Binary symmetry constraints of $\mathcal{N}$-wave interaction equations in $1+1$ and $2+1$ dimensions

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Binary symmetry constraints of the $\mathcal{N}$-wave interaction equations in 1+1 and 2+1 dimensions are proposed to reduce the $\mathcal{N}$-wave interaction equations into finite-dimensional Liouville integrable systems. A new involutive and functionally independent system of polynomial functions is generated from an arbitrary order square matrix Lax operator and used to show the Liouville integrability of the constrained flows of the $\mathcal{N}$-wave interaction equations. The constraints on the potentials resulting from the symmetry constraints give rise to involutive solutions to the $\mathcal{N}$-wave interaction equations, and thus the integrability by quadratures are shown for the $\mathcal{N}$-wave interaction equations by the constrained flows. [DOI: 10.1063/1.1388898]

I. INTRODUCTION

It is a usual practice to utilize the idea of linearization in analyzing nonlinear differential or differential-difference equations (see, for example Refs. 1 and 2). The method of inverse scattering transform is an important application of such an idea to the theory of soliton equations,[3,4] which has been recognized as one of the most significant contributions in the field of applied mathematics in the second half of the last century. The general formulation of Lax pairs is a spectacular tool of realization of inverse scattering transform,[5] by which one can break a nonlinear problem into a couple of linear problems and then handle the resulting linear problems to solve the nonlinear problem.

Recently in the past decade, an unusual way of using the nonlinearization technique arose in the theory of soliton equations.[6–10] Although using the idea of nonlinearization is not normally considered to be a good direction in studying nonlinear equations, one gradually realizes that the nonlinearization technique provides a powerful approach for analyzing soliton equations, especially for showing the integrability by quadratures for soliton equations. The manipulation of nonlinearization not only leads to finite-dimensional Liouville integrable systems,[6–15] but also decomposes infinite-dimensional soliton equations, in whatever dimensions, into finite-dimensional Liouville integrable systems.[16–18] Moreover, it narrows the gap between infinite-dimensional soliton equations and finite-dimensional Liouville integrable systems,[11,16,18] and paves a method of separation of variables for soliton equations,[19,20] which can also be used to analyze the resulting finite-dimensional integrable systems.[21–23] Mathematically speaking, much excitement in the study of nonlinearization comes from a kind of specific symmetry constraints,[24–27] engendered from the variational derivative of the spectral parameter.[26,27] It is due to symmetry constraints that the nonlinearization technique is so powerful in showing the integrability by quadratures for soliton equations.[28,29] The study of symmetry constraints itself is an important part of the kernel of the mathematical theory of nonlinearization, which is also a common conceptional umbrella under which one can manipulate both mono-nonlinearization[6] and binary nonlinearization.[26]

However, all examples of application of the nonlinearization technique, discussed so far, are related to lower-order matrix (here, and in what follows, a matrix is assumed to be square) spectral problems of soliton equations, most of which are only concerned with second-order traceless matrix spectral problems. On the one hand, there appears much difficulty in handling the Liouville integrability[30] of the so-called constrained flows generated from spectral problems, in the case of the third- and fourth-order matrix spectral problems.[28,31,32] It is a challenging task to extend the theory of nonlinearization to the case of higher-order matrix spectral problems. On the other hand, one also notices that mono-nonlinearization cannot be carried out in the cases of odd-order matrix spectral problems and even-order, including the simplest second-order, nontraceless matrix spectral problems. Even for even-order traceless matrix spectral problems, it is not clear how to determine pairs of canonical variables to obtain Hamiltonian structures of the constrained flows while doing mono-nonlinerization. Therefore, one has to take into account adjoint spectral problems and manipulate binary nonlinearization for the case of general matrix spectral problems. In the theory of binary nonlinearization,[33] there exists a natural way for determining symplectic structures to exhibit Hamiltonian forms of the constrained flows.

In this article, we would like to establish a concrete example to apply the nonlinearization technique to the case of higher-order matrix spectral problems, by manipulating binary nonlinearization for arbitrary-order matrix spectral problems associated with the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions. The resulting theory will show a direct way for generating sufficiently many integrals of motion, and more importantly for proving the functional independence of the required integrals of motion, for the Liouville integrability of the constrained flows resulting from higher-order matrix spectral problems.

Let us recall some basic notation on binary nonlinearization (see, for example, Ref. 33 for a detailed description). Let us assume that we have a matrix spectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad U=(U_{ij})_{r\times r}, \quad \phi=(\phi_1,\ldots,\phi_r)^T \tag{1.1}$$

with a spectral parameter λ and a potential $u=(u_1,\ldots,u_q)^T$. Suppose that the compatability conditions

$$U_{t_m} - V_x^{(m)} + [U,V^{(m)}] = 0, \quad m\geqslant 0,$$

of the spectral problem (1.1) and the associated spectral problems

$$\phi_{t_m} = V^{(m)}\phi = V^{(m)}(u,u_x,\ldots;\lambda)\phi, \quad V^{(m)} = (V_{ij}^{(m)})_{r\times r}, \quad m\geqslant 0, \tag{1.2}$$

determine an isospectral ($\lambda_{t_m}=0$) soliton hierarchy

$$u_{t_m} = X_m(u) = JG_m = J\frac{\delta \tilde{H}_m}{\delta u}, \quad m\geqslant 0, \tag{1.3}$$

where J is a Hamiltonian operator and $\tilde{H}_m$ are Hamiltonian functionals. Obviously, the compatability conditions of the adjoint spectral problem

$$\psi_x = -U^T(u,\lambda)\psi, \quad \psi=(\psi_1,\ldots,\psi_r)^T, \tag{1.4}$$

and the adjoint associated spectral problems

$$\psi_{t_m} = -V^{(m)T}\lambda = -V^{(m)T}(u,u_x,\ldots;\lambda)\psi \tag{1.5}$$

still give rise to the same hierarchy $u_{t_m}=X_m(u)$ defined by (1.3). It has been pointed out[16,26] that $J\delta\lambda/\delta u$ is a common symmetry of all equations in the hierarchy (1.3). Introducing N distinct eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_N$, we have

$$\phi_x^{(s)} = U(u,\lambda_s)\phi^{(s)}, \quad \psi_x^{(s)} = -U^T(u,\lambda_s)\psi^{(s)}, \quad 1\leqslant s\leqslant N, \tag{1.6}$$

and

$$\phi^{(s)}_{t_m}=V^{(m)}(u,u_x,\ldots;\lambda_s)\phi^{(s)},\quad \psi^{(s)}_{t_m}=-V^{(m)T}(u,u_x,\ldots;\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N, \tag{1.7}$$

where we set the corresponding eigenfunctions and adjoint eigenfunctions as $\phi^{(s)}$ and $\psi^{(s)}$, $1\leqslant s\leqslant N$. It is assumed that the conserved covariant G_{m_0} does not depend on any derivative of u with respect to x, and thus the so-called general binary Bargmann symmetry constraint reads as

$$X_{m_0}=\sum_{s=1}^{N} E_s\mu_s J\frac{\delta\lambda_s}{\delta u},\quad \text{i.e.,}\quad JG_{m_0}=J\sum_{s=1}^{N}\mu_s\psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)}, \tag{1.8}$$

where μ_s, $1\leqslant s\leqslant N$, are arbitrary nonzero constants, and E_s, $1\leqslant s\leqslant N$, are normalized constants. The right-hand side of the symmetry constraint (1.8) is a linear combination of N symmetries

$$E_sJ\frac{\delta\lambda_s}{\delta u}=J\psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)},\quad 1\leqslant s\leqslant N.$$

Such symmetries are not Lie point, contact or Lie–Bäcklund symmetries, since $\phi^{(s)}$ and $\psi^{(s)}$ cannot be expressed in terms of x, u and derivatives of u with respect to x to some finite order. Suppose that (1.8) has an inverse function

$$u=\tilde{u}=\tilde{u}(\phi^{(1)},\ldots,\phi^{(N)};\psi^{(1)},\ldots,\psi^{(N)}). \tag{1.9}$$

Replacing u with $\tilde{u}$ in the system (1.6) or the system (1.7), we obtain the so-called spatial constrained flow:

$$\phi^{(s)}_{x}=U(\tilde{u},\lambda_s)\phi^{(s)},\quad \psi^{(s)}_{x}=-U^{T}(\tilde{u},\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N, \tag{1.10}$$

or the so-called temporal constrained flows:

$$\phi^{(s)}_{t_m}=V^{(m)}(\tilde{u},\tilde{u}_x,\ldots;\lambda_s)\phi^{(s)},\quad \psi^{(s)}_{t_m}=-V^{(m)T}(\tilde{u},\tilde{u}_x,\ldots;\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N. \tag{1.11}$$

The main problem of nonlinearization is to show that the spatial constrained flow (1.10) and the temporal constrained flows (1.11) under the control of (1.10) are Liouville integrable. Then if $\phi^{(s)}$ and $\psi^{(s)}$, $1\leqslant s\leqslant N$, solve two constrained flows (1.10) and (1.11) simultaneously, $u=\tilde{u}$ will give rise to a solution to the mth soliton equation $u_{t_m}=X_m(u)$. It also follows that the soliton equation $u_{t_m}=X_m(u)$ is decomposed into two finite-dimensional Liouville integrable systems, and $u=\tilde{u}$ presents a Bäcklund transformation between infinite-dimensional soliton equations and finite-dimensional Liouville integrable systems. More generally, if a soliton equation is associated with a set of spectral problems

$$\phi_{x_i}=U^{(i)}(u,\lambda)\phi,\quad 1\leqslant i\leqslant p,$$

then it will be decomposed into $p+1$ finite-dimensional Liouville integrable systems. The above whole process is called binary nonlinearization.[16,33]

This article is structured as follows. In Sec. II, we will present binary symmetry constraints of the $\mathcal{N}$-wave interaction equations in $1+1$ dimensions, and show Hamiltonian structures and Lax presentations of the corresponding constrained flows. In Sec. III, we consider the $2+1$ dimensional case. We will similarly construct binary symmetry constraints of the $\mathcal{N}$-wave interaction equations in $2+1$ dimensions, and discuss some properties of the corresponding constrained flows. In Sec. IV, we go on to propose an involutive system of functionally independent polynomial functions, generated from an arbitrary-order matrix Lax operator, along with an alternative involutive and functionally independent system. An **r**-matrix formulation will be established for the Lax operator, and used to show the involutivity of the obtained system of polynomial functions, together with Newton's identities on elementary symmetric polynomials. A detailed proof will also be made for the functional independence of the system of polynomial functions by using the determinant property of the tensor product of matrices. In Sec. V, two applications of the involutive system engendered in Sec. IV will be given, which verify that all constrained flows associated with the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions are Liouville integrable. Moreover, a kind of involutive solution of the $\mathcal{N}$-wave interaction equations in two cases will be depicted. These also show the integrability by quadratures for the $\mathcal{N}$-wave interaction equations. Finally, in Sec. VI, some concluding remarks will be given, together with conclusions.

II. BINARY SYMMETRY CONSTRAINTS IN 1+1 DIMENSIONS

A. $n\times n$ AKNS hierarchy and 1+1 dimensional $\mathcal{N}$-wave interaction equations

Let n be an arbitrary natural number strictly greater than 2. We begin with the $n\times n$ matrix AKNS spectral problem[34]

$$\phi_x=U\phi=U(u,\lambda)\phi,\quad U(u,\lambda)=\lambda U_0+U_1(u),\quad \phi=(\phi_1,\ldots,\phi_n)^T, \tag{2.1}$$

with a spectral parameter λ and

$$U_0=\mathrm{diag}(\alpha_1,\ldots,\alpha_n),\quad U_1(u)=(u_{ij})_{n\times n}, \tag{2.2}$$

where α_i, $1\leqslant i\leqslant n$, are distinct constants, and $u_{ii}=0$, $1\leqslant i\leqslant n$. The standard AKNS spectral problem, i.e., the spectral problem (2.1) with $n=2$, has been analyzed in Ref. 35, but it cannot generate any $\mathcal{N}$-wave interaction equations and thus it is not discussed here. In order to express related soliton equations in a compact form, we write down the potential u as

$$\begin{gathered} u=\rho(U),\quad \text{i.e., } u=(u_{21},u_{12},u_{13},u_{31},u_{23},u_{32})^T,\quad \text{when } n=3,\\ u=(u_{21},u_{12},u_{13},u_{31},u_{14},u_{41},u_{23},u_{32},\ldots,u_{n,n-1},u_{n-1,n})^T,\quad \text{when } n\geqslant 4, \end{gathered} \tag{2.3}$$

in which we arrange the exponents u_{ij} in a specific way, first from smaller to larger of the integers $k=i+j$ and then symmetrically for each set $\{u_{i,k-i}|1\leqslant i\leqslant k-1\}$.

Let us now consider the construction of the $1+1$ dimensional $\mathcal{N}$-wave interaction equations and its whole isospectral hierarchy associated with the spectral problem (2.1). We first solve the stationary zero-curvature equation for W:

$$W_x-[U,W]=0,\quad W=(W_{ij})_{n\times n}, \tag{2.4}$$

which is equivalent to

$$\begin{gathered} W_{ij,x}+u_{ij}(W_{ii}-W_{jj})+\sum_{\substack{k=1\\k\neq i,j}}^{n}(u_{kj}W_{ik}-u_{ik}W_{kj})-\lambda(\alpha_i-\alpha_j)W_{ij}=0,\quad i\neq j,\\ W_{ii,x}=\sum_{\substack{k=1\\k\neq i}}^{n}(u_{ik}W_{ki}-u_{ki}W_{ik}), \end{gathered} \tag{2.5}$$

where $1\leqslant i,j\leqslant n$. We look for a formal solution of the form

$$W=\sum_{l\geqslant 0}W_l\lambda^{-l},\quad W_l=(W_{ij}^{(l)})_{n\times n}, \tag{2.6}$$

and thus (2.5) becomes the following recursion relation

$$W_{ii,x}^{(0)}=0,\quad W_{ij}^{(0)}=0,\quad i\neq j, \tag{2.7a}$$

$$W_{ij,x}^{(l)}+u_{ij}(W_{ii}^{(l)}-W_{jj}^{(l)})+\sum_{\substack{k=1\\ k\neq i,j}}^{n}(u_{kj}W_{ik}^{(l)}-u_{ik}W_{kj}^{(l)})-(\alpha_i-\alpha_j)W_{ij}^{(l+1)}=0,\quad i\neq j, \tag{2.7b}$$

$$W_{ii,x}^{(l+1)}=\sum_{\substack{k=1\\ k\neq i}}^{n}(u_{ik}W_{ki}^{(l+1)}-u_{ki}W_{ik}^{(l+1)}), \tag{2.7c}$$

where $1\leqslant i,j\leqslant n$ and $l\geqslant 0$. In particular, from the above recursion relation, we have that

$$W_{ii}^{(0)}=\beta_i=\mathrm{const},\quad W_{ij}^{(0)}=0,\quad 1\leqslant i\neq j\leqslant n, \tag{2.8}$$

and

$$W_{ii}^{(1)}=0,\quad W_{ij}^{(1)}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}u_{ij},\quad 1\leqslant i\neq j\leqslant n. \tag{2.9}$$

We require that

$$W_{ij}^{(l)}|_{u=0}=0,\quad 1\leqslant i,j\leqslant n,\quad l\geqslant 1. \tag{2.10}$$

This condition (2.10) means to identify all constants of integration to be zero while using (2.7) to determine W, and thus all W_l, $l\geqslant 1$, will be uniquely determined. For example, we can obtain from (2.7) under (2.10) that

$$\begin{gathered}W_{ij}^{(2)}=\frac{\beta_i-\beta_j}{(\alpha_i-\alpha_j)^2}u_{ij,x}+\frac{1}{\alpha_i-\alpha_j}\sum_{\substack{k=1\\ k\neq i,j}}^{n}\left(\frac{\beta_k-\beta_i}{\alpha_k-\alpha_i}-\frac{\beta_k-\beta_j}{\alpha_k-\alpha_j}\right)u_{ik}u_{kj},\quad 1\leqslant i\neq j\leqslant n,\\ W_{ii}^{(2)}=\sum_{\substack{k=1\\ k\neq i}}^{n}\frac{\beta_k-\beta_i}{(\alpha_k-\alpha_i)^2}u_{ik}u_{ki},\quad 1\leqslant i\leqslant n.\end{gathered} \tag{2.11}$$

