1 \Lambda H^{-1} V_j H\Lambda^{m-j} H^{-1} + V_j H\Lambda^{m-j+1} H^{-1} H_1 H^{-1} - V_j H_1 \Lambda^{m-j+1} H^{-1} - V_{1j} H\Lambda^{m-j+1} H^{-1}),$$

where the two terms involving the underlined expressions cancel each other out. Now, a careful comparison between the above two equalities tells that the sum of S_{1t} and the $(1,2)$th block of $[\bar{S}, \sum_{j=1}^{m} \bar{V}_j \bar{S}^{m-j}]$ is equal to zero. That is, the $(1,2)$th block in the t_m part of the conditions for $\bar{S}$ in (2.27) is zero.

Note that it is a standard result (see, e.g., [25, 26]) that the two diagonal blocks in the conditions for $\bar{S}$ in (2.27) are the same and equal to zero, which precisely means that $D = \lambda I - S$ with $S = H\Lambda H^{-1}$ is a Darboux matrix of the uncoupled spectral problems associated with $u_{t_m} = K_m(u)$:

$$\phi_x = U(u,\lambda)\phi, \quad \phi_{t_m} = V^{[m]}(u,\lambda)\phi, \quad m \geq 0. \tag{2.28}$$

Therefore, the enlarged matrix $\bar{S}$ satisfies the two conditions in (2.27), and further, $\bar{D}$ is a Darboux matrix of the enlarged spectral problems (2.7).

Finally, a simple computation yields that

$$[\bar{J}, \bar{S}] = \begin{bmatrix} [J,S] & [J,S_1] + [J_1,S] \\ 0 & [J,S] \end{bmatrix},$$

where $\bar{J}$ is defined as in (2.6). Therefore, $\bar{P}'$ and $\bar{P}+[\bar{J},\bar{S}]$ have the same matrix form and the transformation (2.19), i.e. $\bar{P}'=\bar{P}+[\bar{J},\bar{S}]$, generates the concrete Bäcklund transformation presented in (2.26). This completes the proof. □

In the following section, we will apply the above generic formulation of Darboux transformations to a kind of integrable couplings of the AKNS equations.

3. Applications to a Kind of Integrable Couplings of the AKNS Equations

3.1. *A kind of integrable couplings of the AKNS equations*

Let us consider a novel enlarged AKNS spectral problem:

$$\bar{\phi}_x=\bar{U}\bar{\phi}=\bar{U}(\bar{u},\lambda)\bar{\phi}, \tag{3.1}$$

where the enlarged spectral matrix is chosen as

$$\bar{U}=\begin{bmatrix} U & U_1 \\ 0 & U \end{bmatrix},\quad U=\begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},\quad U_1=\begin{bmatrix} -\lambda & r \\ s & \lambda \end{bmatrix}, \tag{3.2}$$

and we denote the enlarged potential and eigenfunction by

$$\begin{cases} \bar{u}=(u^T,v^T)^T,\quad u=(p,q)^T,\quad v=(r,s)^T, \\ \bar{\phi}=(\psi^T,\phi^T)^T,\quad \psi=(\psi_1,\psi_2)^T,\quad \phi=(\phi_1,\phi_2)^T. \end{cases} \tag{3.3}$$

We point out that U_1 introduced above depends explicitly on the spectral parameter λ, which is a new try to generate integrable couplings.

Assume that

$$\bar{W}=\begin{bmatrix} W & W_1 \\ 0 & W \end{bmatrix},\quad W=\begin{bmatrix} a & b \\ c & -a \end{bmatrix},\quad W_1=\begin{bmatrix} e & f \\ g & -e \end{bmatrix}. \tag{3.4}$$

Then the enlarged stationary zero curvature equation, $\bar{W}_x=[\bar{U},\bar{W}]$, results in

$$W_x=[U,W],\quad W_{1x}=[U,W_1]+[U_1,W], \tag{3.5}$$

which is equivalent to

$$\begin{cases} a_x=pc-qb, \\ b_x=-2\lambda b-2pa, \\ c_x=2qa+2\lambda c, \end{cases} \tag{3.6}$$

and

$$\begin{cases} e_x=pg-qf+rc-sb, \\ f_x=-2\lambda f-2pe-2\lambda b-2ra, \\ g_x=2qe+2\lambda g+2sa+2\lambda c, \end{cases} \tag{3.7}$$

respectively. Define

$$\begin{cases} W = \sum_{i=0}^{\infty} W_i \lambda^{-i}, \quad W_i = \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix}, \qquad i \geq 0, \\ W_1 = \sum_{i=0}^{\infty} W_{1,i} \lambda^{-i}, \quad W_{1,i} = \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix}, \quad i \geq 0. \end{cases} \tag{3.8}$$

