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J. Differential Equations 264 (2018) 2633–2659

Journal of Differential Equations

www.elsevier.com/locate/jde

Lump solutions to nonlinear partial differential equations via Hirota bilinear forms

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Received 24 July 2017
Available online 15 November 2017

Abstract

Lump solutions are analytical rational function solutions localized in all directions in space. We analyze a class of lump solutions, generated from quadratic functions, to nonlinear partial differential equations. The basis of success is the Hirota bilinear formulation and the primary object is the class of positive multivariate quadratic functions. A complete determination of quadratic functions positive in space and time is given, and positive quadratic functions are characterized as sums of squares of linear functions. Necessary and sufficient conditions for positive quadratic functions to solve Hirota bilinear equations are presented, and such polynomial solutions yield lump solutions to nonlinear partial differential equations under the dependent variable transformations $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$, where x is one spatial variable. Applications are made for a few generalized KP and BKP equations.

MSC: 35Q51; 37K40; 35Q53

Keywords: Soliton; Integrable equation; Hirota bilinear form; Lump solution

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https://doi.org/10.1016/j.jde.2017.10.033
0022-0396/

1. Introduction

The Korteweg–de Vries (KdV) equation and the Kadomtsev–Petviashvili (KP) equation are nonlinear integrable differential equations, and their Hirota bilinear forms play a crucial role in generating their soliton solutions, a kind of exponentially localized solutions, describing diverse nonlinear phenomena [9].

By lump functions, we mean analytical rational functions of spatial and temporal variables, which are localized in all directions in space. In recent years, there has been a growing interest in lump function solutions [4,8,10,22], called lump solutions (see, e.g., [1,7,12,25] for typical examples). The KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0 \tag{1.1}$$

admits the following lump solution

$$u = 4\frac{-[x+ay+3(a^2-b^2)t]^2 + b^2(y+6at)^2 + 1/b^2}{\{[x+ay+3(a^2-b^2)t]^2 + b^2(y+6at)^2 + 1/b^2\}^2}, \tag{1.2}$$

where a and $b \neq 0$ are free real constants [21]. Lump functions provide appropriate prototypes to model rogue wave dynamics in both oceanography [23] and nonlinear optics [27]. There are various discussions on general rational function solutions to integrable equations such as the KdV, KP, Boussinesq and Toda equations [2,3,17–19]. It has become a very interesting topic to search for lump solutions or lump-type solutions, rationally localized solutions in almost all directions in space, to nonlinear partial differential equations, through the Hirota bilinear formulation.

In this paper, we would like to characterize positive quadratic functions and analyze positive quadratic function solutions to Hirota bilinear equations. Such polynomial solutions generate lump or lump-type solutions to nonlinear partial differential equations under the dependent variable transformations $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$, where x is one of the spatial variables. We will present sufficient and necessary conditions for positive quadratic functions to solve Hirota bilinear equations, and apply the resulting theory to a few generalized KP and BKP equations.

2. From Hirota bilinear equations to nonlinear equations

Let M be a natural number and $x = (x_1, x_2, \cdots, x_M)^T$ in $\mathbb{R}^M$ be a column vector of independent variables. For $f, g \in C^\infty(\mathbb{R}^M)$, Hirota bilinear derivatives [9] are defined as follows:

$$D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} f \cdot g := \prod_{i=1}^{M} (\partial_{x_i} - \partial_{x_i'})^{n_i} f(x) g(x')|_{x'=x}, \tag{2.1}$$

where $x' = (x_1', x_2', \cdots, x_M')^T$ and $n_i \geq 0$, $1 \leq i \leq M$. For example, we have the first-order and second-order Hirota bilinear derivatives:

$$D_i f \cdot g = f_{x_i} g - f g_{x_i}, \; D_i D_j f \cdot g = f_{x_i, x_j} g + f g_{x_i, x_j} - f_{x_i} g_{x_j} - f_{x_j} g_{x_i}, \tag{2.2}$$

where $1 \leq i, j \leq M$.