It is easy to see that the recursion relation (2.7) can lead to

$$\begin{aligned}&2u_{ij}\partial^{-1}u_{ij}W_{ji}^{(l)}+(\partial-2u_{ij}\partial^{-1}u_{ji})W_{ij}^{(l)}+\sum_{\substack{k=1\\ k\neq i,j}}^{n}[u_{ij}\partial^{-1}u_{ik}W_{ki}^{(l)}+(u_{kj}-u_{ij}\partial^{-1}u_{ki})W_{ik}^{(l)}]\\ &\quad+\sum_{\substack{k=1\\ k\neq i,j}}^{n}[u_{ij}\partial^{-1}u_{kj}W_{jk}^{(l)}-(u_{ik}+u_{ij}\partial^{-1}u_{jk})W_{kj}^{(l)}]=(\alpha_i-\alpha_j)W_{ij}^{(l+1)},\quad i\neq j,\end{aligned} \tag{2.12}$$

where $1\leqslant i,j\leqslant n$, $l\geqslant 1$, and ∂^{-1} is the inverse operator of $\partial=\partial/\partial x$. This can be written as the Lenard form

$$MG_{l-1}=JG_l,\quad l\geqslant 1, \tag{2.13}$$

where $G_l=\rho(W_{l+1})$ is generated from W_{l+1} in the same way as that for u, and J is a constant operator

$$\begin{cases} J=\mathrm{diag}((\alpha_1-\alpha_2)\sigma_0,(\alpha_1-\alpha_3)\sigma_0,(\alpha_2-\alpha_3)\sigma_0), \quad \text{when} \quad n=3, \\ J=\mathrm{diag}(\underbrace{(\alpha_1-\alpha_2)\sigma_0,(\alpha_1-\alpha_3)\sigma_0,(\alpha_1-\alpha_4)\sigma_0,(\alpha_2-\alpha_3)\sigma_0,\ldots,(\alpha_{n-1}-\alpha_n)\sigma_0}_{n(n-1)/2}), \\ \qquad \text{when} \quad n\geqslant 4, \end{cases} \tag{2.14}$$

with σ_0 being given by

$$\sigma_0=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For example, when $n\geqslant 4$, we have

$$G_{l-1}=(W_{21}^{(l)},W_{12}^{(l)},W_{31}^{(l)},W_{13}^{(l)},W_{41}^{(l)},W_{14}^{(l)},W_{32}^{(l)},W_{23}^{(l)},\ldots,W_{n,n-1}^{(l)},W_{n-1,n}^{(l)})^T, \quad l\geqslant 1, \tag{2.15}$$

the first of which reads as

$$G_0=\left(\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}u_{21},\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}u_{12},\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3}u_{31},\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3}u_{13},\frac{\beta_1-\beta_4}{\alpha_1-\alpha_4}u_{41},\right.$$
$$\left.\frac{\beta_1-\beta_4}{\alpha_1-\alpha_4}u_{14},\ldots,\frac{\beta_{n-1}-\beta_n}{\alpha_{n-1}-\alpha_n}u_{n,n-1},\frac{\beta_{n-1}-\beta_n}{\alpha_{n-1}-\alpha_n}u_{n-1,n}\right)^T. \tag{2.16}$$

The operators J and M are skew-symmetric and can be shown to be a Hamiltonian pair.[36,37]

We proceed to introduce the associated spectral problems with the spectral problem (2.1),

$$\phi_{t_m}=V^{(m)}\phi, \quad V^{(m)}=V^{(m)}(u,\lambda)=(\lambda^m W)_+, \quad m\geqslant 1, \tag{2.17}$$

where the symbol $+$ stands for the choice of the part of non-negative powers of λ. Note that we have

$$W_{lx}=[U_0,W_{l+1}]+[U_1,W_l], \quad l\geqslant 0,$$

and we can compute that

$$\begin{aligned}[U,V^{(m)}]&=\left[\lambda U_0+U_1,\sum_{l=0}^{m}\lambda^{m-l}W_l\right]\\ &=\sum_{l=0}^{m}[U_0,W_l]\lambda^{m+1-l}+\sum_{l=0}^{m}[U_1,W_l]\lambda^{m-l}\\ &=\sum_{l=0}^{m-1}[U_0,W_{l+1}]\lambda^{m-l}+\sum_{l=0}^{m}[U_1,W_l]\lambda^{m-l},\end{aligned}$$

where we have used $[U_0,W_0]=0$. Therefore, under the isospectral conditions

$$\lambda_{t_m}=0, \quad m\geqslant 1, \tag{2.18}$$

the compatibility conditions of the spectral problem (2.1) and the associated spectral problems (2.17), i.e., the zero-curvature equations

$$U_{t_m}-V_x^{(m)}+[U,V^{(m)}]=0,\quad m\geqslant 1,$$

equivalently lead to

$$U_{1t_m}=W_{mx}-[U_1,W_m]=[U_0,W_{m+1}],\quad m\geqslant 1.$$

This gives rise to the so-called $n\times n$ AKNS soliton hierarchy

$$u_{t_m}=X_m:=JG_m,\quad m\geqslant 1, \tag{2.19}$$

where J and $G_m=\rho(W_{m+1})$ are determined by (2.14) and (2.13).

Applying the trace identity[38]

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(W\frac{\partial U}{\partial\lambda}\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right),$$

where γ is a constant to be determined, we can obtain

$$\frac{\delta\tilde{H}_l}{\delta u_{ij}}=W_{ji}^{(l)},\quad \tilde{H}_l:=-\frac{1}{l}\int(\alpha_1W_{11}^{(l+1)}+\alpha_2W_{22}^{(l+1)}+\ldots+\alpha_nW_{nn}^{(l+1)})\,dx,\quad l\geqslant 1, \tag{2.20}$$

in which $1\leqslant i\neq j\leqslant n$ and γ is determined to be zero. In this computation, we need to note that

$$\mathrm{tr}\left(W\frac{\partial U}{\partial\lambda}\right)=\mathrm{tr}(WU_0)=\sum_{l\geqslant 0}(\alpha_1W_{11}^{(l)}+\alpha_2W_{22}^{(l)}+\ldots+\alpha_nW_{nn}^{(l)})\lambda^{-l},$$

and

$$\mathrm{tr}\left(W\frac{\partial U}{\partial u_{ij}}\right)=\mathrm{tr}(WE_{ij})=W_{ji}=\sum_{l\geqslant 0}W_{ji}^{(l)}\lambda^{-l},\quad 1\leqslant i\neq j\leqslant n,$$

where E_{ij} is an $n\times n$ matrix whose (i,j) entry is one but other entries are all zero. Therefore, the isospectral hierarchy (2.19) has a bi-Hamiltonian formulation

$$u_{t_m}=X_m=J\frac{\delta\tilde{H}_{m+1}}{\delta u}=M\frac{\delta\tilde{H}_m}{\delta u},\quad m\geqslant 1. \tag{2.21}$$

The first nonlinear system in the hierarchy (2.19) is the $1+1$ dimensional $\mathcal{N}$-wave interaction equations[39]

$$u_{ij,t_1}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}u_{ij,x}+\sum_{\substack{k=1\\k\neq i,j}}^{n}\left(\frac{\beta_i-\beta_k}{\alpha_i-\alpha_k}-\frac{\beta_k-\beta_j}{\alpha_k-\alpha_j}\right)u_{ik}u_{kj},\quad 1\leqslant i\neq j\leqslant n. \tag{2.22}$$

This system is actually equivalent to the following equation in the matrix form

$$U_{1t_1}=W_{1x}-[U_1,W_1], \tag{2.23}$$

which can be rewritten as

$$P_{t_1}=Q_x-[P,Q],\quad [U_0,Q]=[W_0,P], \tag{2.24}$$

where P and Q are assumed to be two off-diagonal potential matrices. Based on (2.23), a vector field $\rho(\delta P)$ is a symmetry of (2.22) if the matrix δP satisfies the linearized system of (2.22):

$$(\delta P)_{t_1}=(\delta Q)_x-[U_1,\delta Q]-[\delta P,W_1] \tag{2.25}$$

with δQ being determined by

$$[U_0,\delta Q]=[W_0,\delta P]. \tag{2.26}$$

The $\mathcal{N}$-wave interaction equations (2.22) contains a couple of physically important nonlinear models as special reductions,[40] for example, three-wave interaction equations arising in fluid dynamics and plasma physics,[41–43] with U being chosen to be an anti-Hermitian matrix. Its Darboux transformation has been established in Ref. 44, which allows one to construct soliton solutions in a purely algebraic way. The Darboux transformation has also been analyzed for the $\mathcal{N}$-wave interaction equations with additional linear terms.[45]

B. Binary symmetry constraints in 1+1 dimensional case

We would like to present binary symmetry constraints of the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22). To this end, we need to introduce the adjoint spectral problem of (2.1):

$$\psi_x=-U^T(u,\lambda)\psi, \quad \psi=(\psi_1,\dots,\psi_n)^T, \tag{2.27}$$

and the adjoint associated spectral problem of (2.17):

$$\psi_{t_m}=-V^{(m)T}(u,\lambda)\psi, \tag{2.28}$$

where U and $V^{(m)}$ are given as in (2.1) and (2.17), respectively. The compatability condition of (2.27) and (2.28) still gives rise to $u_{t_m}=X_m$ defined by (2.19).

The variational derivative of the spectral parameter λ with respect to the potential u can be calculated by (see Refs. 26, 28, or 16 for a detailed deduction)

$$\frac{\delta\lambda}{\delta u}=E^{-1}\psi^T\frac{\partial U}{\partial u}\phi, \quad \text{i.e.,}\ \frac{\delta\lambda}{\delta u_{ij}}=E^{-1}\phi_i\psi_j, \quad 1\leqslant i\neq j\leqslant n, \tag{2.29}$$

where E is the normalized constant:

$$E=-\int_{-\infty}^{\infty}\psi^T\frac{\partial U}{\partial\lambda}\phi\,dx.$$

A direct calculation can show that the variational derivative satisfies the following equation:

$$M\frac{\delta\lambda}{\delta u}=\lambda J\frac{\delta\lambda}{\delta u}. \tag{2.30}$$

Since λ does not vary with respect to time, we have a specific common symmetry $J(\delta\lambda/\delta u)$ of the hierarchy (2.19). To carry out binary nonlinearization, we take a Lie point symmetry of the $\mathcal{N}$-wave interaction equations (2.22),

$$Y_0:=\rho([\Gamma,U_1]), \quad \Gamma=\mathrm{diag}(\gamma_1,\dots,\gamma_n), \tag{2.31}$$

where $\gamma_1,\gamma_2,\dots,\gamma_n$ are arbitrary distinct constants ($X_0=JG_0$ is an example with $\Gamma=W_0$). It can be easily checked that

$$(\delta P,\delta Q)=([\Gamma,U_1],[\Gamma,W_1])$$

satisfies (2.25), and thus Y_0 is a symmetry of (2.22). Then, make the following binary Bargmann symmetry constraint

$$Y_0=\mu EJ\frac{\delta\lambda}{\delta u}=\mu J\psi^T\frac{\partial U}{\partial u}\phi, \tag{2.32}$$

where μ is an arbitrary nonzero constant, J is defined by (2.14), and ϕ and ψ are the eigenfunction and adjoint eigenfunction of (2.1) and (2.27), respectively. Upon introducing N distinct eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_N$, we obtain a general binary symmetry constraint

$$Y_0=J\sum_{s=0}^{N}\mu_s\psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)}:=Z_0, \tag{2.33}$$

where μ_s, $1\leqslant s\leqslant N$, are N nonzero constants, and $\phi^{(s)}$ and $\psi^{(s)}$, $1\leqslant s\leqslant N$, are eigenfunctions and adjoint eigenfunctions defined by

$$\phi_x^{(s)}=U(u,\lambda_s)\phi^{(s)},\quad \psi_x^{(s)}=-U^T(u,\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N, \tag{2.34}$$

and

$$\phi_{t_1}^{(s)}=V^{(1)}(u,\lambda_s)\phi^{(s)},\quad \psi_{t_1}^{(s)}=-V^{(1)T}(u,\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N. \tag{2.35}$$

Let us rewrite the left-hand side of (2.33) as the matrix form

$$\delta P=\rho^{-1}(Z_0)=\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right], \tag{2.36}$$

which allows us to prove, by a direct computation as in Ref. 46 but more conveniently, that the vector field $Z_0=\rho(\delta P)$ is really a symmetry of the $\mathcal{N}$-wave interaction equations (2.22). Now the symmetry problem is equivalent to showing that

$$(\delta P,\delta Q)=\left(\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right],\left[W_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right) \tag{2.37}$$

satisfies the linearized system (2.25), when $\phi^{(s)}$ and $\psi^{(s)}$, $1\leqslant s\leqslant N$, satisfy (2.34) and (2.35). A detailed proof will be given in Appendix A.

Therefore, we have the following binary symmetry constraint:

$$Y_0=J\sum_{s=0}^{N}\mu_s\psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)},\quad \text{i.e.,}\quad [\Gamma,U_1]=\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]. \tag{2.38}$$

When N and μ_s vary, (2.38) provides us with a set of binary symmetry constraints of the $\mathcal{N}$-wave interaction equations (2.22). Let us assume that

$$\phi^{(s)}=(\phi_{1s},\phi_{2s},\dots,\phi_{ns})^T,\quad \psi^{(s)}=(\psi_{1s},\psi_{2s},\dots,\psi_{ns})^T, \tag{2.39}$$

in order to get an explicit expression for u from the symmetry constraint (2.38), and introduce two diagonal matrices

$$A=\mathrm{diag}(\lambda_1,\dots,\lambda_N),\quad B=\mathrm{diag}(\mu_1,\dots,\mu_N), \tag{2.40}$$

which will be used throughout our discussion. Solving the Bargmann symmetry constraint (2.38) for u, we obtain

$$u_{ij}=\tilde{u}_{ij}:=\frac{\alpha_i-\alpha_j}{\gamma_i-\gamma_j}\langle\Phi_i,B\Psi_j\rangle,\quad 1\leqslant i\neq j\leqslant n, \tag{2.41}$$

where B is given by (2.40), and Φ_i and Ψ_i are defined by

$$\Phi_i=(\phi_{i1},\phi_{i2},\ldots,\phi_{iN})^T,\quad \Psi_i=(\psi_{i1},\psi_{i2},\ldots,\psi_{iN})^T,\quad 1\leqslant i\leqslant n, \tag{2.42}$$

and $\langle\cdot,\cdot\rangle$ denotes the standard inner-product of the Euclidean space $\mathbb{R}^N$.

Note that the compatability condition of (2.34) and (2.35) is still nothing but the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22). Now using (2.41), we nonlinearize the spatial part (2.34) and the temporal part (2.35) of spectral problems and adjoint spectral problems of the $\mathcal{N}$-wave interaction equations (2.22). Namely we replace u_{ij} with $\tilde{u}_{ij}$ in N replicas of the spectral problems and adjoint spectral problems (2.34) and N replicas of the associated spectral problems and adjoint associated spectral problems (2.35), and then obtain two constrained flows for the $\mathcal{N}$-wave interaction equations (2.22):

$$\phi_x^{(s)}=U(\tilde{u},\lambda_s)\phi^{(s)},\quad \psi_x^{(s)}=-U^T(\tilde{u},\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N, \tag{2.43}$$

and

$$\phi_{t_1}^{(s)}=V^{(1)}(\tilde{u},\lambda_s)\phi^{(s)},\quad \psi_{t_1}^{(s)}=-V^{(1)T}(\tilde{u},\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N, \tag{2.44}$$

where $\tilde{u}=\rho((\tilde{u}_{ij})_{n\times n})$ is defined like u. For example, when $n\geqslant 4$, we have

$$\tilde{u}=(\tilde{u}_{21},\tilde{u}_{12},\tilde{u}_{31},\tilde{u}_{13},\tilde{u}_{14},\tilde{u}_{41},\tilde{u}_{23},\tilde{u}_{32},\ldots,\tilde{u}_{n,n-1},\tilde{u}_{n-1,n})^T. \tag{2.45}$$

In order to analyze the Liouville integrability of the above two constrained flows, let us first introduce a symplectic structure

$$\omega^2=\sum_{i=1}^n Bd\Phi_i\wedge d\Psi_i=\sum_{i=1}^n\sum_{s=1}^N \mu_s d\phi_{is}\wedge d\psi_{is} \tag{2.46}$$

over $\mathbb{R}^{2nN}$, and then the corresponding Poisson bracket

$$\begin{aligned}\{f,g\}=\omega^2(Idg,Idf)&=\sum_{i=1}^n\left(\left\langle\frac{\partial f}{\partial\Psi_i},B^{-1}\frac{\partial g}{\partial\Phi_i}\right\rangle-\left\langle\frac{\partial f}{\partial\Phi_i},B^{-1}\frac{\partial g}{\partial\Psi_i}\right\rangle\right)\\&=\sum_{i=1}^n\sum_{s=1}^N\mu_s^{-1}\left(\frac{\partial f}{\partial\psi_{is}}\frac{\partial g}{\partial\phi_{is}}-\frac{\partial f}{\partial\phi_{is}}\frac{\partial g}{\partial\psi_{is}}\right),\quad f,g\in C^\infty(\mathbb{R}^{2nN}),\end{aligned} \tag{2.47}$$

where the vector field Idf is defined by

$$\omega^2(X,Idf)=df(X),\quad X\in T(\mathbb{R}^{2nN}).$$

A Hamiltonian system with a Hamiltonian H defined over the symplectic manifold $(\mathbb{R}^{2nN},\omega^2)$ is given by

$$\Phi_{it}=\{\Phi_i,H\}=-B^{-1}\frac{\partial H}{\partial\Psi_i},\quad \Psi_{it}=\{\Psi_i,H\}=B^{-1}\frac{\partial H}{\partial\Phi_i},\quad 1\leqslant i\leqslant n, \tag{2.48}$$

where t is assumed to be the evolution variable. Second, we need a matrix Lax operator

$$L^{(1)}(\lambda)=C_1+D_1(\lambda), \tag{2.49}$$

with C_1 and $D_1(\lambda)$ being defined by

$$C_1=\Gamma=\mathrm{diag}(\gamma_1,\ldots,\gamma_n),\quad D_1(\lambda)=(D_{ij}^{(1)}(\lambda))_{n\times n},\quad D_{ij}^{(1)}(\lambda)=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}\phi_{is}\psi_{js}, \tag{2.50}$$

where $1\leqslant i,\ j\leqslant n$. Note that upon taking binary nonlinearization, we obtain

$$U(\tilde u,\lambda)=\lambda U_0+U_1(\tilde u)=\lambda U_0+(\tilde u_{ij}),\quad \tilde u_{ij}=\frac{\alpha_i-\alpha_j}{\gamma_i-\gamma_j}\langle\Phi_i,B\Psi_j\rangle, \tag{2.51}$$

$$V^{(1)}(\tilde u,\lambda)=\lambda W_0+W_1(\tilde u)=\lambda W_0+(\tilde v_{ij}),\quad \tilde v_{ij}:=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}\tilde u_{ij}=\frac{\beta_i-\beta_j}{\gamma_i-\gamma_j}\langle\Phi_i,B\Psi_j\rangle, \tag{2.52}$$

where $1\leqslant i,\ j\leqslant n$.