Substituting those into the equations in (3.6) and (3.7), and comparing the coefficients of powers of λ, we obtain the recursion relations to define W and W_1:

$$\begin{cases} a_{0x} = 0, \quad b_0 = 0, \quad c_0 = 0, \\ a_{ix} = pc_i - qb_i, \qquad i \geq 1, \\ b_i = -\frac{1}{2} b_{i-1,x} - pa_{i-1}, \quad i \geq 1, \\ c_i = \frac{1}{2} c_{i-1,x} - qa_{i-1}, \quad i \geq 1, \end{cases} \tag{3.9}$$

and

$$\begin{cases} e_{0x} = 0, \quad f_0 = 0, \quad g_0 = 0, \\ e_{ix} = pg_i - qf_i + rc_i - sb_i, \qquad i \geq 1, \\ f_i = -\frac{1}{2} f_{i-1,x} - pe_{i-1} - b_i - ra_{i-1}, \quad i \geq 1, \\ g_i = \frac{1}{2} g_{i-1,x} - qe_{i-1} - sa_{i-1} - c_i, \quad i \geq 1, \end{cases} \tag{3.10}$$

respectively. We take the initial values

$$a_0 = \alpha, \quad e_0 = \beta, \tag{3.11}$$

where α and β are arbitrary constants, real or complex numbers, and choose the constants of integration to be zero:

$$a_i|_{u=0} = 0, \quad e_i|_{\bar{u}=0} = 0, \quad i \geq 1. \tag{3.12}$$

This way, we can determine the sequence of $a_i, b_i, c_i, e_i, f_i, g_i, i \geq 1$, from (3.9) and (3.10) uniquely. The first few sets can be worked out as follows:

$$b_1 = -\alpha p, \quad c_1 = -\alpha q, \quad a_1 = 0;$$

$$b_2 = \frac{1}{2}\alpha p_x, \quad c_2 = -\frac{1}{2}\alpha q_x, \quad a_2 = -\frac{1}{2}\alpha pq;$$

$$b_3 = -\alpha\left(\frac{1}{4}p_{xx} - \frac{1}{2}p^2 q\right), \quad c_3 = -\alpha\left(\frac{1}{4}q_{xx} - \frac{1}{2}pq^2\right), \quad a_3 = -\frac{1}{4}\alpha(pq_x - qp_x);$$

$$b_4 = \alpha\left(\frac{1}{8}p_{xxx} - \frac{3}{4}pqp_x\right), \quad c_4 = -\alpha\left(\frac{1}{8}q_{xxx} - \frac{3}{4}pqq_x\right),$$

$$a_4 = -\alpha\left(\frac{1}{8}pq_{xx} + \frac{1}{8}qp_{xx} - \frac{1}{8}p_x q_x - \frac{3}{8}p^2 q^2\right);$$

and

$$
\begin{aligned}
f_1 &= \alpha(p-r) - \beta p, \quad g_1 = \alpha(q-s) - \beta q, \quad e_1 = 0;\\
f_2 &= -\alpha\left(p_x - \frac{1}{2}r_x\right) + \frac{1}{2}\beta p_x, \quad g_2 = \alpha\left(q_x - \frac{1}{2}s_x\right) - \frac{1}{2}\beta q_x,\\
e_2 &= \alpha\left(pq - \frac{1}{2}ps - \frac{1}{2}qr\right) - \frac{1}{2}\beta pq;\\
f_3 &= \alpha\left(\frac{3}{4}p_{xx} - \frac{1}{4}r_{xx} - \frac{3}{2}p^2q + \frac{1}{2}p^2s + pqr\right) - \beta\left(\frac{1}{4}p_{xx} - \frac{1}{2}p^2q\right),\\
g_3 &= \alpha\left(\frac{3}{4}q_{xx} - \frac{1}{4}s_{xx} - \frac{3}{2}pq^2 + pqs + \frac{1}{2}q^2r\right) - \beta\left(\frac{1}{4}q_{xx} - \frac{1}{2}pq^2\right),\\
e_3 &= \alpha\left[\frac{3}{4}(pq_x - qp_x) - \frac{1}{4}(ps_x - qr_x + rq_x - sp_x)\right] - \frac{1}{4}\beta(pq_x - qp_x);\\
f_4 &= -\alpha\left(\frac{1}{2}p_{xxx} - \frac{1}{8}r_{xxx} - 3pqp_x + \frac{3}{4}psp_x + \frac{3}{4}pqr_x + \frac{3}{4}qrp_x\right)\\
&\quad + \beta\left(\frac{1}{8}p_{xxx} - \frac{3}{4}pqp_x\right),\\
g_4 &= \alpha\left(\frac{1}{2}q_{xxx} - \frac{1}{8}s_{xxx} - 3pqq_x + \frac{3}{4}psq_x + \frac{3}{4}pqs_x + \frac{3}{4}qrq_x\right)\\
&\quad - \beta\left(\frac{1}{8}q_{xxx} - \frac{3}{4}pqq_x\right),\\
e_4 &= \alpha\left[\frac{1}{2}(pq_{xx} + qp_{xx} - p_xq_x) - \frac{1}{8}(ps_{xx} + sp_{xx} + qr_{xx} + rq_{xx} - p_xs_x - q_xr_x)\right.\\
&\quad \left. - \frac{3}{2}p^2q^2 + \frac{3}{4}(p^2qs + pq^2r)\right] - \beta\left[\frac{1}{8}(pq_{xx} + qp_{xx} - p_xq_x) - \frac{3}{8}p^2q^2\right].
\end{aligned}
$$