One basic property of the Hirota bilinear derivatives is that

$$D_{i_1} D_{i_2} \cdots D_{i_k} f \cdot g = (-1)^k D_{i_1} D_{i_2} \cdots D_{i_k} g \cdot f, \tag{2.3}$$

where $1 \le i_1, i_2, \cdots, i_k \le M$ need not be distinct. It thus follows that if k is odd, we have

$$D_{i_1} D_{i_2} \cdots D_{i_k} f \cdot f = 0. \tag{2.4}$$

We will discuss the following general Hirota bilinear equation

$$P(D) f \cdot f = P(D_1, D_2, \cdots, D_M) f \cdot f = 0, \tag{2.5}$$

where P is a polynomial of M variables and $D = (D_1, D_2, \cdots, D_M)$. Since the terms of odd powers are all zeros, we assume that P is an even polynomial, i.e., $P(-x) = P(x)$, and to generate non-zero polynomial solutions, we require that P has no constant term, i.e., $P(0) = 0$. Moreover, we set

$$P(x) = \sum_{i,j=1}^{M} p_{ij} x_i x_j + \sum_{i,j,k,l=1}^{M} p_{ijkl} x_i x_j x_k x_l + \text{other terms}, \tag{2.6}$$

where p_{ij} and p_{ijkl} are coefficients of terms of second- and fourth-degree, to determine quadratic function solutions.

For convenience's sake, we adopt the index notation for partial derivatives of f:

$$f_{i_1 i_2 \cdots i_k} = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}, \ 1 \le i_1, i_2, \cdots, i_k \le M. \tag{2.7}$$

Using this notation, we have the compact expressions for the second- and fourth-order Hirota bilinear derivatives:

$$D_i D_j f \cdot f = 2(f_{ij} f - f_i f_j), \ 1 \le i, j \le M, \tag{2.8}$$

and

$$\begin{aligned} & D_i D_j D_k D_l (f \cdot f) \\ = {} & 2\big[f_{ijkl} f - f_{ijk} f_l - f_{ijl} f_k - f_{ikl} f_j \\ & - f_{jkl} f_i + f_{ij} f_{kl} + f_{ik} f_{jl} + f_{il} f_{jk} \big], \ 1 \le i, j, k, l \le M. \end{aligned} \tag{2.9}$$

Motivated by Bell polynomial theories on soliton equations [6,14,15], we take the dependent variable transformations:

$$u = 2(\ln f)_{x_1}, \ u = 2(\ln f)_{x_1 x_1}, \tag{2.10}$$

to formulate nonlinear differential equations from Hirota bilinear equations. All integrable nonlinear equations can be generated this way [5,9].

Example 2.1. For the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.11}$$

the transformation $u = 2(\ln f)_{xx}$ provides a link to the bilinear form

$$(D_x D_t + D_x^4) f \cdot f = 0. \tag{2.12}$$

For the KPI and KPII equations

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0, \ \sigma = \mp 1, \tag{2.13}$$

the transformation $u = 2(\ln f)_{xx}$ makes connection with the bilinear form

$$(D_x D_t + D_x^4 + \sigma D_y^2) f \cdot f = 0. \tag{2.14}$$

If a polynomial solution f is positive, then the solution u defined by either of the dependent variable transformations in (2.10) is analytical, and most likely, rationally localized in space, and thus it often presents a lump solution to the corresponding nonlinear differential equation. In what follows, we would like to analyze quadratic function solutions to Hirota bilinear equations to construct lump solutions to nonlinear differential equations.

3. Positive quadratic function solutions to bilinear equations

3.1. Non-negative and positive quadratic functions

Let us consider a general quadratic function

$$f(x) = x^T A x - 2b^T x + c, \ x \in \mathbb{R}^M, \tag{3.1}$$

where $A \in \mathbb{R}^{M\times M}$ is a symmetric matrix, $b \in \mathbb{R}^M$ denotes a column vector, $c \in \mathbb{R}$ is a constant and T denotes transpose.