Theorem 2.1: *Under the symplectic structure (2.46), the spatial constrained flow (2.43) and the temporal constrained flow (2.44) for the 1+1 dimensional $\mathcal{N}$-wave interaction equations (2.22) are Hamiltonian systems with the evolution variables x and t_1, and the Hamiltonians*

$$H_1^x=-\sum_{k=1}^{n}\alpha_k\langle A\Phi_k,B\Psi_k\rangle-\sum_{1\leqslant k<l\leqslant n}\frac{\alpha_k-\alpha_l}{\gamma_k-\gamma_l}\langle\Phi_k,B\Psi_l\rangle\langle\Phi_l,B\Psi_k\rangle, \tag{2.53}$$

$$H_1^{t_1}=-\sum_{k=1}^{n}\beta_k\langle A\Phi_k,B\Psi_k\rangle-\sum_{1\leqslant k<l\leqslant n}\frac{\beta_k-\beta_l}{\gamma_k-\gamma_l}\langle\Phi_k,B\Psi_l\rangle\langle\Phi_l,B\Psi_k\rangle, \tag{2.54}$$

respectively, where A and B are defined by (2.40), and Φ_i and Ψ_i, $1\leqslant i\leqslant n$, are defined by (2.42). Moreover, they possess necessary Lax representations, i.e., we have

$$(L^{(1)}(\lambda))_x=[U(\tilde u,\lambda),L^{(1)}(\lambda)],\quad (L^{(1)}(\lambda))_{t_1}=[V^{(1)}(\tilde u,\lambda),L^{(1)}(\lambda)], \tag{2.55}$$

where $L^{(1)}(\lambda)$, U, and $V^{(1)}(\lambda)$ are given by (2.49)–(2.52), if (2.43) and (2.44) hold, respectively.

Proof: A direct calculation can show the Hamiltonian structures of the spatial constrained flow (2.43) and the temporal constrained flow (2.44) with H_1^x and H_1^t defined by (2.53) and (2.54). Let us then check the Lax representations. By using (2.43), we can compute that

$$\begin{aligned}(L^{(1)}(\lambda))_x&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}(\phi_x^{(s)}\psi^{(s)T}+\phi^{(s)}\psi_x^{(s)T})\\
&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}(U(\tilde u,\lambda_s)\phi^{(s)}\psi^{(s)T}-\phi^{(s)}\psi^{(s)T}U(\tilde u,\lambda_s))\\
&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}[U(\tilde u,\lambda_s),\phi^{(s)}\psi^{(s)T}]\\
&=[U(\tilde u,\lambda),L^{(1)}(\lambda)-C_1]-\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\\
&=[U(\tilde u,\lambda),L^{(1)}(\lambda)]+[C_1,U(\tilde u,\lambda)]-\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\\
&=[U(\tilde u,\lambda),L^{(1)}(\lambda)]+[C_1,U_1(\tilde u)]-\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right].\end{aligned}$$

This implies that $(L^{(1)}(\lambda))_x=[U(\tilde{u},\lambda),L^{(1)}(\lambda)]$ if and only if

$$[C_1,U_1(\tilde{u})]=\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right].$$

The above equality equivalently requires the constraints on the potentials shown in (2.41). Therefore, the spatial constrained flow (2.43) has the necessary Lax representation defined as in (2.55). The proof of the other necessary Lax representation $(L^{(1)}(\lambda))_{t_1}=[V^{(1)}(\tilde{u},\lambda),L^{(1)}(\lambda)]$ is completely similar, and thus we omit it. The proof is finished. ■

We remark that the Lax representations (2.55) are not sufficient. Namely, we cannot obtain the spatial constrained flow (2.43) or the temporal constrained flow (2.44) from the corresponding Lax representation in (2.55). This can be easily observed by considering a special class of solutions of (2.55). For example, either any vector functions $\phi^{(s)}$ with $\psi^{(s)}=0$, $1\leqslant s\leqslant N$, or any vector functions $\psi^{(s)}$ with $\phi^{(s)}=0$, $1\leqslant s\leqslant N$, will solve (2.55), but it is easy to see that they do not always solve (2.43) [or (2.44)] since $\phi^{(s)}$ and $\psi^{(s)}$, $1\leqslant s\leqslant N$, have to solve some ordinary differential equations (ODEs) resulting from (2.43) [or (2.44)].

III. BINARY SYMMETRY CONSTRAINTS IN 2+1 DIMENSIONS

A. 2+1 dimensional $\mathcal{N}$-wave interaction equations

Let n be an arbitrary natural number strictly greater than 2. Similar to the case of the $1+1$ dimensional **N**-wave interaction equations, let us begin with the Lax system

$$F_y=JF_x+PF,\quad F_t=KF_x+QF,\quad F=(f_1,\ldots,f_n)^T \tag{3.1}$$

in $2+1$ dimensions. Here it is assumed that

$$J=\mathrm{diag}(J_1,\ldots,J_n),\quad K=\mathrm{diag}(K_1,\ldots,K_n),\quad J_i\neq J_j,\quad K_i\neq K_j,\quad 1\leqslant i\neq j\leqslant n, \tag{3.2}$$

are two constant diagonal matrices, and P and Q are two $n\times n$ off-diagonal potential matrices

$$P=P(x,y,t)=(p_{ij})_{n\times n},\quad Q=Q(x,y,t)=(q_{ij})_{n\times n}. \tag{3.3}$$

The compatability condition $F_{yt}=F_{ty}$ of the Lax system (3.1) reads as

$$[J,Q]=[K,P],\quad P_t-Q_y+[P,Q]+JQ_x-KP_x=0, \tag{3.4}$$

which is called the $2+1$ dimensional $\mathcal{N}$-wave interaction equations.[47] The equation $[J,Q]=[K,P]$ tells us that Q can be represented by P and vice versa, and so, practically, we have just one of two potential matrices to be solved. The adjoint system of the Lax system (3.1) is given by

$$G_y=JG_x-P^TG,\quad G_t=KG_x-Q^TG,\quad G=(g_1,\ldots,g_n)^T, \tag{3.5}$$

whose compatability condition $G_{yt}=G_{ty}$ still gives rise to the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4).

We first use a symmetry constraint of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4) to change the above problem in $2+1$ dimensions to three problems in $1+1$ dimensions. As made in Refs. 48, and 49, we introduce the spectral problems

$$\phi_x=\Omega^x(F,G,\lambda)\phi=(\lambda\Omega_0^x+\Omega_1^x)\phi=\begin{pmatrix}\lambda I_n & F\\ G^T & 0\end{pmatrix}\phi, \tag{3.6a}$$

$$\phi_y=\Omega^y(P,F,G,\lambda)\phi=(\lambda\Omega_0^y+\Omega_1^y)\phi=\begin{pmatrix}\lambda J+P & JF\\ G^TJ & 0\end{pmatrix}\phi, \tag{3.6b}$$

$$\phi_t=\Omega^t(Q,F,G,\lambda)\phi=(\lambda\Omega_0^t+\Omega_1^t)\phi=\begin{pmatrix}\lambda K+Q & KF\\ G^TK & 0\end{pmatrix}\phi, \tag{3.6c}$$

where I_n is the nth-order identity matrix and $\phi=(\phi_1,\ldots,\phi_n,\phi_{n+1})^T$. The new extended potentials in the above spectral systems consist of not only the original potentials, P and Q, but also the solutions of the Lax system and the adjoint Lax system, F and G. The compatability conditions $\phi_{xy}=\phi_{yx}$, $\phi_{xt}=\phi_{tx}$, and $\phi_{yt}=\phi_{yt}$ give rise to the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4), the original Lax system (3.1) and its adjoint system (3.5), and the nonlinear symmetry constraint of (3.4):

$$P_x=[FG^T,J],\quad Q_x=[FG^T,K]. \tag{3.7}$$

It is easy to check that $(\delta P,\delta Q)=([FG^T,J],[FG^T,K])$ satisfies the linearized system of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4):

$$[J,\delta Q]=[K,\delta P],\quad (\delta P)_t-(\delta Q)_y+[\delta P,Q]+[P,\delta Q]+J(\delta Q)_x-K(\delta P)_x=0, \tag{3.8}$$

when F and G solve the Lax system (3.1) and the adjoint Lax system (3.5), respectively. Therefore, (3.7) is really a symmetry constraint of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4), since both sides of (3.7) are symmetries of (3.4). Now we see that the original problem in $2+1$ dimensions is transformed into three problems in $1+1$ dimensions. The spectral problems (3.6) are our starting point to make a link of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4) to finite-dimensional integrable systems.

B. Binary symmetry constraints in 2+1 dimensional case

Let us start from the spectral problems in (3.6), which are similar to those for the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22). The main difference is that the coefficient matrix of λ in the x-part of the spectral problems (3.6) is

$$\Omega_0^x=\mathrm{diag}(\underbrace{1,\ldots,1}_{n},0), \tag{3.9}$$

whose diagonal entries are not distinct. However, the y-part of the spectral problems (3.6) has the same property as the spectral problem (2.1) in $1+1$ dimensions. Therefore, we use the y-part of the spectral problems (3.6) to compute the variational derivatives of λ:

$$\frac{\delta\lambda}{\delta p_{ij}}=E^{-1}\psi^T\frac{\partial\Omega^y}{\partial p_{ij}}\phi=E^{-1}\phi_i\psi_j,\quad \frac{\delta\lambda}{\delta q_{ij}}=E^{-1}\psi^T\frac{\partial\Omega^y}{\partial q_{ij}}\phi=E^{-1}\frac{J_i-J_j}{K_i-K_j}\phi_i\psi_j,\quad 1\leqslant i\neq j\leqslant n,$$

$$\frac{\delta\lambda}{\delta f_i}=E^{-1}\psi^T\frac{\partial\Omega^y}{\partial f_i}\phi=E^{-1}J_i\phi_{n+1}\psi_i,\quad \frac{\delta\lambda}{\delta g_i}=E^{-1}\psi^T\frac{\partial\Omega^y}{\partial g_i}\phi=E^{-1}J_i\phi_i\psi_{n+1},\quad 1\leqslant i\leqslant n,$$

where E is the normalized constant, and $\psi=(\psi_1,\ldots,\psi_n,\psi_{n+1})^T$ is an adjoint eigenfunction of the adjoint spectral problems

$$\psi_x=-(\Omega^x(F,G,\lambda))^T\psi=-(\lambda(\Omega_0^x)^T+(\Omega_1^x)^T)\psi=-\begin{pmatrix}\lambda I_n & G\\ F^T & 0\end{pmatrix}\psi, \tag{3.10a}$$

$$\psi_y=-(\Omega^y(P,F,G,\lambda))\psi=-(\lambda(\Omega_0^y)+(\Omega_1^y)^T)\psi=-\begin{pmatrix}\lambda J+P^T & JG\\ F^TJ & 0\end{pmatrix}\psi, \tag{3.10b}$$

$$\psi_t=-(\Omega^t(Q,F,G,\lambda))^T\psi=-(\lambda(\Omega_0^t)^T+(\Omega_1^t)^T)\psi=-\begin{pmatrix}\lambda K+Q^T & KG\\ F^TK & 0\end{pmatrix}\psi. \tag{3.10c}$$

These variational derivatives of λ give us a conserved covariant and also a clue to compute a required symmetry, expressed in terms of eigenfunctions and adjoint eigenfunctions.

As in the $1+1$ dimensional case, upon introducing N distinct eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_N$, we have

$$\phi_x^{(s)}=\Omega^x(u,\lambda_s)\phi^{(s)},\quad \phi_y^{(s)}=\Omega^y(u,\lambda_s)\phi^{(s)},\quad \phi_t^{(s)}=\Omega^t(u,\lambda_s)\phi^{(s)},\quad 1\leqslant s\leqslant N, \tag{3.11}$$

and

$$\psi_x^{(s)}=-(\Omega^x)^T(u,\lambda_s)\psi^{(s)},\quad \psi_y^{(s)}=-(\Omega^y)^T(u,\lambda_s)\psi^{(s)},\quad \psi_t^{(s)}=-(\Omega^t)^T(u,\lambda_s)\psi^{(s)},\quad 1\leqslant s\leqslant N, \tag{3.12}$$

where $\phi^{(s)}$ and $\psi^{(s)}$ are $n+1$ dimensional vector functions:

$$\phi^{(s)}=(\phi_{1s},\ldots,\phi_{ns},\phi_{n+1,s})^T,\quad \psi^{(s)}=(\psi_{1s},\ldots,\psi_{ns},\psi_{n+1,s})^T,\quad 1\leqslant s\leqslant N. \tag{3.13}$$

To carry out binary nonlinearization, we need to construct two special symmetries, one of which is a Lie point symmetry, and the other of which is not a Lie point, contact or Lie–Bäcklund symmetry, but generated from (3.11) and (3.12). Let us choose a set of $n+1$ arbitrary distinct constants $\delta_1,\ldots,\delta_n,\delta_{n+1}$, and set

$$\Delta=\mathrm{diag}(\delta_1,\ldots,\delta_n). \tag{3.14}$$

Similar to the $1+1$ dimensional case, it can be directly shown that

$$(\delta P,\delta Q,\delta F,\delta G)=([\Delta,P],[\Delta,Q],\Delta F-\delta_{n+1}F,\Delta G-\delta_{n+1}G) \tag{3.15}$$

and

$$\begin{gathered}\delta p_{ij}=(J_i-J_j)\langle\Phi_i,B\Psi_j\rangle,\quad \delta q_{ij}=(K_i-K_j)\langle\Phi_i,B\Psi_j\rangle,\quad 1\leqslant i\neq j\leqslant n,\\ \delta f_i=\langle\Phi_i,B\Psi_{n+1}\rangle,\quad \delta g_i=\langle\Phi_{n+1},B\Psi_i\rangle,\quad 1\leqslant i\leqslant n,\end{gathered} \tag{3.16}$$

are two symmetries of the equations (3.4), (3.1) and (3.5). That is to say, that they satisfy the linearized system of the equations (3.4), (3.1) and (3.5): the first subsystem (3.8) and the second subsystem

$$\begin{gathered}(\delta F)_y=J(\delta F)_x+(\delta P)F+P\,\delta F,\quad (\delta F)_t=K(\delta F)_x+(\delta Q)F+Q\,\delta F,\\ (\delta G)_y=J(\delta G)_x-(\delta P)^TG-P^T\delta G,\quad (\delta G)_t=K(\delta G)_x-(\delta Q)^TG-Q^T\delta G,\end{gathered} \tag{3.17}$$

for all solutions (P,Q,F,G) of (3.4), (3.1) and (3.5). Here we remind that

$$B=\mathrm{diag}(\mu_1,\ldots,\mu_N)^T$$

is defined by (2.40), $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^N$, and Φ_i and Ψ_i are similarly defined as

$$\Phi_i=(\phi_{i1},\phi_{i2},\ldots,\phi_{iN})^T,\quad \Psi_i=(\psi_{i1},\psi_{i2},\ldots,\psi_{iN})^T,\quad 1\leqslant i\leqslant n+1. \tag{3.18}$$

Now a binary Bargmann symmetry constraint of (3.4), (3.1) and (3.5) can be taken as

$$([\Delta,P])_{ij}=(J_i-J_j)\langle\Phi_i,B\Psi_j\rangle,\quad ([\Delta,Q])_{ij}=(K_i-K_j)\langle\Phi_i,B\Psi_j\rangle,\quad 1\leqslant i\neq j\leqslant n, \tag{3.19}$$

$$(\Delta F-\delta_{n+1}F)_i=\langle\Phi_i,B\Psi_{n+1}\rangle,\quad (\Delta G-\delta_{n+1}G)_i=\langle\Phi_{n+1},B\Psi_i\rangle,\quad 1\leqslant i\leqslant n. \tag{3.20}$$

This symmetry constraint gives us the following choice for the constraints on the extended potentials

$$p_{ij}=\tilde{p}_{ij}:=\frac{J_i-J_j}{\delta_i-\delta_j}\langle\Phi_i,B\Psi_j\rangle,\quad q_{ij}=\tilde{q}_{ij}:=\frac{K_i-K_j}{\delta_i-\delta_j}\langle\Phi_i,B\Psi_j\rangle,\quad 1\leqslant i\neq j\leqslant n, \tag{3.21}$$

$$f_i=\tilde{f}_i:=\frac{1}{\delta_i-\delta_{n+1}}\langle\Phi_i,B\Psi_{n+1}\rangle,\quad g_i=\tilde{g}_i:=\frac{1}{\delta_i-\delta_{n+1}}\langle\Phi_{n+1},B\Psi_i\rangle,\quad 1\leqslant i\leqslant n. \tag{3.22}$$

One can express the above symmetry constraint in another way. Actually, it can be proved that

$$(\delta P,\delta Q)=([\Delta,P],[\Delta,Q]),$$

and under the constraint (3.22),

$$\delta p_{ij}=(J_i-J_j)\langle\Phi_i,B\Psi_j\rangle,\quad \delta q_{ij}=(K_i-K_j)\langle\Phi_i,B\Psi_j\rangle,\quad 1\leqslant i\neq j\leqslant n,$$

are two symmetries of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4).

Now plug the above expressions for the extended potentials, (3.21) and (3.22), into the spectral problems (3.6) and the adjoint spectral problems (3.10), and then we get the constrained flows

$$\phi_x^{(s)}=\Omega^x(\tilde{F},\tilde{G},\lambda_s)\phi^{(s)},\quad \psi_x^{(s)}=-(\Omega^x(\tilde{F},\tilde{G},\lambda_s))^T\psi^{(s)}, \tag{3.23}$$

$$\phi_y^{(s)}=\Omega^y(\tilde{P},\tilde{F},\tilde{G},\lambda_s)\phi^{(s)},\quad \psi_y^{(s)}=-(\Omega^y(\tilde{P},\tilde{F},\tilde{G},\lambda_s))^T\psi^{(s)}, \tag{3.24}$$

$$\phi_t^{(s)}=\Omega^t(\tilde{Q},\tilde{F},\tilde{G},\lambda_s)\phi^{(s)},\quad \psi_t^{(s)}=-(\Omega^t(\tilde{Q},\tilde{F},\tilde{G},\lambda_s))^T\psi^{(s)}, \tag{3.25}$$

where

$$\tilde{P}=(\tilde{p}_{ij})_{n\times n},\quad \tilde{Q}=(\tilde{q}_{ij})_{n\times n},\quad \tilde{F}=(\tilde{f}_1,\ldots,\tilde{f}_n)^T,\quad \tilde{G}=(\tilde{g}_1,\ldots,\tilde{g}_n)^T. \tag{3.26}$$

All these three constrained flows are systems of ordinary differential equations of ϕ_{is} and ψ_{is}, $1\leqslant i\leqslant n+1$, $1\leqslant s\leqslant N$.