Now, we take the enlarged Lax matrices as

$$
\bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_1^{[m]} \\ 0 & V^{[m]} \end{bmatrix}, \quad m \geq 0, \tag{3.13}
$$

where the matrix blocks are defined by

$$
\begin{cases}
V^{[m]} = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & -a^{[m]} \end{bmatrix} = (\lambda^m W)_+ = \displaystyle\sum_{i=0}^{m} W_i \lambda^{m-i},\\
V_1^{[m]} = \begin{bmatrix} e^{[m]} & f^{[m]} \\ g^{[m]} & -e^{[m]} \end{bmatrix} = (\lambda^m W_1)_+ = \displaystyle\sum_{i=0}^{m} W_{1,i} \lambda^{m-i},
\end{cases}
\quad m \geq 0, \tag{3.14}
$$

to introduce the time evolution of the enlarged eigenfunction

$$\bar{\phi}_{t_m} = \bar{V}^{[m]}\bar{\phi} = \bar{V}^{[m]}(\bar{u}, \lambda)\bar{\phi}, \quad m \geq 0. \tag{3.15}$$

Then based on the recursion relations in (3.9) and (3.10), the compatibility conditions of the enlarged spectral problems (3.1) and (3.15), i.e. the enlarged zero curvature equations:

$$\bar{U}_{t_m} - \bar{V}^{[m]}_x + [\bar{U}, \bar{V}^{[m]}] = 0, \quad m \geq 0, \tag{3.16}$$

which is equivalent to

$$\begin{cases} U_{t_m} - V^{[m]}_x + [U, V^{[m]}] = 0, \\ U_{1t_m} - V^{[m]}_{1x} + [U, V^{[m]}_1] + [U_1, V^{[m]}] = 0, \end{cases} \quad m \geq 0,$$

generate a hierarchy of AKNS integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} - 2b_{m+1} \\ 2g_{m+1} + 2c_{m+1} \end{bmatrix} = \bar{\Phi}^m \begin{bmatrix} 2\alpha p \\ -2\alpha q \\ 2(\beta p + \alpha r) \\ -2(\beta q + \alpha s) \end{bmatrix}, \quad m \geq 0, \tag{3.17}$$

where the enlarged hereditary recursion operator $\bar{\Phi}$ reads

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_c - \Phi & \Phi \end{bmatrix}, \tag{3.18}$$

with Φ and Φ_c being determined by

$$\Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}q \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix},$$

$$\Phi_c = \begin{bmatrix} r\partial^{-1}q + p\partial^{-1}s & r\partial^{-1}p + p\partial^{-1}r \\ -s\partial^{-1}q - q\partial^{-1}s & -s\partial^{-1}p - q\partial^{-1}r \end{bmatrix}. \tag{3.19}$$

The first few sets of integrable couplings in (3.17) can be worked out as follows:

$$\bar{u}_{t_1} = \bar{K}_1(\bar{u}) = \begin{bmatrix} -\alpha p_x \\ -\alpha q_x \\ \alpha(p_x - r_x) - \beta p_x \\ \alpha(q_x - s_x) - \beta q_x \end{bmatrix}, \tag{3.20}$$

$$\bar{u}_{t_2} = \bar{K}_2(\bar{u}) = \begin{bmatrix} \alpha\left(\frac{1}{2}p_{xx} - p^2 q\right) \\ -\alpha\left(\frac{1}{2}q_{xx} - q^2 p\right) \\ -\alpha\left(p_{xx} - \frac{1}{2}r_{xx} - 2p^2 q + p^2 s + 2pqr\right) + \beta\left(\frac{1}{2}p_{xx} - p^2 q\right) \\ \alpha\left(q_{xx} - \frac{1}{2}s_{xx} - 2pq^2 + q^2 r + 2pqs\right) - \beta\left(\frac{1}{2}q_{xx} - pq^2\right) \end{bmatrix}, \tag{3.21}$$

$$\begin{aligned} \bar{u}_{t_3} &= \bar{K}_3(\bar{u}) \\ &= \begin{bmatrix} -\alpha\left(\frac{1}{4}p_{xxx} - \frac{3}{2}pqp_x\right) \\ -\alpha\left(\frac{1}{4}q_{xxx} - \frac{3}{2}qpq_x\right) \\ \alpha\left(\frac{3}{4}p_{xxx} - \frac{1}{4}r_{xxx} - \frac{9}{2}pqp_x + \frac{3}{2}psp_x + \frac{3}{2}pqr_x + \frac{3}{2}qrp_x\right) \\ \quad - \beta\left(\frac{1}{4}p_{xxx} - \frac{3}{2}pqp_x\right) \\ \alpha\left(\frac{3}{4}q_{xxx} - \frac{1}{4}s_{xxx} - \frac{9}{2}pqq_x + \frac{3}{2}psq_x + \frac{3}{2}pqs_x + \frac{3}{2}qrq_x\right) \\ \quad - \beta\left(\frac{1}{4}q_{xxx} - \frac{3}{2}pqq_x\right) \end{bmatrix}. \end{aligned} \tag{3.22}$$