We say that a polynomial f is non-negative (or positive) if $f(x) \geq 0$, $\forall x \in \mathbb{R}^M$ (or $f(x) > 0$, $\forall x \in \mathbb{R}^M$). We need the pseudoinverse of a matrix to determine the non-negativity (or positivity) of a quadratic function.

For a matrix $A \in \mathbb{R}^{N\times M}$, we call a matrix $A^+ \in \mathbb{R}^{M\times N}$ the Moore–Penrose pseudoinverse of A if

$$AA^+A = A, \ A^+AA^+ = A^+, \ (AA^+)^T = AA^+, \ (A^+A)^T = A^+A, \tag{3.2}$$

which uniquely defines A^+ for any given matrix A [24]. Obviously, the Moore–Penrose pseudoinverse of a zero matrix is its transpose and $(A^+)^T = (A^T)^+$, which implies that if A is symmetric, then so is A^+. When a square matrix A is non-singular, i.e., $|A| = \det(A) \neq 0$, we have $A^+ = A^{-1}$, A^{-1} being the inverse of A.

Suppose that a non-zero matrix $A \in \mathbb{R}^{N\times M}$ has its singular value decomposition

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{3.3}$$

where $U \in \mathbb{R}^{N \times N}$ and $V \in \mathbb{R}^{M \times M}$ are orthogonal matrices, and Σ reads

$$\Sigma = \mathrm{diag}(d_1, \cdots, d_r), \; d_1 \geq \cdots \geq d_r > 0, \; r = \mathrm{rank}(A). \tag{3.4}$$

Then the Moore–Penrose pseudoinverse of A is given by

$$A^+ = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T. \tag{3.5}$$

The Moore–Penrose pseudoinverse can be applied to analysis of linear systems [24]. A linear system $A\alpha = b$ is consistent if and only if $AA^+b = b$. Moreover, if it is consistent, then its solution set is given by

$$\{\alpha = A^+b + (I_M - A^+A)\beta \,|\, \beta \in \mathbb{R}^M\},$$

where I_M is the identity matrix of size M.

Lemma 3.1. *Let $A \in \mathbb{R}^{M \times M}$ be symmetric and $b \in \mathbb{R}^M$ be arbitrary. If $\alpha \in \mathbb{R}^M$ solves $A\alpha = b$, then*

$$\alpha^T A \alpha = b^T A^+ b, \tag{3.6}$$

and further,

$$(x - \alpha)^T A (x - \alpha) = (x - A^+b)^T A (x - A^+b). \tag{3.7}$$

Proof. Recalling the first property in (3.2) and using $A^T = A$, we have

$$\alpha^T A \alpha = \alpha^T A A^+ A \alpha = \alpha^T A^T A^+ A \alpha = b^T A^+ b.$$

Therefore, (3.6) holds. Now, noting that

$$\alpha^T A x = (A\alpha)^T x = b^T x, \; x^T (A\alpha) = x^T b = b^T x,$$
$$(A^+b)^T A x = b^T A^+ A x = \alpha^T A A^+ A x = \alpha^T A x = (A\alpha)^T x = b^T x,$$

and

$$x^T A (A^+b) = x^T A A^+ A \alpha = x^T A \alpha = x^T b = b^T x,$$

we see that (3.7) follows directly from (3.6). □

We denote a positive-semidefinite (or positive-definite) matrix $A \in \mathbb{R}^{M \times M}$ by $A \geq 0$ (or $A > 0$). Namely, $A \geq 0$ (or $A > 0$) means that $x^T A x \geq 0$ for all $x \in \mathbb{R}^M$ (or $x^T A x > 0$ for all non-zero $x \in \mathbb{R}^M$). The following theorem gives a description of non-negative (or positive) quadratic functions.