We introduce the symplectic structure

$$\omega^2=\sum_{i=1}^{n+1} Bd\Phi_i\wedge d\Psi_i=\sum_{i=1}^{n+1}\sum_{s=1}^{N}\mu_s d\phi_{is}\wedge d\psi_{is} \tag{3.27}$$

over $\mathbb{R}^{2(n+1)N}$. The corresponding Poisson bracket and the corresponding Hamiltonian form with the Hamiltonian H and the evolution variable t are similarly taken as

$$\{f,g\}=\sum_{i=1}^{n+1}\left(\left\langle\frac{\partial f}{\partial\Psi_i},B^{-1}\frac{\partial g}{\partial\Phi_i}\right\rangle-\left\langle\frac{\partial f}{\partial\Phi_i},B^{-1}\frac{\partial g}{\partial\Psi_i}\right\rangle\right),\quad f,g\in C^\infty(\mathbb{R}^{2(n+1)N}), \tag{3.28}$$

$$\Phi_{it}=\{\Phi_i,H\}=-B^{-1}\frac{\partial H}{\partial\Psi_i},\quad \Psi_{it}=\{\Psi_i,H\}=B^{-1}\frac{\partial H}{\partial\Phi_i},\quad 1\leqslant i\leqslant n+1. \tag{3.29}$$

Similar to Theorem 2.1, we have the following.

Theorem 3.1: *Under the symplectic structure (3.27), three constrained flows (3.23), (3.24) and (3.25) are Hamiltonian systems with the evolution variables x, y and t, and the Hamiltonians*

$$H_2^x=-\sum_{k=1}^{n}\langle A\Phi_k,B\Psi_k\rangle-\sum_{k=1}^{n}\frac{1}{\delta_k-\delta_{n+1}}\langle\Phi_k,B\Psi_{n+1}\rangle\langle\Phi_{n+1},B\Psi_k\rangle, \tag{3.30}$$

$$H_2^y=-\sum_{k=1}^{n} J_k\langle A\Phi_k ,B\Psi_k\rangle-\sum_{1\leqslant k<l\leqslant n}\frac{J_k-J_l}{\delta_k-\delta_l}\langle \Phi_k ,B\Psi_l\rangle\langle \Phi_l ,B\Psi_k\rangle$$
$$-\sum_{k=1}^{n}\frac{J_k}{\delta_k-\delta_{n+1}}\langle \Phi_k ,B\Psi_{n+1}\rangle\langle \Phi_{n+1} ,B\Psi_k\rangle, \tag{3.31}$$

$$H_2^t=-\sum_{k=1}^{n} K_k\langle A\Phi_k ,B\Psi_k\rangle-\sum_{1\leqslant k<l\leqslant n}\frac{K_k-K_l}{\delta_k-\delta_l}\langle \Phi_k ,B\Psi_l\rangle\langle \Phi_l ,B\Psi_k\rangle$$
$$-\sum_{k=1}^{n}\frac{K_k}{\delta_k-\delta_{n+1}}\langle \Phi_k ,B\Psi_{n+1}\rangle\langle \Phi_{n+1} ,B\Psi_k\rangle, \tag{3.32}$$

respectively, where A and B are defined by (2.40), Φ_i *and* Ψ_i, $1\leqslant i\leqslant n+1$, *are defined by (3.18). Moreover, they possess the necessary Lax representations*

$$(L^{(2)}(\lambda))_x=[\Omega^x(\tilde F,\tilde G,\lambda),L^{(2)}(\lambda)], \tag{3.33}$$

$$(L^{(2)}(\lambda))_y=[\Omega^y(\tilde P,\tilde F,\tilde G,\lambda),L^{(2)}(\lambda)], \tag{3.34}$$

$$(L^{(2)}(\lambda))_t=[\Omega^t(\tilde Q,\tilde F,\tilde G,\lambda),L^{(2)}(\lambda)], \tag{3.35}$$

respectively, where $\tilde P$, $\tilde Q$, $\tilde F$ *and* $\tilde G$ *are given by (3.26), (3.21) and (3.22), and* $L^{(2)}(\lambda)$ *is defined by*

$$\begin{aligned} &L^{(2)}(\lambda)=C_2+D_2(\lambda),\quad C_2=\mathrm{diag}(\Delta,\delta_{n+1})=\mathrm{diag}(\delta_1,\ldots,\delta_n,\delta_{n+1}),\\ &D_2=(D_{ij}^{(2)})_{n+1,n+1},\quad D_{ij}^{(2)}=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}\phi_{is}\psi_{js},\quad 1\leqslant i,j\leqslant n+1.\end{aligned} \tag{3.36}$$

Proof: It can be verified by a direct calculation that all three constrained flows (3.23)–(3.25) have the Hamiltonian structures under the symplectic structure (3.27) with the Hamiltonian functions H_2^x, H_2^y and H_2^t shown in (3.30)–(3.32). Let us now check three Lax representations (3.33)–(3.35). Since the proofs are similar for all three cases, we just show the second case, i.e., the Lax representation of the constrained flow (3.24). By using (3.24), we can compute that

$$\begin{aligned}(L^{(2)}(\lambda))_y&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}(\phi_y^{(s)}\psi^{(s)T}+\phi^{(s)}\psi_y^{(s)T})\\
&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}(\Omega^y(\tilde P,\tilde F,\tilde G,\lambda_s)\phi^{(s)}\psi^{(s)T}-\phi^{(s)}\psi^{(s)T}\Omega^y(\tilde P,\tilde F,\tilde G,\lambda_s))\\
&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}[\Omega^y(\tilde P,\tilde F,\tilde G,\lambda_s),\phi^{(s)}\psi^{(s)T}]\\
&=\sum_{s=1}^{N}\frac{\mu_s}{\lambda-\lambda_s}([\Omega^y(\tilde P,\tilde F,\tilde G,\lambda),\phi^{(s)}\psi^{(s)T}]-[\Omega^y(\tilde P,\tilde F,\tilde G,\lambda)\\
&\quad-\Omega^y(\tilde P,\tilde F,\tilde G,\lambda_s),\phi^{(s)}\psi^{(s)T}])\\
&=[\Omega^y(\tilde P,\tilde F,\tilde G,\lambda),L^{(2)}(\lambda)-C_2]-\left[\Omega_0^y,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\\
&=[\Omega^y(\tilde P,\tilde F,\tilde G,\lambda),L^{(2)}(\lambda)]-[\Omega_1^y(\tilde P,\tilde F,\tilde G),C_2]-\left[\Omega_0^y,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right].\end{aligned}$$

Therefore, it follows that $(L^{(2)}(\lambda))_y=[\Omega^y(\tilde{P},\tilde{F},\tilde{G},\lambda),L^{(2)}(\lambda)]$ if and only if

$$[C_2,\Omega_1^y(\tilde{P},\tilde{F},\tilde{G})]=\left[\Omega_0^y,\sum_{s=1}^N \mu_s\phi^{(s)}\psi^{(s)T}\right].$$

This equality equivalently requires the nonlinear constraints on the potentials defined by (3.21) and (3.22). Therefore, the constrained flow (3.24) has the necessary Lax representation shown in (3.34). The proof is finished. ■

We also remark that the Lax representations (3.33)–(3.35) are not sufficient to generate the corresponding constrained flows defined by (3.23)–(3.25), since the Gateaux derivative operators of the Lax operators Ω^x, Ω^y and Ω^t given in (3.23)–(3.25) are not injective. However, it will be shown that they are good enough in generating integrals of motion of the constrained flows.

IV. AN INVOLUTIVE AND FUNCTIONALLY INDEPENDENT SYSTEM OF POLYNOMIAL FUNCTIONS

Let m be an arbitrary natural number. We start from an mth-order matrix Lax operator

$$L(\lambda)=L(\lambda;c_1,\ldots,c_m)=C+D(\lambda), \tag{4.1}$$

with C and $D(\lambda)$ being defined by

$$C=\mathrm{diag}(c_1,\ldots,c_m),\quad D(\lambda)=(D_{ij}(\lambda))_{m\times m},\quad D_{ij}(\lambda)=\sum_{s=1}^N \frac{\mu_s}{\lambda-\lambda_s}\phi_{is}\psi_{js},\quad 1\leqslant i,j\leqslant m. \tag{4.2}$$

Here c_i, λ_s, and μ_s are arbitrary constants satisfying

$$\prod_{s=1}^N \mu_s\neq 0,\quad \lambda_i\neq\lambda_j,\quad 1\leqslant i\neq j\leqslant N, \tag{4.3}$$

and ϕ_{is} and ψ_{js} are pairs of canonical variables of the symplectic manifold $(\mathbb{R}^{2mN},\omega^2)$ with the symplectic structure

$$\omega^2=\sum_{i=1}^m\sum_{s=1}^N \mu_s d\phi_{is}\wedge d\psi_{is}. \tag{4.4}$$

The corresponding Poisson bracket reads as

$$\{f,g\}=\omega^2(Idg,Idf)=\sum_{i=1}^m\sum_{s=1}^N \mu_s^{-1}\left(\frac{\partial f}{\partial\psi_{is}}\frac{\partial g}{\partial\phi_{is}}-\frac{\partial f}{\partial\phi_{is}}\frac{\partial g}{\partial\psi_{is}}\right),\quad f,g\in C^\infty(\mathbb{R}^{2mN}). \tag{4.5}$$

A. r-matrix formulation

As usual, two special matrices defined by the tensor product of matrices are chosen as

$$L_1(\lambda)=L(\lambda)\otimes I_m,\quad L_2(\mu)=I_m\otimes L(\mu), \tag{4.6}$$

where I_m is the mth-order identity matrix, and

$$(A\otimes B)_{ij,kl}=a_{ik}b_{jl}\ \text{ if }\ A=(a_{ij})\ \text{ and }\ B=(b_{ij}). \tag{4.7}$$

We want to find an $m^2\times m^2$ matrix $\mathbf{r}=\mathbf{r}(\lambda,\mu)$ so that we have an $\mathbf{r}$-matrix formulation[50,51]

$$\{L(\lambda)\overset{\otimes}{,}L(\mu)\}=[\mathbf{r}(\lambda,\mu),L_1(\lambda)+L_2(\mu)], \tag{4.8}$$

with the Poisson bracket $\{L(\lambda)\overset{\otimes}{,}L(\mu)\}$ being defined by

$$(\{L(\lambda)\overset{\otimes}{,}L(\mu)\})_{ij,kl}=\{L_{ik}(\lambda),L_{jl}(\mu)\}=\omega^2(IdL_{jl}(\mu),IdL_{ik}(\lambda)),\quad 1\leqslant i,j,k,l\leqslant m, \tag{4.9}$$

where $L=(L_{ij})_{m\times m}$ is assumed. Let us first compute $\{L_{ij}(\lambda),L_{kl}(\mu)\}$. When $i\neq l$ and $j\neq k$, it is easy to obtain $\{L_{ij}(\lambda),L_{kl}(\mu)\}=0$. When $i\neq l$ and $j=k$, we have

$$\begin{aligned}\{L_{ij}(\lambda),L_{jl}(\mu)\}&=\sum_{s=1}^{N}\mu_s\frac{\phi_{is}}{\lambda-\lambda_s}\frac{\psi_{ls}}{\mu-\lambda_s}\\&=\sum_{s=1}^{N}\frac{1}{\mu-\lambda}\left(\frac{\mu_s}{\lambda-\lambda_s}-\frac{\mu_s}{\mu-\lambda_s}\right)\phi_{is}\psi_{ls}\\&=\frac{1}{\mu-\lambda}(L_{il}(\lambda)-L_{il}(\mu)).\end{aligned}$$

Similarly, when $i=l$ and $j\neq k$, we have

$$\{L_{ij}(\lambda),L_{ki}(\mu)\}=-\sum_{s=1}^{N}\mu_s\frac{\psi_{js}}{\lambda-\lambda_s}\frac{\phi_{ks}}{\mu-\lambda_s}=\frac{1}{\mu-\lambda}(L_{kj}(\mu)-L_{kj}(\lambda)),$$

and when $i=l$ and $j=k$, we have

$$\begin{aligned}\{L_{ij}(\lambda),L_{ji}(\mu)\}&=\sum_{s=1}^{N}\mu_s\frac{\phi_{is}}{\lambda-\lambda_s}\frac{\psi_{is}}{\mu-\lambda_s}-\sum_{s=1}^{N}\mu_s\frac{\psi_{js}}{\lambda-\lambda_s}\frac{\phi_{js}}{\mu-\lambda_s}\\&=\frac{1}{\mu-\lambda}[(L_{ii}(\lambda)-L_{ii}(\mu))-(L_{jj}(\lambda)-L_{jj}(\mu))].\end{aligned}$$

Therefore, we obtain

$$\{L_{ij}(\lambda),L_{kl}(\mu)\}=\begin{cases}0,\quad \text{when}\ \ i\neq l,\ \ j\neq k;\\ \dfrac{1}{\mu-\lambda}(L_{kj}(\mu)-L_{kj}(\lambda)),\quad \text{when}\ \ i=l,\ \ j\neq k;\\ \dfrac{1}{\mu-\lambda}(L_{il}(\lambda)-L_{il}(\mu)),\quad \text{when}\ \ i\neq l,\ \ j=k;\\ \dfrac{1}{\mu-\lambda}[(L_{ii}(\lambda)-L_{ii}(\mu))-(L_{jj}(\lambda)-L_{jj}(\mu))],\quad \text{when}\ \ i=l,\ \ j=k.\end{cases} \tag{4.10}$$

In view of this property, we claim that

$$\mathbf{r}(\lambda,\mu)=\frac{1}{\mu-\lambda}\mathcal{P},\quad \mathcal{P}=\sum_{p,q=1}^{m}E_{pq}\otimes E_{qp}, \tag{4.11}$$

where E_{pq} is an $m\times m$ matrix with the (p,q) entry being one but the others, zero. Let us second compute that

$$\begin{aligned}
&\left(\left[\frac{1}{\mu-\lambda}\mathcal{P},L_1(\lambda)+L_2(\mu)\right]\right)_{ij,kl}\\
&=\frac{1}{\mu-\lambda}([\mathcal{P},L_1(\lambda)]+[\mathcal{P},L_2(\mu)])_{ij,kl}\\
&=\frac{1}{\mu-\lambda}\sum_{p,q=1}^{m}([E_{pq},L(\lambda)]\otimes E_{qp}+E_{qp}\otimes[E_{pq},L(\mu)])_{ij,kl}\\
&=\frac{1}{\mu-\lambda}\sum_{p,q=1}^{m}([E_{pq},L(\lambda)])_{ik}(E_{qp})_{jl}+(E_{qp})_{ik}[E_{pq},L(\mu)]_{jl}\\
&=\frac{1}{\mu-\lambda}([E_{lj},L(\lambda)]_{ik}+[E_{ki},L(\mu))]_{jl},
\end{aligned}$$

where we have used $(A\otimes B)(A'\otimes B')=(AA')\otimes(BB')$. Further noting that

$$[E_{pq},L]=E_{pq}L-LE_{pq}=\begin{array}{c}\\ \\ p\text{th}\\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} & & q\text{th} & & \end{array}\\ \begin{bmatrix} 0 & \cdots & -L_{1p} & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ L_{q1} & \cdots & L_{qq}-L_{pp} & \cdots & L_{qm}\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & -L_{mp} & \cdots & 0 \end{bmatrix}\end{array},$$

we have

$$\begin{aligned}
&\left(\left[\frac{1}{\mu-\lambda}\mathcal{P},L_1(\lambda)+L_2(\mu)\right]\right)_{ij,kl}\\
&=\begin{cases} 0, \quad \text{when } i\neq l,\ j\neq k;\\ \frac{1}{\mu-\lambda}(L_{jk}(\lambda)-L_{jk}(\mu)), \quad \text{when } i=l,\ j\neq k;\\ \frac{1}{\mu-\lambda}(-L_{il}(\lambda)+L_{il}(\mu)), \quad \text{when } i\neq l,\ j=k;\\ \frac{1}{\mu-\lambda}[(L_{jj}(\lambda)-L_{ii}(\lambda))+(L_{ii}(\mu)-L_{jj}(\mu))], \quad \text{when } i=l,\ j=k. \end{cases}
\end{aligned} \tag{4.12}$$

Now (4.10) and (4.12) shed right on the following theorem.

Theorem 4.1: *If $L(\lambda)=L(\lambda;c_1,\ldots,c_m)$ is defined by (4.1) and (4.2), then the* **r***-matrix formulation*

$$\{L(\lambda)\overset{\otimes}{,}L(\mu)\}=[\mathbf{r}(\lambda,\mu),L(\lambda)\otimes I_m+I_m\otimes L(\mu)],\quad \mathbf{r}=\frac{1}{\mu-\lambda}\sum_{i,j=1}^{m}E_{ij}\otimes E_{ji} \tag{4.13}$$

holds for arbitrary constants $c_1,c_2,\ldots,c_m$.