3.2. *Darboux transformations of the AKNS integrable couplings*

We would here like to apply Theorem 2.2 on Darboux transformations to the integrable couplings in (3.17). In this case, we have

$$U = U(u,\lambda) = \lambda J + P, \quad U_1 = U_1(\bar{u},\lambda) = \lambda J_1 + P_1, \tag{3.23}$$

$$J = J_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}. \tag{3.24}$$

Take two different eigenvalues λ_1 and λ_2, and denote

$$\phi_{jk} = \phi_j(\lambda_k), \quad \psi_{jk} = \psi_j(\lambda_k), \quad j,k = 1,2. \tag{3.25}$$

Then, we have

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad H = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad H_1 = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}, \tag{3.26}$$

which allows us to compute that

$$S = H\Lambda H^{-1}, \quad S_1 = -H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1}. \tag{3.27}$$

The Darboux matrix of first-order in λ presented in Theorem 2.2 reads

$$\bar{D} = \lambda\bar{I} - \bar{S}, \quad \bar{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \bar{S} = \begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \tag{3.28}$$

where $I = \text{diag}(1,1)$ and the corresponding Darboux transformation is given by

$$\bar{\phi}' = \bar{D}\bar{\phi}, \quad \bar{P}' = \bar{P} + [\bar{J}, \bar{S}], \tag{3.29}$$

which leads to the associated Bäcklund transformation:

$$P' = P + [J, S], \quad P_1' = P_1 + [J_1, S] + [J, S_1]. \tag{3.30}$$

Reformulate an initial solution (P, P_1) and eigenfunction $\bar{\phi}$ as

$$P = P^{[0]} = \begin{bmatrix} 0 & p^{[0]} \\ q^{[0]} & 0 \end{bmatrix}, \quad P_1 = P_1^{[0]} = \begin{bmatrix} 0 & r^{[0]} \\ s^{[0]} & 0 \end{bmatrix}, \quad \bar{\phi} = \bar{\phi}^{[0]} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \phi_1 \\ \phi_2 \end{bmatrix}, \tag{3.31}$$

and a new solution (P', P_1') and eigenfunction $\bar{\phi}'$ as

$$P' = P^{[1]} = \begin{bmatrix} 0 & p^{[1]} \\ q^{[1]} & 0 \end{bmatrix}, \quad P_1' = P_1^{[1]} = \begin{bmatrix} 0 & r^{[1]} \\ s^{[1]} & 0 \end{bmatrix}, \quad \bar{\phi}' = \bar{\phi}^{[1]} = \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix}. \tag{3.32}$$

Then, we arrive at the Bäcklund transformation generating new solutions from known ones:

$$P^{[1]} = P^{[0]} + [J, S], \quad P_1^{[1]} = P_1^{[0]} + [J_1, S] + [J, S_1], \tag{3.33}$$

which precisely defines

$$\begin{cases} p^{[1]} = p^{[0]} + \frac{2(\lambda_1-\lambda_2)\phi_{11}\phi_{12}}{\phi_{11}\phi_{22}-\phi_{12}\phi_{21}}, \\ q^{[1]} = q^{[0]} + \frac{2(\lambda_1-\lambda_2)\phi_{21}\phi_{22}}{\phi_{11}\phi_{22}-\phi_{12}\phi_{21}}, \\ r^{[1]} = r^{[0]} + \frac{2(\lambda_1-\lambda_2)\left(\phi_{11}{}^2\phi_{12}\phi_{22}-\phi_{11}{}^2\phi_{12}\psi_{22}+\phi_{11}{}^2\phi_{22}\psi_{12}-\phi_{11}\phi_{12}{}^2\phi_{21}+\phi_{11}\phi_{12}{}^2\psi_{21}-\phi_{12}{}^2\phi_{21}\psi_{11}\right)}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\ s^{[1]} = s^{[0]} + \frac{2(\lambda_1-\lambda_2)\left(\phi_{11}\phi_{21}\phi_{22}{}^2+\phi_{11}\phi_{22}{}^2\psi_{21}-\phi_{12}\phi_{21}{}^2\phi_{22}-\phi_{12}\phi_{21}{}^2\psi_{22}+\phi_{21}{}^2\phi_{22}\psi_{12}-\phi_{21}\phi_{22}{}^2\psi_{11}\right)}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}. \end{cases} \tag{3.34}$$

To obtain Darboux transformations of higher-order in λ, we can iterate the Darboux transformation established in Theorem 2.2 a few times. This also means that we can further compute a newer solution $(p^{[2]}, q^{[2]}, r^{[2]}, s^{[2]})$ associated with two different eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ from the solution $(p^{[1]}, q^{[1]}, r^{[1]}, s^{[1]})$ just computed:

$$P^{[2]} = P^{[1]} + [J, \tilde{S}], \quad P_1^{[2]} = P_1^{[1]} + [J_1, \tilde{S}] + [J, \tilde{S}_1], \tag{3.35}$$

where $\tilde{S}$ and $\tilde{S}_1$ are defined similarly by

$$\tilde{S} = \tilde{H}\tilde{\Lambda}\tilde{H}^{-1}, \quad \tilde{S}_1 = -\tilde{H}\tilde{\Lambda}\tilde{H}^{-1}\tilde{H}_1\tilde{H}^{-1} + \tilde{H}_1\tilde{\Lambda}\tilde{H}^{-1}, \tag{3.36}$$

through a new set of matrices

$$\tilde{\Lambda} = \begin{bmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{\phi}_{13} & \tilde{\phi}_{14} \\ \tilde{\phi}_{23} & \tilde{\phi}_{24} \end{bmatrix}, \quad \tilde{H}_1 = \begin{bmatrix} \tilde{\psi}_{13} & \tilde{\psi}_{14} \\ \tilde{\psi}_{23} & \tilde{\psi}_{24} \end{bmatrix}, \tag{3.37}$$

where $\tilde{\phi}_{jk} = \tilde{\phi}_j(\tilde{\lambda}_k)$ and $\tilde{\psi}_{jk} = \tilde{\psi}_j(\tilde{\lambda}_k), j, k = 1, 2$.