Theorem 3.2. *Let a quadratic function f be defined by (3.1). Then (a) if $b \in \mathrm{range}(A)$, then*

$$f(x) = (x-\alpha)^T A(x-\alpha) + c - \alpha^T A\alpha \\ = (x - A^+b)^T A(x - A^+b) + c - b^T A^+ b, \tag{3.8}$$

where $\alpha \in \mathbb{R}^M$ solves $A\alpha = b$; and (b) f is non-negative (or positive) if and only if $A \geq 0$, $b \in \mathrm{range}(A)$ and

$$d = c - b^T A^+ b \tag{3.9}$$

is non-negative (or positive).

Proof. (a) First, based on Lemma 3.1, it is sufficient to show that

$$f(x) = x^T Ax - 2\alpha^T Ax + c = (x-\alpha)^T A(x-\alpha) + c - \alpha^T A\alpha, \tag{3.10}$$

where we have made use of $b = A\alpha$ and $A^T = A$.

(b) Second, we prove part (b).

($\Leftarrow$) This directly follows from the second equality of (3.8) in part (a).

($\Rightarrow$) Suppose that $A \geq 0$ is false. Then there exists a vector $\beta \in \mathbb{R}^M$ such that $\beta^T A\beta < 0$, and further for $r \in \mathbb{R}$, we have

$$f(r\beta) = r^2\beta^T A\beta - 2rb^T\beta + c \to -\infty, \text{ as } r \to \pm\infty.$$

This is a contradiction to the assumption on f that f is non-negative (or positive). Therefore, we have $A \geq 0$.

Now let $b = b^{(1)} + b^{(2)}$ with $b^{(1)} \in \mathrm{range}(A)$ and $b^{(2)} \in \mathrm{range}(A)^\perp$. Assume that $\alpha \in \mathbb{R}^M$ satisfies $A\alpha = b^{(1)}$. Consider $x = \alpha + rb^{(2)}$, with r being a positive number. Then we can have

$$\begin{aligned} f(x) &= x^T Ax - 2\alpha^T Ax - 2b^{(2)T}x + c \\ &= (x-\alpha)^T A(x-\alpha) - 2b^{(2)T}x + c - \alpha^T A\alpha \\ &= r^2 b^{(2)T}Ab^{(2)} - 2b^{(2)T}\alpha - 2rb^{(2)T}b^{(2)} + c - \alpha^T A\alpha \\ &= -2rb^{(2)T}b^{(2)} - 2b^{(2)T}\alpha + c - \alpha^T A\alpha \to -\infty, \text{ as } r \to \infty, \end{aligned}$$

if $b^{(2)} \neq 0$. Therefore $b^{(2)} = 0$, since f is non-negative (or positive). This implies $b \in \mathrm{range}(A)$. Further, $d = f(\alpha) \geq 0$ (or > 0). The proof is finished. □

Any two solutions $\alpha^{(1)}$ and $\alpha^{(2)}$ to $A\alpha = b$ satisfy $A(\alpha^{(1)} - \alpha^{(2)}) = A\alpha^{(1)} - A\alpha^{(2)} = 0$, which means that $\alpha^{(1)} - \alpha^{(2)} \in \ker(A)$ and thus

$$\alpha^{(1)T}A\alpha^{(1)} = \alpha^{(1)T}A\alpha^{(2)} = \alpha^{(2)T}A\alpha^{(1)} = \alpha^{(2)T}A\alpha^{(2)}.$$

This is just a consequence of (3.6). We also point out that all the results presented in an earlier paper [11] are consequences of our results in Theorem 3.2. For example, we can have the last two theorems in [11], i.e., Theorems 6 and 7 in [11]: a quadratic function f is bounded from below if and only if f reaches its minimum at a point $x_0 \in \mathbb{R}^M$ if and only if $A \geq 0$ and $Ax_0 = b$, where f is assumed to be given by (3.1). Actually, Theorem 3.2 also tells that f achieves its minimum at any point $\alpha \in \mathbb{R}^M$, where α is a solution to $A\alpha = b$, and its minimum is $c - b^T A^+ b$. We proves the result on the extreme value as follows.