It follows from (4.13) that

$$\{L^k(\lambda)\overset{\otimes}{,}L^l(\mu)\}=[\mathbf{r}^{k,l}(\lambda,\mu),L_1(\lambda)+L_2(\mu)],\quad k,l\geqslant 1, \tag{4.14}$$

where $\mathbf{r}^{k,l}(\lambda,\mu)$ is given by[52]

$$\mathbf{r}^{k,l}(\lambda,\mu)=\sum_{i=1}^{k}\sum_{j=1}^{l} L_1^{k-i}(\lambda)L_2^{l-j}(\mu)\mathbf{r}(\lambda,\mu)L_1^{i-1}(\lambda)L_2^{j-1}(\mu). \tag{4.15}$$

Since for $A=(a_{ij})_{m\times m}$ and $B=(b_{ij})_{m\times m}$ we have

$$\mathrm{tr}\{A \overset{\otimes}{,} B\}=\sum_{i,j=1}^{m}\{A \overset{\otimes}{,} B\}_{ij,ij}=\sum_{i,j=1}^{m}\{a_{ii},b_{jj}\}=\{\mathrm{tr}\,A,\mathrm{tr}\,B\}, \tag{4.16}$$

we can compute, based on (4.14), that

$$\begin{aligned}&\{\mathrm{tr}\,L^k(\lambda),\mathrm{tr}\,L^l(\mu)\}\\ &=\mathrm{tr}\{L^k(\lambda)\overset{\otimes}{,}L^l(\mu)\}=\mathrm{tr}[\mathbf{r}^{k,l}(\lambda,\mu),L_1(\lambda)+L_2(\mu)]=0,\ \ k,l\geqslant 1.\end{aligned} \tag{4.17}$$

This will be used to generate an involutive system of functions defined over the symplectic manifold $(\mathbb{R}^{2mN},\omega^2)$ for any natural number m.

B. An involutive and functionally independent system

Let us begin to construct an involutive system of polynomial functions by expanding

$$\det(\nu I_m-L(\lambda))=\nu^m-\mathcal{F}_\lambda^{(1)}\nu^{m-1}+\mathcal{F}_\lambda^{(2)}\nu^{m-2}+\cdots+(-1)^m\mathcal{F}_\lambda^{(m)},\quad \nu=\mathrm{const}, \tag{4.18}$$

where $\mathcal{F}_\lambda^{(k)}$, $1\leqslant k\leqslant m$, must read as

$$\mathcal{F}_\lambda^{(k)}=\mathcal{F}_\lambda^{(k)}(c_1,\ldots,c_m)=\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant m}\begin{vmatrix} L_{j_1j_1} & L_{j_1j_2} & \cdots & L_{j_1j_k}\\ L_{j_2j_1} & L_{j_2j_2} & \cdots & L_{j_2j_k}\\ \vdots & \vdots & \ddots & \vdots\\ L_{j_kj_1} & L_{j_kj_2} & \cdots & L_{j_kj_k}\end{vmatrix},\quad 1\leqslant k\leqslant m. \tag{4.19}$$

Here we mention once more that $L=(L_{ij})_{m\times m}$ is assumed. We define bilinear functions $\overset{ij}{Q}_\lambda$ on $\mathbb{R}^N$

$$\overset{ij}{Q}_\lambda=\sum_{s=1}^{N}\mu_s\frac{\phi_{is}\psi_{js}}{\lambda-\lambda_s}=\sum_{l\geqslant 0}\langle A^l\Phi_i,B\Psi_j\rangle\lambda^{-l-1},\quad 1\leqslant i,j\leqslant m, \tag{4.20}$$

where A and B are given by (2.40), and Φ_i and Ψ_i are defined as before,

$$\Phi_i=(\phi_{i1},\phi_{i2},\ldots,\phi_{iN})^T,\quad \Psi_i=(\psi_{i1},\psi_{i2},\ldots,\psi_{iN})^T,\quad 1\leqslant i\leqslant m. \tag{4.21}$$

Then we have

$$L_{ij}=\sum_{l\geqslant 0}\langle A^l\Phi_i,B\Psi_j\rangle\lambda^{-l-1}=\overset{ij}{Q}_\lambda,\quad 1\leqslant i\neq j\leqslant m,$$

$$L_{ii}=c_i+\sum_{l\geqslant 0}\langle A^l\Phi_i,B\Psi_i\rangle\lambda^{-l-1}=c_i+\overset{ii}{Q}_\lambda,\quad 1\leqslant i\leqslant m.$$

Therefore, the system of functions $\mathcal{F}_\lambda^{(k)}$ is transformed into

$$\mathcal{F}_\lambda^{(k)}=\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant m}\begin{vmatrix} c_{j_1}+Q_\lambda^{j_1j_1} & Q_\lambda^{j_1j_2} & \cdots & Q_\lambda^{j_1j_k} \\ Q_\lambda^{j_2j_1} & c_{j_2}+Q_\lambda^{j_2j_2} & \cdots & Q_\lambda^{j_2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ Q_\lambda^{j_kj_1} & Q_\lambda^{j_kj_2} & \cdots & c_{j_k}+Q_\lambda^{j_kj_k}\end{vmatrix}, \quad 1\leqslant k\leqslant m. \tag{4.22}$$

A set of more concrete formulas for computing $\mathcal{F}_\lambda^{(k)}$ will be given in Appendix B. Now we further expand $\mathcal{F}_\lambda^{(k)}$ as a power series of $1/\lambda$:

$$\mathcal{F}_\lambda^{(k)}=\mathcal{F}_\lambda^{(k)}(c_1,\ldots,c_m)=\sum_{l\geqslant 0}F_{kl}(c_1,\ldots,c_m)\lambda^{-l}, \quad 1\leqslant k\leqslant m. \tag{4.23}$$

Based on the formulas of $\mathcal{F}_\lambda^{(k)}$ in Appendix B, it is not difficult to find that

$$F_{k0}=F_{k0}(c_1,\ldots,c_m)=\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant m}\prod_{p=1}^{k}c_{j_p},$$

$$\begin{aligned}F_{kl}&=F_{kl}(c_1,\ldots,c_m)\\ &=\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant m}\sum_{r=1}^{\min(k,l)}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant k}\prod_{\substack{p=1\\ p\neq i_1,i_2,\cdots,i_r}}^{k}c_{j_p}\\ &\quad\times\sum_{\substack{p_1+p_2+\cdots+p_r=l-r\\ p_1,p_2,\cdots,p_r\geqslant 0}}\begin{vmatrix}\langle A^{p_1}\Phi_{j_{i_1}},B\Psi_{j_{i_1}}\rangle & \langle A^{p_2}\Phi_{j_{i_2}},B\Psi_{j_{i_1}}\rangle & \cdots & \langle A^{p_r}\Phi_{j_{i_r}},B\Psi_{j_{i_1}}\rangle\\ \langle A^{p_1}\Phi_{j_{i_1}},B\Psi_{j_{i_2}}\rangle & \langle A^{p_2}\Phi_{j_{i_2}},B\Psi_{j_{i_2}}\rangle & \cdots & \langle A^{p_r}\Phi_{j_{i_r}},B\Psi_{j_{i_2}}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle A^{p_1}\Phi_{j_{i_1}},B\Psi_{j_{i_r}}\rangle & \langle A^{p_2}\Phi_{j_{i_2}},B\Psi_{j_{i_r}}\rangle & \cdots & \langle A^{p_r}\Phi_{j_{i_r}},B\Psi_{j_{i_r}}\rangle\end{vmatrix}, \quad l\geqslant 1,\end{aligned} \tag{4.24}$$

which are all polynomials in the canonical variables ϕ_{is} and ψ_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$.

Theorem 4.2: *For all constants $c_1,c_2,\ldots,c_m$, the polynomial functions in ϕ_{is} and ψ_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$: $F_{il}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $l\geqslant 1$, defined by (4.24), are in involution in pair with respect to the Poisson bracket (4.5).*

Proof: On the one hand, by using Newton's identities on elementary symmetric polynomials[53]

$$\zeta_k(\lambda)-\mathcal{F}_\lambda^{(1)}\zeta_{k-1}(\lambda)+\mathcal{F}_\lambda^{(2)}\zeta_{k-2}(\lambda)+\cdots+(-1)^{k-1}\mathcal{F}_\lambda^{(k-1)}\zeta_1(\lambda)+(-1)^k k\mathcal{F}_\lambda^{(k)}=0,$$

where $1\leqslant k\leqslant m$ and

$$\zeta_i(\lambda)=\mathrm{tr}\, L^i(\lambda), \quad 1\leqslant i\leqslant m,$$

we can have

$$\mathcal{F}_\lambda^{(k)}=\mathcal{F}_\lambda^{(k)}(\zeta_1(\lambda),\zeta_2(\lambda),\ldots,\zeta_k(\lambda)), \quad 1\leqslant k\leqslant m. \tag{4.25}$$

Therefore, we can compute that

$$\{\mathcal{F}_\lambda^{(k)},\mathcal{F}_\mu^{(i)}\}=\{\mathcal{F}_\lambda^{(k)}(\zeta_1(\lambda),\zeta_2(\lambda),\ldots,\zeta_k(\lambda)),\mathcal{F}_\mu^{(i)}(\zeta_1(\mu),\zeta_2(\mu),\ldots,\zeta_i(\mu))\}$$

$$=\sum_{l=1}^{k}\sum_{j=1}^{i}\frac{\partial\mathcal{F}_\lambda^{(k)}}{\partial\zeta_l(\lambda)}\frac{\mathcal{F}_\mu^{(i)}}{\partial\zeta_j(\mu)}\{\mathrm{tr}L^l(\lambda),\mathrm{tr}\,L^j(\mu)\}=0,\quad 1\leqslant k,i\leqslant m.$$

The last equality is a consequence of the involutivity of $\zeta_i(\lambda)$, $1\leqslant i\leqslant m$, shown in (4.17). On the other hand, we have

$$\{\mathcal{F}_\lambda^{(k)},\mathcal{F}_\mu^{(i)}\}=\sum_{l,j\geqslant 0}\{F_{kl},F_{ij}\}\lambda^{-l}\mu^{-j}.$$

It follows that the polynomial functions $F_{il}=F_{il}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $l\geqslant 1$, are in involution in pair with respect to the Poisson bracket (4.5). ■

Let us now go on to show the functional independence of the polynomial functions $F_{is}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$.

Theorem 4.3: *If all constants $c_1,c_2,\ldots,c_m$ are distinct, then the polynomial functions in ϕ_{is} and ψ_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$: $F_{is}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, defined by (4.24), are functionally independent over a dense open subset of $\mathbb{R}^{2mN}$.*

Proof: Let P_0 be a point of $\mathbb{R}^{2mN}$ satisfying

$$\phi_{is}=\varepsilon,\quad 1\leqslant i\leqslant m,\quad 1\leqslant s\leqslant N,$$

where ε is a small constant. Keep (4.24) in mind, and then at this point P_0, we obviously have

$$\frac{\partial F_{is_1}}{\partial\psi_{js_2}}=\frac{\partial}{\partial\psi_{js_2}}\sum_{1\leqslant j_1<j_2<\cdots<j_i\leqslant m}\sum_{q=1}^{i}\prod_{\substack{p=1\\p\neq q}}^{i}c_{j_p}\langle A^{s_1-1}\Phi_{j_q},B\Psi_{j_q}\rangle+\mathrm{O}(\varepsilon^2)$$

$$=\varepsilon\sum_{\substack{1\leqslant j_1<j_2<\cdots<j_{i-1}\leqslant m\\ j_1,j_2,\cdots,j_{i-1}\neq j}}c_{j_1}c_{j_2}\cdots c_{j_{i-1}}\lambda_{s_2}^{s_1-1}\mu_{s_2}+\mathrm{O}(\varepsilon^2),\tag{4.26}$$

where $1\leqslant i,j\leqslant m$, $1\leqslant s_1,s_2\leqslant N$. In the above computation, only the term with $r=1$ in the expression (4.24) of F_{is} contributes to the first-order term of ε. Let the matrix Θ_N be defined by

$$\Theta_N=(\Theta_{ij}^{(N)})_{N\times N},\quad \Theta_{ij}^{(N)}=\lambda_i^{j-1}\mu_i,\quad 1\leqslant i,j\leqslant N,$$

whose determinant is easily found to be

$$\det(\Theta_N)=\prod_{i=1}^{N}\mu_i\prod_{1\leqslant i<j\leqslant N}(\lambda_j-\lambda_i).$$

Then at the point P_0, the Jacobian of the functions F_{is_1} with respect to ψ_{js_2} can be computed as follows

$$\frac{\partial(F_{11},\cdots,F_{1N},F_{21},\cdots,F_{2N},\cdots,F_{m1},\cdots,F_{mN})}{\partial(\psi_{11},\cdots,\psi_{1N},\psi_{21},\cdots,\psi_{2N},\cdots,\psi_{m1},\cdots,\psi_{mN})}$$

$$=\varepsilon^{mN}\begin{vmatrix} \Theta_N & \sum_{i=2}^{m} c_i\Theta_N & \sum_{2\leqslant i<j\leqslant m} c_ic_j\Theta_N & \cdots & \prod_{i=2}^{m} c_i\Theta_N \\ \Theta_N & \sum_{\substack{i=1\\ i\neq 2}}^{m} c_i\Theta_N & \sum_{\substack{1\leqslant i<j\leqslant m\\ i,j\neq 2}} c_ic_j\Theta_N & \cdots & \prod_{\substack{i=1\\ i\neq 2}}^{m} c_i\Theta_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_N & \sum_{i=1}^{m-1} c_i\Theta_N & \sum_{1\leqslant i<j\leqslant m-1} c_ic_j\Theta_N & \cdots & \prod_{i=1}^{m-1} c_i\Theta_N \end{vmatrix} + \mathrm{O}(\varepsilon^{mN+1})$$

$$=\varepsilon^{mN}\det(\Omega_m\otimes\Theta_N)+\mathrm{O}(\varepsilon^{mN+1})$$

$$=\varepsilon^{mN}(\det(\Omega_m))^N(\det(\Theta_N))^m+\mathrm{O}(\varepsilon^{mN+1})$$

$$=\varepsilon^{mN}\prod_{1\leqslant i<j\leqslant m}(c_i-c_j)^N\prod_{i=1}^{N}\mu_i\prod_{1\leqslant i<j\leqslant N}(\lambda_j-\lambda_i)^m+\mathrm{O}(\varepsilon^{mN+1}),$$

where we have used the determinant property of the tensor product of matrices and the determinant result of the matrix Ω_m in Appendix C. This allows us to conclude that if the constants $c_1,c_2,\ldots,c_m$ are distinct, the above Jacobian is not zero at P_0 when $\varepsilon\neq 0$ is small enough. Since the Jacobian is a polynomial function of ϕ_{is} and ψ_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, it is not zero over a dense open subset of $\mathbb{R}^{2mN}$. Therefore, the functions F_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, are functionally independent over that dense open subset of $\mathbb{R}^{2mN}$. The proof is completed. ■

C. An alternative involutive system to the F_{is}'s

We would like to express the involutive system of the polynomial functions F_{is} in another way, and so we introduce

$$s_0(v_1,\cdots,v_m)=1, \tag{4.27a}$$

$$s_k(v_1,\cdots,v_m)=\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant m} v_{j_1}\cdots v_{j_k},\quad 1\leqslant k\leqslant m, \tag{4.27b}$$

$$s_k(v_1,\cdots,v_m)=0,\quad \text{when}\ \ k\geqslant m+1\ \ \text{or}\ \ k\leqslant -1, \tag{4.27c}$$

where $v_1,v_2,\ldots,v_m$ are m numbers. Obviously, for $m\geqslant 2$, we have the following relation:

$$s_k(v_1,\ldots,v_m)=v_ms_{k-1}(v_1,\ldots,v_{m-1})+s_k(v_1,\ldots,v_{m-1}),\ \ k\in\mathbb{Z}. \tag{4.28}$$

Let us now define

$$E_{1l}=F_{1l},\ \ E_{il}=(-1)^{i+1}F_{il}+\sum_{j=1}^{i-1}(-1)^{j+1}s_j(c_1,\ldots,c_m)E_{i-j,l},\quad i\geqslant 2,\ \ l\geqslant 1. \tag{4.29}$$

From (4.29), we can have

$$F_{il}=\sum_{j=0}^{i-1}(-1)^{i-j+1}s_j(c_1,\ldots,c_m)E_{i-j,l},\quad i,l\geqslant 1. \tag{4.30}$$

Therefore, by Proposition D.2 in Appendix D, we obtain

$$E_{il}=E_{il}(c_1,\ldots,c_m)$$
$$=\sum_{r=1}^{\min(i,l)}(-1)^{r+1}\sum_{1\leqslant j_1<j_2<\cdots<j_r\leqslant m}\ \sum_{\substack{l_1+l_2+\cdots+l_r=i-r\\ l_1,l_2,\cdots,l_r\geqslant 0}} c_{j_1}^{l_1}c_{j_2}^{l_2}\cdots c_{j_r}^{l_r}$$
$$\times\sum_{\substack{p_1+p_2+\cdots+p_r=l-r\\ p_1,p_2,\cdots,p_r\geqslant 0}}\begin{vmatrix}\langle A^{p_1}\Phi_{j_1},B\Psi_{j_1}\rangle & \langle A^{p_2}\Phi_{j_2},B\Psi_{j_1}\rangle & \cdots & \langle A^{p_r}\Phi_{j_r},B\Psi_{j_1}\rangle\\ \langle A^{p_1}\Phi_{j_1},B\Psi_{j_2}\rangle & \langle A^{p_2}\Phi_{j_2},B\Psi_{j_2}\rangle & \cdots & \langle A^{p_r}\Phi_{j_r},B\Psi_{j_2}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle A^{p_1}\Phi_{j_1},B\Psi_{j_r}\rangle & \langle A^{p_2}\Phi_{j_2},B\Psi_{j_r}\rangle & \cdots & \langle A^{p_r}\Phi_{j_r},B\Psi_{j_r}\rangle\end{vmatrix}, \tag{4.31}$$

where $1\leqslant i\leqslant m$ and $l\geqslant 1$. Obviously, each E_{il} is a linear combination of the F_{il}'s, and hence $\{E_{ik},E_{jl}\}=0$ holds for all $1\leqslant i,j\leqslant m$ and $k,l\geqslant 1$. This means that the polynomial functions E_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, are also in involution in pair.