While computing examples below, we denote the time variable only by t, but not t_m, for convenience's sake.

Example 1. The Second-Order System

Let us firstly consider the integrable coupling system $\bar{u}_t = \bar{K}_2$ defined by (3.21). Note that from (3.14), we have

$$\begin{cases} V^{[2]} = \begin{bmatrix} a^{[2]} & b^{[2]} \\ c^{[2]} & -a^{[2]} \end{bmatrix} = \sum_{i=0}^{2} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{2-i}, \\ V_1^{[2]} = \begin{bmatrix} e^{[2]} & f^{[2]} \\ g^{[2]} & -e^{[2]} \end{bmatrix} = \sum_{i=0}^{2} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{2-i}, \end{cases} \tag{3.38}$$

where

$$\begin{cases} a^{[2]} = \alpha\left(\lambda^2 - \frac{1}{2}pq\right), \quad b^{[2]} = \alpha\left(-p\lambda + \frac{1}{2}p_x\right), \quad c^{[2]} = \alpha\left(-q\lambda - \frac{1}{2}q_x\right), \\ e^{[2]} = \alpha\left(pq - \frac{1}{2}ps - \frac{1}{2}qr\right) + \beta\left(\lambda^2 - \frac{1}{2}pq\right), \\ f^{[2]} = \alpha\left[(p-r)\lambda - p_x + \frac{1}{2}r_x\right] + \beta\left(-p\lambda + \frac{1}{2}p_x\right), \\ g^{[2]} = \alpha\left[(q-s)\lambda + q_x - \frac{1}{2}s_x\right] + \beta\left(-q\lambda - \frac{1}{2}q_x\right). \end{cases} \tag{3.39}$$

Now, starting from the zero seed solution and solving the corresponding spectral problems (3.1) and (3.15) generates the following eigenfunctions associated with an eigenvalue λ:

$$\begin{cases} \psi_1 = \chi_1(\lambda) = (\beta\lambda^2\mu_1 t - \lambda\mu_1 x + \mu_3)e^{\lambda(\alpha\lambda t - x)}, \\ \psi_2 = \chi_2(\lambda) = -(\beta\lambda^2\mu_2 t - \lambda\mu_2 x - \mu_4)e^{-\lambda(\alpha\lambda t - x)}, \\ \phi_1 = \chi_3(\lambda) = \mu_1 e^{\lambda(\alpha\lambda t - x)}, \\ \phi_2 = \chi_4(\lambda) = \mu_2 e^{-\lambda(\alpha\lambda t - x)}, \end{cases} \tag{3.40}$$

where $\mu_i, 1 \le i \le 4$, are arbitrary constants. To obtain analytical solutions by the proposed Darboux transformation, we choose the following two vectors of eigenfunctions:

$$\begin{aligned} \psi_1(\lambda_1) = \chi_1(\lambda_1), \quad \psi_2(\lambda_1) = \chi_2(\lambda_1), \\ \phi_1(\lambda_1) = \chi_3(\lambda_1), \quad \phi_2(\lambda_1) = \chi_4(\lambda_1), \end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\psi_1(\lambda_2) &= -\chi_1(\lambda_2), \quad \psi_2(\lambda_2) = \chi_2(\lambda_2),\\ \phi_1(\lambda_2) &= -\chi_3(\lambda_2), \quad \phi_2(\lambda_2) = \chi_4(\lambda_2),\end{aligned} \tag{3.42}$$

associated with two different eigenvalues λ_1 and λ_2, respectively. Then by the Bäcklund transformation (3.34), we obtain a one-soliton-like solution to the AKNS integrable coupling system (3.21):

$$\begin{cases} p = -\dfrac{\mu_1}{\mu_2}(\lambda_1 - \lambda_2)e^{\alpha({\lambda_1}^2+{\lambda_2}^2)t-(\lambda_1+\lambda_2)x}\mathrm{sech}\xi,\\ q = \dfrac{\mu_2}{\mu_1}(\lambda_1 - \lambda_2)e^{-\alpha({\lambda_1}^2+{\lambda_2}^2)t+(\lambda_1+\lambda_2)x}\mathrm{sech}\xi,\\ r = -\dfrac{1}{2\mu_2^2}(\lambda_1-\lambda_2)\eta\,\mathrm{sech}^2\xi,\\ s = -\dfrac{1}{2\mu_1^2}(\lambda_1-\lambda_2)\zeta\,\mathrm{sech}^2\xi, \end{cases} \tag{3.43}$$

where

$$\begin{cases} \xi = (\lambda_1-\lambda_2)[\alpha(\lambda_1+\lambda_2)t - x],\\ \eta = [\mu_1\mu_2(2\beta{\lambda_2}^2 t - 2\lambda_2 x + 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{2\lambda_1(\alpha\lambda_1 t - x)}\\ \qquad + [\mu_1\mu_2(2\beta{\lambda_1}^2 t - 2\lambda_1 x + 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{2\lambda_2(\alpha\lambda_2 t - x)},\\ \zeta = [\mu_1\mu_2(2\beta{\lambda_2}^2 t - 2\lambda_2 x - 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{-2\lambda_1(\alpha\lambda_1 t - x)}\\ \qquad + [\mu_1\mu_2(2\beta{\lambda_1}^2 t - 2\lambda_1 x - 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{-2\lambda_2(\alpha\lambda_2 t - x)}. \end{cases}$$

Example 2. The Third-Order System

Let us secondly consider the integrable coupling system $\bar{u}_t = \bar{K}_3$ defined by (3.22). Note that from (3.14), we have

$$\begin{cases} V^{[3]} = \begin{bmatrix} a^{[3]} & b^{[3]} \\ c^{[3]} & -a^{[3]} \end{bmatrix} = \displaystyle\sum_{i=0}^{3} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{3-i},\\ V_1^{[3]} = \begin{bmatrix} e^{[3]} & f^{[3]} \\ g^{[3]} & -e^{[3]} \end{bmatrix} = \displaystyle\sum_{i=0}^{3} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{3-i}, \end{cases} \tag{3.44}$$

where

$$\begin{cases} a^{[3]} = \alpha\left[\lambda^3 - \dfrac{1}{2}pq\lambda + \dfrac{1}{4}(p_x q - p q_x)\right],\\ b^{[3]} = \alpha\left(-p\lambda^2 + \dfrac{1}{2}p_x\lambda - \dfrac{1}{4}p_{xx} + \dfrac{1}{2}p^2 q\right),\\ c^{[3]} = \alpha\left(-q\lambda^2 - \dfrac{1}{2}q_x\lambda - \dfrac{1}{4}q_{xx} + \dfrac{1}{2}pq^2\right), \end{cases} \tag{3.45}$$

and

$$\begin{cases} e^{[3]} = \alpha\left[\left(pq - \frac{1}{2}ps - \frac{1}{2}qr\right)\lambda + \frac{3}{4}(pq_x - qq_x) - \frac{1}{4}(ps_x - qr_x + rq_x - sp_x)\right] \\ \qquad + \beta\left[\lambda^3 - \frac{1}{2}pq\lambda - \frac{1}{4}(pq_x - qp_x)\right], \\ f^{[3]} = \alpha\left[(p-r)\lambda^2 - \left(p_x - \frac{1}{2}r_x\right)\lambda + \frac{3}{4}p_{xx} - \frac{1}{4}r_{xx} - \frac{3}{2}p^2q + \frac{1}{2}p^2s + pqr\right] \\ \qquad + \beta\left(-p\lambda^2 + \frac{1}{2}p_x\lambda - \frac{1}{4}p_{xx} + \frac{1}{2}p^2q\right), \\ g^{[3]} = \alpha\left[(q-s)\lambda^2 + \left(q_x - \frac{1}{2}s_x\right)\lambda + \frac{3}{4}q_{xx} - \frac{1}{4}s_{xx} - \frac{3}{2}pq^2 + \frac{1}{2}q^2r + pqs\right] \\ \qquad + \beta\left(-q\lambda^2 - \frac{1}{2}q_x\lambda - \frac{1}{4}q_{xx} + \frac{1}{2}pq^2\right). \end{cases} \tag{3.46}$$

Now, starting from the zero seed solution, we have the eigenfunctions of the corresponding spectral problems (3.1) and (3.15) associated with an eigenvalue λ:

$$\begin{cases} \psi_1 = (\beta\lambda^3\mu_1 t - \lambda\mu_1 x + \mu_3)e^{\lambda(\alpha\lambda^2 t - x)}, \\ \psi_2 = -(\beta\lambda^3\mu_2 t - \lambda\mu_2 x - \mu_4)e^{-\lambda(\alpha\lambda^2 t - x)}, \\ \phi_1 = \mu_1 e^{\lambda(\alpha\lambda^2 t - x)}, \\ \phi_2 = \mu_2 e^{-\lambda(\alpha\lambda^2 t - x)}, \end{cases} \tag{3.47}$$

where $\mu_i, 1 \leq i \leq 4$, are arbitrary constants. In the proposed Darboux transformation, we take the same type of vector eigenfunctions associated with two different eigenvalues λ_1 and λ_2 as did in the previous example. Again by the Bäcklund transformation (3.34), we obtain a one-soliton-like solution of the AKNS integrable coupling system (3.22):