In order to show the functional independence of E_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, similar to the proof of Theorem 4.3, let P_0 be a point of $\mathbb{R}^{2mN}$ satisfying $\phi_{is}=\varepsilon$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, where ε is a small constant. Then at this point P_0, we have

$$\frac{\partial E_{is_1}}{\partial\psi_{js_2}}=\varepsilon c_j^{i-1}\lambda_{s_2}^{s_1-1}\mu_{s_2}+\mathrm{O}(\varepsilon^2),\quad 1\leqslant i,j\leqslant m,\ \ 1\leqslant s_1,s_2\leqslant N. \tag{4.32}$$

Hence a direct argument can give rise to

$$\frac{\partial(E_{11},\cdots,E_{1N},E_{21},\cdots,E_{2N},\cdots,E_{m1},\cdots,E_{mN})}{\partial(\psi_{11},\cdots,\psi_{1N},\psi_{21},\cdots,\psi_{2N},\cdots,\psi_{m1},\cdots,\psi_{mN})}$$
$$=\varepsilon^{mN}\prod_{i=1}^{N}\mu_i\prod_{1\leqslant i<j\leqslant N}(\lambda_j-\lambda_i)^m\prod_{1\leqslant i<j\leqslant m}(c_j-c_i)^N+\mathrm{O}(\varepsilon^{mN+1}). \tag{4.33}$$

Therefore, if $c_1,c_2,\ldots,c_m$ are distinct, the above Jacobian is not zero at P_0 when $\varepsilon\neq 0$ is small enough. This implies that the functions E_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, are functionally independent over a dense open subset of $\mathbb{R}^{2mN}$.

Let us sum up these results in the following theorem.

Theorem 4.4: *All polynomial functions in ϕ_{is} and ψ_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$: $E_{il}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $l\geqslant 1$, defined by (4.31), are in involution in pair with respect to the Poisson bracket (4.5) for all constants $c_1,c_2,\ldots,c_m$. Moreover, among them the polynomial functions $E_{is}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, are functionally independent over a dense open subset of $\mathbb{R}^{2mN}$ for distinct constants $c_1,c_2,\ldots,c_m$.*

Note that all polynomial functions F_{il} are also linear combinations of the E_{il}'s. The above theorem actually shows us an alternative to the involutive and functionally independent system of the polynomial functions F_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$. The E_{is}'s have the compact form for the constants $c_1,c_2,\ldots,c_m$, and thus it is more convenient to deal with them.

V. LIOUVILLE INTEGRABILITY AND INVOLUTIVE SOLUTIONS

Let us now turn to establish the Liouville integrability of the obtained constrained flows, and to present involutive solutions of the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions. The involutive system of the polynomial functions

$$F_{is}=F_{is}(c_1,\ldots,c_m),\quad 1\leqslant i\leqslant m,\quad 1\leqslant s\leqslant N,$$

alternatively

$$E_{is}=E_{is}(c_1,\ldots,c_m),\quad 1\leqslant i\leqslant m,\quad 1\leqslant s\leqslant N,$$

will play an extremely important role in the following discussion.

A. Liouville integrability of the constrained flows

For the $1+1$ dimensional case, we have the matrix Lax operator as defined by (2.49) and (2.50), i.e.,

$$L^{(1)}(\lambda)=L^{(1)}(\lambda;\gamma_1,\ldots,\gamma_n)=C_1(\gamma_1,\ldots,\gamma_n)+D_1(\lambda),$$

where C_1 and $D_1(\lambda)$ are given by (2.50). Note that

$$\gamma_i\neq\gamma_j,\quad 1\leqslant i\neq j\leqslant n.$$

According to Theorems 4.2 and 4.3 for the case $m=n$ and $c_i=\gamma_i$, $1\leqslant i\leqslant n$, we know that $F_{is}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant N$, defined by (4.24), are functionally independent over a dense open subset of $\mathbb{R}^{2nN}$ and in involution in pair with respect to the Poisson bracket (2.47), i.e.,

$$\{f,g\}=\sum_{i=1}^{n}\left(\left\langle\frac{\partial f}{\partial\Psi_i},B^{-1}\frac{\partial g}{\partial\Phi_i}\right\rangle-\left\langle\frac{\partial f}{\partial\Phi_i},B^{-1}\frac{\partial g}{\partial\Psi_i}\right\rangle\right),\quad f,g\in C^{\infty}(\mathbb{R}^{2nN}).$$

Theorem 5.1: *Let $\gamma_1,\gamma_2,\ldots,\gamma_n$ be n distinct numbers. Then the spatial constrained flow (2.43) and the temporal constrained flow (2.44) of the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22) are Liouville integrable Hamiltonian systems, which possess involutive and functionally independent integrals of motion*

$$F_{is}(\gamma_1,\ldots,\gamma_n),\quad 1\leqslant i\leqslant n,\quad 1\leqslant s\leqslant N,$$

defined by (4.24) in the case

$$m=n,\quad c_i=\gamma_i,\quad 1\leqslant i\leqslant n.$$

Proof: From the necessary Lax representations of the spatial constrained flow (2.43) and the temporal constrained flow (2.44),

$$(L^{(1)}(\lambda))_x=[U(\tilde{u},\lambda),L^{(1)}(\lambda)],\quad (L^{(1)}(\lambda))_{t_1}=[V^{(1)}(\tilde{u},\lambda),L^{(1)}(\lambda)],$$

which are shown in Theorem 2.1, we can obtain[26]

$$((L^{(1)}(\lambda))^i)_x=[U(\tilde{u},\lambda),(L^{(1)}(\lambda))^i],\quad ((L^{(1)}(\lambda))^j)_{t_1}=[V^{(1)}(\tilde{u},\lambda),(L^{(1)}(\lambda))^j],\quad i,j\geqslant 1,$$

and thus we have

$$(\mathrm{tr}(L^{(1)}(\lambda))^i)_x=\mathrm{tr}((L^{(1)}(\lambda))^i)_x=\mathrm{tr}[U(\tilde{u},\lambda),(L^{(1)}(\lambda))^i]=0,\quad i\geqslant 1,$$

$$(\mathrm{tr}(L^{(1)}(\lambda))^j)_{t_1}=\mathrm{tr}((L^{(1)}(\lambda))^j)_{t_1}=\mathrm{tr}[V^{(1)}(\tilde{u},\lambda),(L^{(1)}(\lambda))^j]=0,\quad j\geqslant 1.$$

Therefore, $\mathcal{F}_{\lambda}^{(k)}(\gamma_1,\ldots,\gamma_n)$ are all generating functions of integrals of motion of (2.43) and (2.44) in the light of the expression (4.25) determined by Newton's identities. It follows that $F_{is}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant N$, are all integrals of motion of the spatial constrained flow (2.43) and the temporal constrained flow (2.44). Note that all constants $\gamma_1,\gamma_2,\ldots,\gamma_n$ are distinct. Therefore, Theorems 4.2 and 4.3 in the case of $m=n$ and $c_i=\gamma_i$, $1\leqslant i\leqslant n$, together with Theorem 2.1, show that the spatial constrained flow (2.43) and the temporal constrained flow (2.44) are Liouville integrable Hamiltonian systems, which possess the involutive and functionally independent integrals of motion $F_{is}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant N$. The proof is finished. ■

We remark that from the Lax representations shown in Theorem 2.1, we have

$$(\nu I_n - L^{(1)}(\lambda))_x = [U(\tilde{u},\lambda), \nu I_n - L^{(1)}(\lambda)],$$

$$(\nu I_n - L^{(1)}(\lambda))_{t_1} = [V^{(1)}(\tilde{u},\lambda), \nu I_n - L^{(1)}(\lambda)]$$

for any constant ν. It follows[54] that $\det(\nu I_n - L^{(1)}(\lambda))$ is a common generating function of integrals of motion of the constrained flows (2.43) and (2.44), and thus so are $\mathcal{F}_\lambda^{(k)}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant k\leqslant n$. This is an alternative proof for showing that $\mathcal{F}_\lambda^{(k)}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant k\leqslant n$, are the generating functions of integrals of motion of (2.43) and (2.44).

For the $2+1$ dimensional case, a completely similar argument can give rise to the following theorem on the Liouville integrability of the constrained flows (3.23)–(3.25) of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4).

Theorem 5.2: *Let $\delta_1,\ldots,\delta_n,\delta_{n+1}$ be $n+1$ distinct numbers. Then all three constrained flows (3.23)–(3.25) of the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4) are Liouville integrable Hamiltonian systems, which possess the involutive and functionally independent integrals of motion*

$$F_{is}(\delta_1,\ldots,\delta_n,\delta_{n+1}), \quad 1\leqslant i\leqslant n+1, \quad 1\leqslant s\leqslant N,$$

defined by (4.24) in the case

$$m=n+1, \quad c_i=\delta_i, \quad 1\leqslant i\leqslant n+1.$$

B. Involutive solutions of the $\mathcal{N}$-wave interaction equations

We would like to show that the constrained flows provide involutive solutions to the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions. For the $1+1$ dimensional case, we have the following result.

Theorem 5.3: *If $\phi_{is}(x,t_1)$ and $\psi_{is}(x,t_1)$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant N$, solve the spatial constrained flow (2.43) and the temporal constrained flow (2.44) simultaneously, then*

$$u_{ij}(x,t_1) = \frac{\alpha_i-\alpha_j}{\gamma_i-\gamma_j}\langle \Phi_i(x,t_1), B\Psi_j(x,t_1)\rangle, \quad 1\leqslant i\neq j\leqslant n, \tag{5.1}$$

with $\Phi_i(x,t_1)$ and $\Psi_i(x,t_1)$ being given by

$$\Phi_i(x,t_1) = (\phi_{i1}(x,t_1),\ldots,\phi_{iN}(x,t_1))^T, \quad \Psi_i(x,t_1) = (\psi_{i1}(x,t_1),\ldots,\psi_{iN}(x,t_1))^T, \quad 1\leqslant i\leqslant n,$$

solve the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22).

Proof: Note that the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22) is the compatability condition of the spectral problem (2.1) and the associated spectral problem (2.17) with $m=1$ or the adjoint spectral problem (2.27) and the adjoint associated spectral problem (2.28) with $m=1$ for whatever potential u. Therefore, the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22) are also the compatability condition of the spatial constrained flow (2.43) and the temporal constrained flow (2.44) under the constraint (2.41). Now $\phi_{is}(x,t_1)$ and $\psi_{is}(x,t_1)$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant N$, are assumed to solve (2.43) and (2.44) simultaneously, and thus the potential defined by (5.1) must satisfy the compatability condition of the spatial constrained flow (2.43) and the temporal constrained flow (2.44). This means that the potential defined by (5.1) must be a solution to the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22). The proof is finished. ∎

We remark that a direct computation can also show the above theorem. For the $2+1$ dimensional case, a similar deduction can give rise to the following theorem.

Theorem 5.4: *If $\phi_{is}(x,t)$ and $\psi_{is}(x,t)$, $1\leqslant i\leqslant n+1$, $1\leqslant s\leqslant N$, solve the constrained flows (3.23)–(3.25) simultaneously, then*

$$p_{ij}(x,y,t)=\frac{J_i-J_j}{\delta_i-\delta_j}\langle \Phi_i(x,y,t),B\Psi_j(x,y,t)\rangle, \quad 1\leqslant i\neq j\leqslant n,$$
$$q_{ij}(x,y,t)=\frac{K_i-K_j}{\delta_i-\delta_j}\langle \Phi_i(x,y,t),B\Psi_j(x,y,t)\rangle, \quad 1\leqslant i\neq j\leqslant n, \tag{5.2}$$

with $\Phi_i(x,t)$ and $\Psi_i(x,t)$ being given by

$$\Phi_i(x,t)=(\phi_{i1}(x,t),\ldots,\phi_{iN}(x,t))^T, \quad \Psi_i(x,t)=(\psi_{i1}(x,t),\ldots,\psi_{iN}(x,t))^T, \quad 1\leqslant i\leqslant n+1,$$

solve the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4).

Also, one can find that

$$f_i=\frac{1}{\delta_i-\delta_{n+1}}\langle \Phi_i,B\Psi_{n+1}\rangle, \quad g_i=\frac{1}{\delta_i-\delta_{n+1}}\langle \Phi_{n+1},B\Psi_i\rangle, \quad 1\leqslant i\leqslant n \tag{5.3}$$

provide a solution to the Lax system (3.1) and the adjoint Lax system (3.5) with the potentials given by (5.2). What's more, (5.2) and (5.3) automatically satisfy our first symmetry constraint (3.7).

In the following theorem, the solutions given in Theorems 5.3 and 5.4 are shown to be involutive.

Theorem 5.5: *The Hamiltonians H_1^x and $H_1^{t_1}$ of the constrained flows in $1+1$ dimensions, defined by (2.53) and (2.54), are the second-order polynomial functions of $E_{il}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant i\leqslant n$, $l=1,2$, and thus they commute, i.e.,*

$$\{H_1^x,H_1^{t_1}\}=0, \tag{5.4}$$

where the Poisson bracket $\{\cdot,\cdot\}$ is defined by (2.47). The Hamiltonians H_2^x, H_2^y and H_2^t of the constrained flows in $2+1$ dimensions, defined by (3.30)–(3.32), are also the second-order polynomial functions of $E_{il}(\delta_1,\ldots,\delta_n,\delta_{n+1})$, $1\leqslant i\leqslant n+1$, $l=1,2$, and thus they commute with each other, i.e.,

$$\{H_2^x,H_2^y\}=\{H_2^x,H_2^t\}=\{H_2^y,H_2^t\}=0, \tag{5.5}$$

where the Poisson bracket $\{\cdot,\cdot\}$ is defined by (3.28).

Proof: Directly from the explicit expression (4.31) of the E_{is}'s, we have

$$E_{i1}=\sum_{j=1}^m c_j^{i-1}\langle \Phi_j,B\Psi_j\rangle, \quad 1\leqslant i\leqslant m, \tag{5.6}$$

$$\begin{aligned} E_{i2}&=\sum_{j=1}^m c_j^{i-1}\langle A\Phi_j,B\Psi_j\rangle \\ &\quad -\sum_{1\leqslant j<k\leqslant m}\frac{c_j^{i-1}-c_k^{i-1}}{c_j-c_k}(\langle \Phi_j,B\Psi_j\rangle\langle \Phi_k,B\Psi_k\rangle-\langle \Phi_j,B\Psi_k\rangle\langle \Phi_k,B\Psi_j\rangle) \\ &=\sum_{j=1}^m c_j^{i-1}\mathcal{E}_j-\sum_{\substack{j,k=1\\ j\neq k}}^m \frac{c_j^{i-1}}{c_j-c_k}\langle \Phi_j,B\Psi_j\rangle\langle \Phi_k,B\Psi_k\rangle, \quad 1\leqslant i\leqslant m, \end{aligned} \tag{5.7}$$

where the $\mathcal{E}_j$'s are defined as follows:

$$\mathcal{E}_j=\langle A\Phi_j,B\Psi_j\rangle+\sum_{\substack{k=1\\k\neq j}}^{m}\frac{1}{c_j-c_k}\langle\Phi_j,B\Psi_k\rangle\langle\Phi_k,B\Psi_j\rangle,\quad 1\leqslant j\leqslant m. \tag{5.8}$$

Now solving (5.6) for $\langle\Phi_i,B\Psi_i\rangle$, $1\leqslant i\leqslant m$, leads to

$$\langle\Phi_i,B\Psi_i\rangle=\left(\prod_{\substack{r=1\\r\neq i}}^{m}\frac{1}{c_i-c_r}\right)\sum_{j=1}^{m}(-1)^{m-j}s_{m-j}(c_1,\ldots,c_{i-1},\hat{c}_i,c_{i+1},\ldots,c_m)E_{j1},\quad 1\leqslant i\leqslant m, \tag{5.9}$$

where the s_j's are defined by (4.27) and $\hat{c}_i$ means that c_i does not appear. Therefore, each $\langle\Phi_i,B\Psi_i\rangle$ can be expressed as a linear combination of E_{i1}, $1\leqslant i\leqslant m$. Similarly, solving (5.7) for $\mathcal{E}_j$, $1\leqslant j\leqslant m$, leads to

$$\begin{aligned}\mathcal{E}_i=&\left(\prod_{\substack{r=1\\r\neq i}}^{m}\frac{1}{c_i-c_r}\right)\sum_{j=1}^{m}(-1)^{m-j}s_{m-j}(c_1,\ldots,c_{i-1},\hat{c}_i,c_{i+1},\ldots,c_m)\\&\times\left(E_{j2}+\sum_{\substack{k,l=1\\k\neq l}}^{m}\frac{c_k^{j-1}}{c_k-c_l}\langle\Phi_k,B\Psi_k\rangle\langle\Phi_l,B\Psi_l\rangle\right),\quad 1\leqslant i\leqslant m.\end{aligned} \tag{5.10}$$

This expression together with (5.9) implies that each $\mathcal{E}_j$ can be expressed as a linear combination of E_{i1} and E_{i2}, $1\leqslant i\leqslant m$.

In the $1+1$ dimensional case, we have $m=n$, $c_j=\gamma_j$, $1\leqslant j\leqslant n$. Hence

$$\mathcal{E}_j=\langle A\Phi_j,B\Psi_j\rangle+\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{\gamma_j-\gamma_k}\langle\Phi_j,B\Psi_k\rangle\langle\Phi_k,B\Psi_j\rangle,\quad 1\leqslant j\leqslant n. \tag{5.11}$$

The Hamiltonians H_1^x and $H_1^{t_1}$ in Theorem 2.1 can be easily expressed as

$$H_1^x=-\sum_{k=1}^{n}\alpha_k\mathcal{E}_k,\quad H_1^{t_1}=-\sum_{k=1}^{n}\beta_k\mathcal{E}_k, \tag{5.12}$$

where the $\mathcal{E}_k$'s are defined by (5.11).