$$\begin{cases} p = -\frac{\mu_1}{\mu_2}(\lambda_1 - \lambda_2)e^{(\lambda_1+\lambda_2)[\alpha({\lambda_1}^2 - \lambda_1\lambda_2 + {\lambda_2}^2)t - x]}\mathrm{sech}\xi, \\ q = \frac{\mu_2}{\mu_1}(\lambda_1 - \lambda_2)e^{-(\lambda_1+\lambda_2)[\alpha({\lambda_1}^2 - \lambda_1\lambda_2 + {\lambda_2}^2)t + x]}\mathrm{sech}\xi, \\ r = -\frac{1}{2\mu_2^2}(\lambda_1 - \lambda_2)\eta\,\mathrm{sech}^2\xi, \\ s = -\frac{1}{2\mu_1^2}(\lambda_1 - \lambda_2)\zeta\,\mathrm{sech}^2\xi, \end{cases} \tag{3.48}$$

where

$$\begin{cases} \xi = (\lambda_1 - \lambda_2)[\alpha(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)t - x], \\ \eta = [\mu_1\mu_2(2\beta{\lambda_2}^3 t - 2\lambda_2 x + 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{2\lambda_1(\alpha{\lambda_1}^2 t - x)} \\ \qquad + [\mu_1\mu_2(2\beta{\lambda_1}^3 t - 2\lambda_1 x + 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{2\lambda_2(\alpha{\lambda_2}^2 t - x)}, \\ \zeta = [\mu_1\mu_2(2\beta{\lambda_2}^3 t - 2\lambda_2 x - 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{-2\lambda_1(\alpha{\lambda_1}^2 t - x)} \\ \qquad + [\mu_1\mu_2(2\beta{\lambda_1}^3 t - 2\lambda_1 x - 1) - \mu_1\mu_4 + \mu_2\mu_3]e^{-2\lambda_2(\alpha{\lambda_2}^2 t - x)}. \end{cases}$$

3.3. *Reductions to the NLS and MKdV equations*

Let us make the following type of reduction:

$$p = -q^*, \quad r = -s^*, \tag{3.49}$$

where the superscript $*$ denotes the complex conjugate, and assume that

$$V^{[m]T}(-\lambda^*) = -(V^{[m]}(\lambda))^*, \quad V_1^{[m]T}(-\lambda^*) = -(V_1^{[m]}(\lambda))^*, \tag{3.50}$$

which precisely means that

$$\begin{cases} a^{[m]}(-\lambda^*) = -(a^{[m]}(\lambda))^*, \quad b^{[m]}(-\lambda^*) = -(c^{[m]}(\lambda))^*, \\ e^{[m]}(-\lambda^*) = -(e^{[m]}(\lambda))^*, \quad f^{[m]}(-\lambda^*) = -(g^{[m]}(\lambda))^*. \end{cases}$$

Then, if $\bar{\phi} = (\psi_1, \psi_2, \phi_1, \phi_2)^T$ is an eigenfunction of the spectral problems (3.1) and (3.15) associated with $\lambda = \lambda_1$, we can see that $\bar{\psi} = (\psi_2^*, -\psi_1^*, \phi_2^*, -\phi_1^*)^T$ is an eigenfunction of the spectral problems (3.1) and (3.15) associated with $\lambda = -\lambda_1^*$. Therefore, upon taking

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1^* \end{bmatrix}, \quad H = \begin{bmatrix} \phi_{11} & \phi_{21}^* \\ \phi_{21} & -\phi_{11}^* \end{bmatrix}, \quad H_1 = \begin{bmatrix} \psi_{11} & \psi_{21}^* \\ \psi_{21} & -\psi_{11}^* \end{bmatrix},$$

we can work out the associated Bäcklund transformation

$$\begin{cases} p^{[1]} = p^{[0]} - \frac{2(\lambda_1+\lambda_1^*)\phi_{11}\phi_{21}^*}{\phi_{11}\phi_{11}^*+\phi_{21}^*\phi_{21}}, \\ r^{[1]} = r^{[0]} + \frac{2(\lambda_1+\lambda_1^*)(-{\phi_{11}}^2\phi_{21}^*\phi_{11}^*+{\phi_{11}}^2\phi_{21}^*\psi_{11}^*-{\phi_{11}}^2\phi_{11}^*\psi_{21}^*-\phi_{11}{\phi_{21}^*}^2\phi_{21}+\phi_{11}{\phi_{21}^*}^2\psi_{21}-{\phi_{21}^*}^2\phi_{21}\psi_{11})}{(\phi_{11}\phi_{11}^*+\phi_{21}^*\phi_{21})^2}, \end{cases} \tag{3.51}$$

for the reduced AKNS integrable couplings under the reduction (3.49).

To achieve the restrictions in (3.50), we take the initial value conditions

$$a_0 = \alpha = -\alpha^*, \quad e_0 = \beta = -\beta^*, \quad \text{when } m \text{ - even}, \tag{3.52}$$

or

$$a_0 = \alpha = \alpha^*, \quad e_0 = \beta = \beta^*, \quad \text{when } m \text{ - odd}. \tag{3.53}$$

Particularly, under the reduction (3.49) with the conditions in (3.52) and (3.53), the AKNS integrable coupling systems (3.21) and (3.22) reduce to integrable couplings of the NLS and MKdV equations, respectively. To obtain exact analytical solutions to those two integrable coupling systems from the Bäcklund transformation (3.51), we set

$$\lambda_1 = \iota + i\kappa, \quad \lambda_2 = -\lambda_1^* = -\iota + i\kappa. \tag{3.54}$$

We list two examples below to show the resulting one-soliton-like solutions for the reduced second and third-order AKNS integrable coupling systems.