Likewise, in the $2+1$ dimensional case, we have $m=n+1$, $c_j=\delta_j$, $1\leqslant j\leqslant n+1$. Hence

$$\mathcal{E}_j=\langle A\Phi_j,B\Psi_j\rangle+\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{\delta_j-\delta_k}\langle\Phi_j,B\Psi_k\rangle\langle\Phi_k,B\Psi_j\rangle+\frac{1}{\delta_j-\delta_{n+1}}\langle\Phi_j,B\Psi_{n+1}\rangle\langle\Phi_{n+1},B\Psi_j\rangle,$$
$$1\leqslant j\leqslant n, \tag{5.13}$$

$$\mathcal{E}_{n+1}=\langle A\Phi_{n+1},B\Psi_{n+1}\rangle+\sum_{k=1}^{n}\frac{1}{\delta_{n+1}-\delta_k}\langle\Phi_{n+1},B\Psi_k\rangle\langle\Phi_k,B\Psi_{n+1}\rangle. \tag{5.14}$$

The Hamiltonians H_2^x, H_2^y and H_2^t in Theorem 3.1 can be expressed as

$$H_2^x=-\sum_{k=1}^{n}\mathcal{E}_k,\quad H_2^y=-\sum_{k=1}^{n}J_k\mathcal{E}_k,\quad H_2^t=-\sum_{k=1}^{n}K_k\mathcal{E}_k, \tag{5.15}$$

where the $\mathcal{E}_k$'s are defined by (5.13).

Therefore, H_1^x and $H_1^{t_1}$ are linear combinations of $E_{il}(\gamma_1,\ldots,\gamma_n)$, $1\leqslant i\leqslant n$, $l=1,2$, and H_2^x, H_2^y and H_2^t are linear combinations of $E_{il}(\delta_1,\ldots,\delta_n,\delta_{n+1})$, $1\leqslant i\leqslant n+1$, $l=1,2$. It follows from Theorem 4.4 that H_1^x and $H_1^{t_1}$ are in involution, and H_2^x, H_2^y and H_2^t are in involution in pair, too. The proof is finished. ■

We remark that a direct computation can also give a proof for the involutive property of the Hamiltonians of the constrained flows in both $1+1$ and $2+1$ dimensions. Only a new set of equalities

$$\frac{a_j-a_i}{c_j-c_i}\,\frac{b_k-b_i}{c_k-c_i}-\frac{a_k-a_i}{c_k-c_i}\,\frac{b_j-b_i}{c_j-c_i}+\text{cycle}(i,j,k)=0,\quad 1\leqslant i,j,k\leqslant n,$$

has to be utilized, where a_i, b_i, and c_i, $1\leqslant i\leqslant n$, are arbitrary constants. This just needs a direct check, too. However, the proof of Theorem 5.5 also gives rise to the explicit expressions for all Hamiltonians of the constrained flows in both $1+1$ and $2+1$ dimensions, in terms of the integrals of motion E_{is}.

Now if we denote the Hamiltonian flows of the spatial constrained flow (2.43) and the temporal constrained flow (2.44) by $g_x^{H_1^x}$ and $g_t^{H_1^{t_1}}$, respectively, then the above theorems present a kind of involutive solution to the $1+1$ dimensional $\mathcal{N}$-wave interaction equations (2.22):

$$\begin{aligned} u_{ij}(x,t_1)&=\frac{\alpha_i-\alpha_j}{\gamma_i-\gamma_j}\langle g_x^{H_1^x}g_t^{H_1^{t_1}}\Phi_{i0},g_x^{H_1^x}g_t^{H_1^{t_1}}B\Psi_{j0}\rangle\\ &=\frac{\alpha_i-\alpha_j}{\gamma_i-\gamma_j}\langle g_t^{H_1^{t_1}}g_x^{H_1^x}\Phi_{i0},g_t^{H_1^{t_1}}g_x^{H_1^x}B\Psi_{j0}\rangle,\quad 1\leqslant i\neq j\leqslant n, \end{aligned}\tag{5.16}$$

where the initial values Φ_{i0} and Ψ_{i0} of Φ_i and Ψ_i can be taken to be any arbitrary constant vectors of the Euclidean space $\mathbb{R}^N$. Similarly, if we denote the Hamiltonian flows of the constrained flows (3.23)–(3.25) by $g_x^{H_2^x}$, $g_y^{H_2^y}$, and $g_t^{H_2^t}$, respectively, then the above theorems present a kind of involutive solutions to the $2+1$ dimensional $\mathcal{N}$-wave interaction equations (3.4):

$$\begin{aligned} p_{ij}(x,t)&=\frac{J_i-J_j}{\delta_i-\delta_j}\langle g_x^{H_2^x}g_y^{H_2^y}g_t^{H_2^t}\overline{\Phi}_{i0},g_x^{H_2^x}g_y^{H_2^y}g_t^{H_2^t}B\overline{\Psi}_{j0}\rangle\\ &=\frac{J_i-J_j}{\delta_i-\delta_j}\langle g_y^{H_2^y}g_t^{H_2^t}g_x^{H_2^x}\overline{\Phi}_{i0},g_y^{H_2^y}g_t^{H_2^t}g_x^{H_2^x}B\overline{\Psi}_{j0}\rangle\\ &=\frac{J_i-J_j}{\delta_i-\delta_j}\langle g_t^{H_2^t}g_x^{H_2^x}g_y^{H_2^y}\overline{\Phi}_{i0},g_t^{H_2^t}g_x^{H_2^x}g_y^{H_2^y}B\overline{\Psi}_{j0}\rangle\\ &=\cdots,\quad 1\leqslant i\neq j\leqslant n, \end{aligned}\tag{5.17}$$

$$\begin{aligned} q_{ij}(x,t)&=\frac{K_i-K_j}{\delta_i-\delta_j}\langle g_x^{H_2^x}g_y^{H_2^y}g_t^{H_2^t}\overline{\Phi}_{i0},g_x^{H_2^x}g_y^{H_2^y}g_t^{H_2^t}B\overline{\Psi}_{j0}\rangle\\ &=\frac{K_i-K_j}{\delta_i-\delta_j}\langle g_y^{H_2^y}g_t^{H_2^t}g_x^{H_2^x}\overline{\Phi}_{i0},g_y^{H_2^y}g_t^{H_2^t}g_x^{H_2^x}B\overline{\Psi}_{j0}\rangle\\ &=\frac{K_i-K_j}{\delta_i-\delta_j}\langle g_t^{H_2^t}g_x^{H_2^x}g_y^{H_2^y}\overline{\Phi}_{i0},g_t^{H_2^t}g_x^{H_2^x}g_y^{H_2^y}B\overline{\Psi}_{j0}\rangle\\ &=\cdots,\quad 1\leqslant i\neq j\leqslant n, \end{aligned}\tag{5.18}$$

where the initial values $\overline{\Phi}_{i0}$ and $\overline{\Psi}_{i0}$ of Φ_i and Ψ_i can also be taken to be any arbitrary constant vectors of the Euclidean space $\mathbb{R}^N$.

Note that all constrained flows in both $1+1$ and $2+1$ dimensions are Liouville integrable, and that the initial values of Φ_i and Ψ_i, $1\leqslant i\leqslant n$, can be arbitrarily chosen. Therefore, together with Theorems 5.1 and 5.2, the above involutive solutions also show us the richness of solutions and the integrability by quadratures for the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions. Of importance is of course that binary symmetry constraints decompose the

$\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions into finite-dimensional Liouville integrable Hamiltonian systems, and the resulting involutive solutions present the Bäcklund transformations between the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions and these finite-dimensional Liouville integrable Hamiltonian systems.

VI. CONCLUSIONS AND REMARKS

We have introduced a class of special symmetry constraints, (2.38) in the $1+1$ dimensional case, and (3.19) and (3.20) in the $2+1$ dimensional case, for the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions. These symmetry constraints nonlinearize the $n\times n$ spectral problem and adjoint spectral problem, (2.34) and (2.35), and the $(n+1)\times(n+1)$ spectral problem and adjoint spectral problem, (3.11) and (3.12), into finite-dimensional Liouville integrable Hamiltonian systems, and decompose the $\mathcal{N}$-wave interaction equations in both $1+1$ and $2+1$ dimensions into these finite-dimensional Liouville integrable Hamiltonian systems. A general involutive and functionally independent system of the polynomial functions $F_{is}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, or alternatively $E_{is}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, associated with an arbitrarily higher-order matrix Lax operator, was presented and used to show the Liouville integrability of the resulting constrained flows. The nonlinear constraints on the potentials, resulting from the symmetry constraints, also provide us with a class of Bäcklund transformations from the $\mathcal{N}$-wave interaction equations to the obtained finite-dimensional Liouville integrable systems. The involutive solutions to the $\mathcal{N}$-wave interaction equations are given through the constrained flows, and thus the integrability by quadratures has been exhibited for the $\mathcal{N}$-wave interaction equations. The special case with $\Gamma=W_0$, i.e., $\mathrm{diag}(\gamma_1,\ldots,\gamma_n)=\mathrm{diag}(\beta_1,\ldots,\beta_n)$ of two reductions of $n=3$ and $n=4$ in $1+1$ dimensions presents all results established in Refs. 31 and 32.

We point out that for a more general matrix Lax operator $L=C+D$ with any constant matrix $C=(c_{ij})_{m\times m}$ and the matrix D defined by (4.2), the **r**-matrix formulation (4.13) still holds. Therefore, an involutive system of polynomial functions can be generated, but we do not know what conditions on the matrix C can ensure the functional independence of that involutive system. We are also curious about other examples of higher-order matrix Lax operators which lead to involutive and functionally independent systems. Our crucial techniques to present the involutive and functionally independent system F_{is}, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, are the **r**-matrix formulation, Newton's identities on elementary symmetric polynomials, and the determinant property of tensor products of matrices; and the whole process of their applications provides an efficient way to show the involutive property and the functional independence.

Of course, one of the important results in binary nonlinearization is the integrability of soliton equations by quadratures, which implies that one can integrate soliton equations themselves by quadratures. However, the potentials obtained by symmetry constraints can be proved to belong to a kind of finite-gap-type solutions containing multi-soliton solutions, and thus they may not present solutions to given initial value and/or boundary problems of soliton equations. It is a challenging problem to establish a general theory of complete integrability for nonlinear differential and differential-difference equations, which should state what mathematical properties the equations must possess so that their solutions to initial value and/or boundary problems can also be determined by quadratures.

Symmetry constraints yield nonlinear constraints on potentials of soliton equations, and put linear spectral problems (linear with respect to eigenfunctions) into nonlinear constrained flows (nonlinear again with respect to eigenfunctions), which makes it more complicated to solve soliton equations. However, since spectral problems are overdetermined, one needs additional conditions (compatability conditions) to guarantee the existence of eigenfunctions of spectral problems. The symmetry property brings us the Liouville integrability for nonlinear constrained flows. Thus, symmetry constraints make up for the disadvantage of nonlinearization in manipulating binary nonlinearization. Of special interest in the study of symmetry constraints are creating new classical integrable systems,[55] which supplement the known class of integrable systems,[56] and exposing the integrability by quadratures for soliton equations by using constrained flows.[33]

The idea of binary nonlinearization is quite similar to that of using adjoint symmetries to generate conservation laws for differential equations, both Lagrangian and non-Lagrangian.[57] In binary nonlinearization, we adopt adjoint spectral problems to formulate Hamiltonian structures for constrained flows so that finite-dimensional Liouville integrable systems result. Note that there exist also some special symmetry constraints which do not yield Hamiltonian structures with constant coefficient symplectic forms, including both canonical and noncanonical ones, for constrained flows.[46] Therefore, it will be particularly interesting and important to classify symmetry constraints which exhibit Hamiltonian structures with constant and variable coefficient symplectic forms for constrained flows.

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APPENDIX A: NON-LIE SYMMETRIES

Proposition A.1: If $\phi^{(s)}$ *and* $\psi^{(s)}$, $1\leqslant s\leqslant N$, *satisfy (2.34) and (2.35), then the vector field*

$$Z_0=J\sum_{s=1}^{N}\mu_s\psi^{(s)T}\frac{\partial U(u,\lambda_s)}{\partial u}\phi^{(s)}=\rho\left(\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right) \tag{A1}$$

is a symmetry of the $1+1$ *dimensional* $\mathcal{N}$*-wave interaction equations (2.22).*

Proof: It is required to show that

$$(\delta P,\delta Q)=\left(\left[U_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right],\left[W_0,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right) \tag{A2}$$

satisfies the linearized system (2.25). By using (2.34) and (2.35), we can first compute that

$$\begin{aligned}\left(\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right)_{t_1}&=\sum_{s=1}^{N}\mu_s\phi^{(s)}_{t_1}\psi^{(s)T}+\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}_{t_1}\\&=\sum_{s=1}^{N}\mu_sV^{(1)}(u,\lambda_s)\phi^{(s)}\psi^{(s)T}-\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}V^{(1)}(u,\lambda_s)\\&=\sum_{s=1}^{N}\mu_s[V^{(1)}(u,\lambda_s),\phi^{(s)}\psi^{(s)T}]\\&=\sum_{s=1}^{N}\lambda_s\mu_s[W_0,\phi^{(s)}\psi^{(s)T}]+\left[W_1,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right],\end{aligned}$$

and, similarly, we can have

$$\left(\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right)_x=\sum_{s=1}^{N}\lambda_s\mu_s[U_0,\phi^{(s)}\psi^{(s)T}]+\left[U_1,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right].$$

Thus, noting the Jacobi identity, it follows that

$$
\begin{aligned}
(\delta P)_{t_1}-(\delta Q)_x &= \left[U_0,\left(\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right)_{t_1}\right]-\left[W_0,\left(\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right)_x\right]\\
&= \sum_{s=1}^{N}\lambda_s\mu_s([U_0,[W_0,\phi^{(s)}\psi^{(s)T}]]-[W_0,[U_0,\phi^{(s)}\psi^{(s)T}]])\\
&\quad+\left[U_0,\left[W_1,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right]-\left[W_0,\left[U_1,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right]\\
&= \left[U_0,\left[W_1,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right]-\left[W_0,\left[U_1,\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T}\right]\right],
\end{aligned}
$$

where δP and δQ are defined by (A2). Then, again noting the Jacobi identity, we can have

$$
\begin{aligned}
&(\delta P)_{t_1}-(\delta Q)_x+[U_1,\delta Q]+[\delta P,W_1]\\
&\quad=\left[\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T},[U_0,W_1]\right]-\left[\sum_{s=1}^{N}\mu_s\phi^{(s)}\psi^{(s)T},[W_0,U_1]\right]=0,
\end{aligned}
$$

in the last step of which we have used $[U_0,W_1]=[W_0,U_1]$. The proof is finished. ■

All of the symmetries presented in this proposition are not Lie point, contact, or Bäcklund symmetries, since they cannot be written in terms of the potentials u_{ij} and their spatial derivatives.

APPENDIX B: FORMULAS FOR COMPUTING $\mathcal{F}_\lambda^{(k)}$

Immediately from the expressions of $\mathcal{F}_\lambda^{(k)}$ in (4.22), we can obtain the following more concrete formulas for computing $\mathcal{F}_\lambda^{(k)}$:

$$
\mathcal{F}_\lambda^{(1)}=\sum_{i=1}^{m}(c_i+\overset{ii}{Q}_\lambda),
$$

$$
\mathcal{F}_\lambda^{(2)}=\sum_{1\le i<j\le m}\left(c_ic_j+c_j\overset{ii}{Q}_\lambda+c_i\overset{jj}{Q}_\lambda+\begin{vmatrix}\overset{ii}{Q}_\lambda & \overset{ij}{Q}_\lambda\\ \overset{ji}{Q}_\lambda & \overset{jj}{Q}_\lambda\end{vmatrix}\right),
$$

$$
\begin{aligned}
\mathcal{F}_\lambda^{(3)}=&\sum_{1\le i<j<k\le m}(c_ic_jc_k+c_ic_k\overset{jj}{Q}_\lambda+c_jc_k\overset{ii}{Q}_\lambda+c_ic_j\overset{kk}{Q}_\lambda)\\
&+\sum_{1\le i<j<k\le m}\left(c_i\begin{vmatrix}\overset{jj}{Q}_\lambda & \overset{jk}{Q}_\lambda\\ \overset{kj}{Q}_\lambda & \overset{kk}{Q}_\lambda\end{vmatrix}+c_j\begin{vmatrix}\overset{ii}{Q}_\lambda & \overset{ik}{Q}_\lambda\\ \overset{ki}{Q}_\lambda & \overset{kk}{Q}_\lambda\end{vmatrix}+c_k\begin{vmatrix}\overset{ii}{Q}_\lambda & \overset{ij}{Q}_\lambda\\ \overset{ji}{Q}_\lambda & \overset{jj}{Q}_\lambda\end{vmatrix}+\begin{vmatrix}\overset{ii}{Q}_\lambda & \overset{ij}{Q}_\lambda & \overset{ik}{Q}_\lambda\\ \overset{ji}{Q}_\lambda & \overset{jj}{Q}_\lambda & \overset{jk}{Q}_\lambda\\ \overset{ki}{Q}_\lambda & \overset{kj}{Q}_\lambda & \overset{kk}{Q}_\lambda\end{vmatrix}\right),
\end{aligned}
$$