Example 1. The Reduced Second-Order System

Let us firstly take the spectial initial value conditions

$$a_0 = \alpha = -2i, \quad e_0 = \beta = 0. \tag{3.55}$$

Then, the AKNS integrable coupling system (3.21) reduces to an integrable coupling of the NLS equation:

$$\begin{cases} ip_t - p_{xx} - 2|p|^2 p = 0, \\ ir_t - r_{xx} + 2p_{xx} - 2p^2 r^* - 4|p|^2 r + 4|p|^2 p = 0. \end{cases} \tag{3.56}$$

The resulting one-soliton-like solution (3.43) with $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$ becomes

$$\begin{cases} p = -2\iota e^{-2\,i[2(\iota^2-\kappa^2)t+\kappa x]}\text{sech}[2\iota(4\kappa t - x)], \\ r = \iota e^{-2\,i[2(\iota^2-\kappa^2)t+\kappa x]}\eta\,\text{sech}^2[2\iota(4\kappa t - x)], \end{cases} \tag{3.57}$$

where

$$\eta = (2\,i\kappa x + 2\iota x - 1)e^{-8\iota\kappa t + 2\iota x} + (2\,i\kappa x - 2\iota x - 1)e^{8\iota\kappa t - 2\iota x}.$$

This solution can also be computed directly by using the associated Bäcklund transformation (3.51).

Example 2. The Reduced Third-Order System

Let us secondly take the special initial value conditions

$$a_0 = \alpha = -2, \quad e_0 = \beta = 0. \tag{3.58}$$

Then, the AKNS integrable coupling system (3.22) reduces to an integrable coupling of the MKdV equation:

$$\begin{cases} p_t - \dfrac{1}{2}p_{xxx} - 3|p|^2 p_x = 0, \\ r_t + \dfrac{3}{2}p_{xxx} - \dfrac{1}{2}r_{xxx} + 9|p|^2 p_x - 3|p|^2 r_x - 3pr^* p_x - 3p^* r p_x = 0. \end{cases} \tag{3.59}$$

Similarly, the resulting one-soliton-like solution (3.48) with $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$ becomes

$$\begin{cases} p = -2\iota e^{-2\,i\kappa[2(3\iota^2-\kappa^2)t+x]}\text{sech}\{2\iota[2(\iota^2 - 3\kappa^2)t + x]\}, \\ r = \iota e^{-2\,i\kappa[2(3\iota^2-\kappa^2)t+x]}\eta\,\text{sech}^2\{2\iota[2(\iota^2 - 3\kappa^2)t + x]\}, \end{cases} \tag{3.60}$$

where

$$\eta = (2\,i\kappa x - 2\iota x - 1)e^{-2\iota[2(\iota^2 t - 3\kappa^2)t + x]} + (2\,i\kappa x + 2\iota x - 1)e^{2\iota[2(\iota^2 - 3\kappa^2)t + x]},$$

and this solution can also be worked out directly by using the associated Bäcklund transformation (3.51).

4. Concluding Remarks

Based on the algebraic structures of non-semisimple matrix Lie algebras, we formulated a theory of Darboux transformations for integrable couplings. The resulting Darboux transformation theory was applied to construction of solutions to a kind of integrable couplings of the AKNS equations. Similar Darboux transformations to other integrable couplings can be computed, based on our idea proposed in this paper. It is expected that other solution methods could be developed to solve integrable couplings.

There has been a growing interest in generating hierarchies of integrable couplings [35], from matrix spectral problems associated with non-semisimple matrix loop algebras [14]. Integrable couplings show rich mathematical structures, bringing us inspiring thoughts and ideas to classify multi-component integrable systems [38]. Bi-integrable couplings [39] and tri-integrable couplings [40] do exhibit diverse structures on recursion operators in specific block matrix forms [34, 38, 41]. It should also be important to explore concrete mathematical structures on Darboux transformations for bi-integrable couplings and tri-integrable couplings.

There are many other interesting questions on integrable couplings, which are worthy of further investigation. For example, is there any Hamiltonian structure for the bi-integrable coupling

$$u_t = K(u), \quad v_t = K'(u)[v], \quad w_t = K'(u)[w],$$

where K' denotes the Gateaux derivative of K, when $u_t = K(u)$ is assumed to be Hamiltonian? How can one generally solve the perturbation system

$$u_t = K(u), \quad v_t = K'(u)[v]?$$

A special case associated with the KdV equation $u_y = 6uu_x + u_{xxx}$ is

$$u_t = 6uu_x + u_{xxx}, \quad v_t = 6(uv)_x + v_{xxx},$$

where the second equation is a variable-coefficient linear third-order partial differential equation for v. It is particularly interesting to us how to solve any initial-boundary value problems of this linearized KdV equation once u is given. There exist, though, plenty of particular solutions to the perturbation system, and one class of immediate solutions is to take v as a symmetry of $u_t = K(u)$. Hirota bilinear forms of the perturbation systems of different orders can also be used to present exact solutions [42].

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