$$
\ldots,
$$

$$
\mathcal{F}_\lambda^{(k)}=\sum_{1\le j_1<j_2<\ldots<j_k\le m}\left(\prod_{p=1}^{k}c_{j_p}+\sum_{i=1}^{k}\prod_{\substack{p=1\\p\ne i}}^{k}c_{j_p}\overset{j_ij_i}{Q}_\lambda+\sum_{1\le i_1<i_2\le k}\prod_{\substack{p=1\\p\ne i_1,i_2}}^{k}c_{j_p}\begin{vmatrix}\overset{j_{i_1}j_{i_1}}{Q}_\lambda & \overset{j_{i_1}j_{i_2}}{Q}_\lambda\\ \overset{j_{i_2}j_{i_1}}{Q}_\lambda & \overset{j_{i_2}j_{i_2}}{Q}_\lambda\end{vmatrix}\right)
$$

$$
+\sum_{1\leqslant j_1<j_2<\ldots<j_k\leqslant m}\ \sum_{1\leqslant i_1<i_2<i_3\leqslant k}\ \prod_{\substack{p=1\\ p\neq i_1,i_2,i_3}}^{k} c_{j_p}\begin{vmatrix} Q_\lambda^{j_{i_1}j_{i_1}} & Q_\lambda^{j_{i_1}j_{i_2}} & Q_\lambda^{j_{i_1}j_{i_3}}\\ Q_\lambda^{j_{i_2}j_{i_1}} & Q_\lambda^{j_{i_2}j_{i_2}} & Q_\lambda^{j_{i_2}j_{i_3}}\\ Q_\lambda^{j_{i_3}j_{i_1}} & Q_\lambda^{j_{i_3}j_{i_2}} & Q_\lambda^{j_{i_3}j_{i_3}}\end{vmatrix}+\ldots
$$

$$
+\sum_{1\leqslant j_1<j_2<\ldots<j_k\leqslant m}\begin{vmatrix} Q_\lambda^{j_1j_1} & Q_\lambda^{j_1j_2} & \cdots & Q_\lambda^{j_1j_k}\\ Q_\lambda^{j_2j_1} & Q_\lambda^{j_2j_2} & \cdots & Q_\lambda^{j_2j_k}\\ \vdots & \vdots & \ddots & \vdots\\ Q_\lambda^{j_kj_1} & Q_\lambda^{j_kj_2} & \cdots & Q_\lambda^{j_kj_k}\end{vmatrix},
$$

$$
\ldots,
$$

$$
\mathcal{F}_\lambda^{(m)}=\prod_{p=1}^{m} c_p+\sum_{i=1}^{m}\prod_{\substack{p=1\\ p\neq i}}^{m} c_p Q_\lambda^{ii}+\sum_{1\leqslant i<j\leqslant m}\ \prod_{\substack{p=1\\ p\neq i,j}}^{m} c_p\begin{vmatrix} Q_\lambda^{ii} & Q_\lambda^{ij}\\ Q_\lambda^{ji} & Q_\lambda^{jj}\end{vmatrix}
$$

$$
+\sum_{1\leqslant i<j<k\leqslant m}\ \prod_{\substack{p=1\\ p\neq i,j,k}}^{k} c_p\begin{vmatrix} Q_\lambda^{ii} & Q_\lambda^{ij} & Q_\lambda^{ik}\\ Q_\lambda^{ji} & Q_\lambda^{jj} & Q_\lambda^{jk}\\ Q_\lambda^{ki} & Q_\lambda^{kj} & Q_\lambda^{kk}\end{vmatrix}+\ldots+\begin{vmatrix} Q_\lambda^{11} & Q_\lambda^{12} & \cdots & Q_\lambda^{1m}\\ Q_\lambda^{21} & Q_\lambda^{22} & \cdots & Q_\lambda^{2r}\\ \vdots & \vdots & \ddots & \vdots\\ Q_\lambda^{m1} & Q_\lambda^{r2} & \cdots & Q_\lambda^{mm}\end{vmatrix}.
$$

APPENDIX C: THE DETERMINANT OF Ω_m

The following proposition has been used while showing the functional independence of the polynomial functions $F_{is}(c_1,\ldots,c_m)$, $1\leqslant i\leqslant m$, $1\leqslant s\leqslant N$, which is of interest itself.

Proposition C.1: Let $m\geqslant 2$, and $c_1,c_2,\ldots,c_m$ be constants. Then

$$
\det(\Omega_m)=\begin{vmatrix} 1 & \sum_{i=2}^{m} c_i & \sum_{2\leqslant i<j\leqslant m} c_ic_j & \sum_{2\leqslant i<j<k\leqslant m} c_ic_jc_k & \cdots & \prod_{i=2}^{m} c_i\\ 1 & \sum_{\substack{i=1\\ i\neq 2}}^{m} c_i & \sum_{\substack{1\leqslant i<j\leqslant m\\ i,j\neq 2}} c_ic_j & \sum_{\substack{1\leqslant i<j<k\leqslant m\\ i,j,k\neq 2}} c_ic_jc_k & \cdots & \prod_{\substack{i=1\\ i\neq 2}}^{m} c_i\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \sum_{i=1}^{m-1} c_i & \sum_{1\leqslant i<j\leqslant m-1} c_ic_j & \sum_{1\leqslant i<j<k\leqslant m-1} c_ic_jc_k & \cdots & \prod_{i=1}^{m-1} c_i\end{vmatrix}
$$

$$
=\prod_{1\leqslant i<j\leqslant m}(c_i-c_j). \tag{C1}
$$

Proof: We prove this proposition by the principle of mathematical induction. It is obvious that (C1) is true when $m=2$. Suppose that (C1) is true when $m=l$. Let us verify that (C1) is also true when $m=l+1$. Note that

$$\sum_{\substack{1\leqslant i_1<i_2<\dots<i_k\leqslant l+1\\ i_1,i_2,\dots,i_k\neq j}} c_{i_1}c_{i_2}\dots c_{i_k} - \sum_{\substack{1\leqslant i_1<i_2<\dots<i_k\leqslant l+1\\ i_1,i_2,\dots,i_k\neq i}} c_{i_1}c_{i_2}\dots c_{i_k}$$

$$=(c_i-c_j)\sum_{\substack{1\leqslant i_1<i_2<\dots<i_{k-1}\leqslant l+1\\ i_1,i_2,\dots,i_{k-1}\neq i,j}} c_{i_1}c_{i_2}\dots c_{i_{k-1}},$$

$$1\leqslant i,j\leqslant l+1, \quad 1\leqslant k\leqslant l.$$

For each $2\leqslant j\leqslant l+1$, we subtract

$$\sum_{2\leqslant i_1<i_2<\dots<i_{j-1}\leqslant l+1}^{l+1} c_{i_1}c_{i_2}\dots c_{i_{j-1}}\times \text{the first column of } \det(\Omega_{l+1})$$

from the jth column of $\det(\Omega_{l+1})$, and then we have

$$\det(\Omega_{l+1})$$

$$=\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & c_1-c_2 & (c_1-c_2)\sum_{i=3}^{l+1} c_i & (c_1-c_2)\sum_{3\leqslant i<j\leqslant l+1} c_ic_j & \dots & (c_1-c_2)\prod_{i=3}^{l+1} c_i \\ 1 & c_1-c_3 & (c_1-c_3)\sum_{\substack{i=2\\ i\neq 3}}^{l+1} c_i & (c_1-c_3)\sum_{\substack{2\leqslant i<j\leqslant l+1\\ i,j\neq 3}} c_ic_j & \dots & (c_1-c_3)\prod_{\substack{i=2\\ i\neq 3}}^{l+1} c_i \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_1-c_{l+1} & (c_1-c_{l+1})\sum_{i=2}^{l} c_i & (c_1-c_{l+1})\sum_{2\leqslant i<j\leqslant l} c_ic_j & \dots & (c_1-c_{l+1})\prod_{i=2}^{l} c_i \end{vmatrix}$$

$$=\prod_{j=2}^{l+1}(c_1-c_j)\begin{vmatrix} 1 & \sum_{i=3}^{l+1} c_i & \sum_{3\leqslant i<j\leqslant l+1} c_ic_j & \dots & \prod_{i=3}^{l+1} c_i \\ 1 & \sum_{\substack{i=2\\ i\neq 3}}^{l+1} c_i & \sum_{\substack{2\leqslant i<j\leqslant l+1\\ i,j\neq 3}} c_ic_j & \dots & \prod_{\substack{i=2\\ i\neq 3}}^{l+1} c_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sum_{i=2}^{l} c_i & \sum_{2\leqslant i<j\leqslant l} c_ic_j & \dots & \prod_{i=2}^{l} c_i \end{vmatrix} = \prod_{1\leqslant i<j\leqslant l+1}(c_i-c_j),$$

in the last step of which we have used the inductive assumption. This means that (C1) is also true when $m=l+1$, i.e., the inductive step is satisfied. Therefore, the formula (C1) is always true by the principle of mathematical induction. The proof is finished. ∎

APPENDIX D: TWO IDENTITIES ON SYMMETRIC POLYNOMIALS

Let the s_j's be symmetric polynomials defined by (4.27).

Proposition D.1: For any integers r and i with $i\geqslant r\geqslant 1$, and any numbers $c_1,\dots,c_r$, we have

$$\sum_{j=0}^{i-r}(-1)^j s_j(c_1,\ldots,c_r)\sum_{\substack{l_1+\ldots+l_r=i-r-j\\ l_1,\ldots,l_r\geqslant 0}} c_1^{l_1}\ldots c_r^{l_r}=\begin{cases}1, & \text{if } i=r,\\ 0, & \text{if } i>r.\end{cases} \tag{D1}$$

Proof: Use the principle of mathematical induction on r. When $r=1$ and $i=1$, the left-hand side of (D1) is 1. When $r=1$ and $i>1$, the left-hand side of (D1) is 0. Hence (D1) holds when $r=1$.

Now suppose that (D1) holds when $r=k$, i.e.,

$$\sum_{j=0}^{i-k}(-1)^j s_j(c_1,\ldots,c_k)\sum_{\substack{l_1+\ldots+l_k=i-k-j\\ l_1,\ldots,l_k\geqslant 0}} c_1^{l_1}\ldots c_k^{l_k}=\begin{cases}1, & \text{if } i=k,\\ 0, & \text{if } i>k.\end{cases} \tag{D2}$$

Then, when $r=k+1$, the left-hand side of (D1) is

$$\sum_{j=0}^{i-k-1}(-1)^j s_j(c_1,\ldots,c_{k+1})\sum_{\substack{l_1+\cdots+l_{k+1}=i-k-j-1\\ l_1,\cdots,l_{k+1}\geqslant 0}} c_1^{l_1}\cdots c_{k+1}^{l_{k+1}}. \tag{D3}$$

By using (4.28), it equals

$$\begin{aligned}
&\sum_{j=0}^{i-k-1}(-1)^j\sum_{l_{r+1}=0}^{i-k-j-1} c_{k+1}^{l_{k+1}+1}s_{j-1}(c_1,\ldots,c_k)\sum_{\substack{l_1+\cdots+l_k=i-k-j-1-l_{k+1}\\ l_1,\ldots,l_k\geqslant 0}} c_1^{l_1}\cdots c_k^{l_k}\\
&\quad+\sum_{j=0}^{i-k-1}(-1)^j\sum_{l_{r+1}=0}^{i-k-j-1} c_{k+1}^{l_{k+1}}s_j(c_1,\ldots,c_k)\sum_{\substack{l_1+\cdots+l_k=i-k-j-1-l_{k+1}\\ l_1,\cdots,l_k\geqslant 0}} c_1^{l_1}\cdots c_k^{l_k}\\
&\quad=\sum_{l_{r+1}=0}^{i-k-1} c_{k+1}^{l_{k+1}+1}\sum_{j=0}^{i-k-l_{k+1}-2}(-1)^{j+1}s_j(c_1,\ldots,c_k)\sum_{\substack{l_1+\cdots+l_k=i-k-j-l_{k+1}-2\\ l_1,\cdots,l_k\geqslant 0}} c_1^{l_1}\cdots c_k^{l_k}\\
&\quad\quad+\sum_{l_{r+1}=0}^{i-k-1} c_{k+1}^{l_{k+1}}\sum_{j=0}^{i-k-l_{k+1}-1}(-1)^{j}s_j(c_1,\ldots,c_k)\sum_{\substack{l_1+\cdots+l_k=i-k-j-l_{k+1}-1\\ l_1,\cdots,l_k\geqslant 0}} c_1^{l_1}\cdots c_k^{l_k},
\end{aligned} \tag{D4}$$

where an empty sum is understood to be zero.

When $i=k+1$, it is easy to see that (D4) equals 1. If $i>k+1$, then by (D2), the first sum equals

$$-c_{k+1}^{l_{k+1}+1}\big|_{l_{k+1}=i-k-2}=-c_{k+1}^{i-k-1}, \tag{D5}$$

and again, by (D2), the second sum equals

$$c_{k+1}^{l_{k+1}}\big|_{l_{k+1}=i-k-1}=c_{k+1}^{i-k-1}. \tag{D6}$$

Hence (D4) equals 0 if $i>k+1$, which implies that (D1) holds when $r=k+1$. Therefore, (D1) always holds by the principle of mathematical induction. The proposition is proved. ∎

Proposition D.2: For any integers m, r, i with $i\geqslant r+1\geqslant 2$, m numbers $c_1,\ldots,c_m$, and r integers $j_1,\ldots,j_r$ with $1\leqslant j_1<\ldots<j_r\leqslant m$, we have

$$\sum_{j=0}^{i-r} (-1)^{i-r-j} s_j(c_1,\ldots,c_m) \sum_{\substack{l_1+\cdots+l_r=i-r-j\\ l_1,\ldots,l_r\geqslant 0}} c_{j_1}^{l_1}\cdots c_{j_r}^{l_r} = \sum_{\substack{1\leqslant \rho_1<\cdots<\rho_{i-r}\leqslant m\\ \rho_\alpha\neq j_\beta \text{ for all } \alpha,\beta}} c_{\rho_1}\cdots c_{\rho_{i-r}}. \tag{D7}$$

Proof: Without loss of generality, suppose that $j_i=i$ when $i=1,\ldots,r$, since each $s_j(c_1,\ldots,c_m)$ is symmetric with respect to $c_1,\ldots,c_m$. Then, (D7) becomes

$$\sum_{j=0}^{i-r} (-1)^{i-r-j} s_j(c_1,\ldots,c_m) \sum_{\substack{l_1+\cdots+l_r=i-r-j\\ l_1,\cdots,l_r\geqslant 0}} c_1^{l_1}\cdots c_r^{l_r} = \sum_{r+1\leqslant \rho_1<\cdots<\rho_{i-r}\leqslant m} c_{\rho_1}\cdots c_{\rho_{i-r}}. \tag{D8}$$

Obviously, for any fixed j with $r+1\leqslant j\leqslant m$, both sides of (D8) are linear with respect to c_j.

We use the principle of mathematical induction on i to prove (D8). When $i=r+1$, both sides of (D8) equal $c_{r+1}+\cdots+c_m$.

Suppose that (D8) holds when $i=k$ $(k>r)$. Then, when $i=k+1$, the left-hand side of (D8) reads as

$$R := \sum_{j=0}^{k+1-r} (-1)^{k+1-r-j} s_j(c_1,\ldots,c_m) \sum_{\substack{l_1+\cdots+l_r=k+1-r-j\\ l_1,\cdots,l_r\geqslant 0}} c_1^{l_1}\cdots c_r^{l_r}$$

$$= \sum_{j=-1}^{k-r} (-1)^{k-r-j} s_{j+1}(c_1,\ldots,c_m) \sum_{\substack{l_1+\cdots+l_r=k-r-j\\ l_1,\cdots,l_r\geqslant 0}} c_1^{l_1}\cdots c_r^{l_r}. \tag{D9}$$

Then by (4.28), we have

$$\frac{\partial R}{\partial c_m} = \sum_{j=0}^{k-r} (-1)^{k-r-j} s_j(c_1,\ldots,c_{m-1}) \sum_{\substack{l_1+\cdots+l_r=k-r-j\\ l_1,\cdots,l_r\geqslant 0}} c_1^{l_1}\cdots c_r^{l_r}. \tag{D10}$$

By the inductive assumption, it becomes

$$\frac{\partial R}{\partial c_m} = \sum_{r+1\leqslant \rho_1<\ldots<\rho_{k-r}\leqslant m-1} c_{\rho_1}\cdots c_{\rho_{k-r}}. \tag{D11}$$

Hence we obtain

$$R = \sum_{r+1\leqslant \rho_1<\cdots<\rho_{k-r}\leqslant m-1} c_{\rho_1}\cdots c_{\rho_{k-r}} c_m + R_1(c_1,\ldots,c_{m-1}), \tag{D12}$$

where R_1 is a polynomial. Since R is symmetric with respect to $c_{r+1},\ldots,c_m$, we have

$$R = \sum_{r+1\leqslant \rho_1<\cdots<\rho_{k+1-r}\leqslant m} c_{\rho_1}\cdots c_{\rho_{k+1-r}} + R_0(c_1,\ldots,c_r), \tag{D13}$$

where by setting $c_{r+1}=\cdots=c_m=0$ in (D9), R_0 is determined to be

$$R_0(c_1,\ldots,c_r) = \sum_{j=0}^{k+1-r} (-1)^{k+1-r-j} s_j(c_1,\ldots,c_r) \sum_{\substack{l_1+\cdots+l_r=k+1-r-j\\ l_1,\ldots,l_r\geqslant 0}} c_1^{l_1}\cdots c_r^{l_r}. \tag{D14}$$

By Proposition D.1, $R_0=0$ since $k+1=i>r$. Hence

$$R = \sum_{r+1 \leqslant \rho_1 < \cdots < \rho_{k+1-r} \leqslant m} c_{\rho_1} \cdots c_{\rho_{k+1-r}}, \tag{D15}$$

which implies that (D8) holds when $i=k+1$. Therefore, (D8) holds for all $i>r$ by the principle of mathematical induction. The proof is completed. ■

The identity (D7) is needed in presenting an alternative involutive system E_{is}'s to the F_{is}'s in Sec. IV C.

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