Corollary 3.3. *If a quadratic function defined by (3.1) reaches its minimum or maximum, then its extreme value is $c - b^T A^+ b$.*

Proof. If f reaches its maximum γ, then $g = f - \gamma$ is non-negative and by Theorem 3.2, we have $g(x) = (x-\alpha)^T A(x-\alpha) + (c-\gamma) - b^T A^+ b$ with $A \geq 0$, which says that $f(x) \geq c - b^T A^+ b$ and $f(\alpha) = c - b^T A^+ b$, and so the minimum value of f is $c - b^T A^+ b$. If f reaches its maximum, then $g = -f$ reaches its minimum. Therefore, as we just proved, g achieves the minimum value $-c + b^T A^+ b$, and so f has the maximum value $c - b^T A^+ b$, which completes the proof. □

3.2. Positive quadratic function solutions

Let $\alpha = (\alpha_1, \cdots, \alpha_M)^T \in \mathbb{R}^M$ be a fixed vector. Consider a quadratic function defined as follows:

$$f(x) = (x-\alpha)^T A(x-\alpha) + d = \sum_{i,j=1}^{M} a_{ij}(x_i - \alpha_i)(x_j - \alpha_j) + d, \tag{3.11}$$

where the real matrix $A = (a_{ij})_{M\times M}$ is symmetric and $d \in \mathbb{R}$ is a constant. Theorem 3.2 guarantees that when $A \geq 0$ and $d > 0$, this presents the class of positive quadratic functions.

Obviously, we have

$$D_{i_1} D_{i_2} \cdots D_{i_k} f \cdot f = 0,\ 1 \leq i_j \leq M,\ 1 \leq j \leq k,\ k > 4,$$

for any quadratic function f. Moreover, because all odd-order Hirota bilinear derivative terms in the Hirota bilinear equation (2.5) are zero, the bilinear equation (2.5) is reduced to

$$Q(D) f \cdot f = 0, \tag{3.12}$$

where

$$Q(x) = \sum_{i,j=1}^{M} p_{ij} x_i x_j + \sum_{i,j,k,l=1}^{M} p_{ijkl} x_i x_j x_k x_l, \tag{3.13}$$

since $Q(D) f \cdot f = P(D) f \cdot f$ for P defined by (2.6).

Now we compute the second- and fourth-order Hirota bilinear derivatives of a positive quadratic function defined by (3.11). Note that

$$f_i = 2\sum_{k=1}^{M} a_{ik}(x_k - \alpha_k) = 2A_i^T (x-\alpha),\ \ f_{ij} = 2a_{ij},\ 1 \leq i, j \leq M,$$

where A_i is the ith column vector of A for $1 \le i \le M$. We denote $y = x - \alpha$. Then using (2.8), we have

$$\sum_{i,j=1}^{M} p_{ij} D_i D_j f \cdot f = 4 \sum_{i,j=1}^{M} p_{ij} a_{ij} f - 8 \sum_{i,j=1}^{M} p_{ij} y^T A_i A_j^T y$$

$$= 4d \sum_{i,j=1}^{M} p_{ij} a_{ij} + 4y^T \Big[\sum_{i,j=1}^{M} p_{ij} (a_{ij} A - A_i A_j^T - A_j A_i^T) \Big] y. \tag{3.14}$$

By (2.9), the fourth-order Hirota bilinear derivatives of f in (3.11) read

$$D_i D_j D_k D_l f \cdot f = 2(f_{ij} f_{kl} + f_{ik} f_{jl} + f_{il} f_{jk}) = 8(a_{ij} a_{kl} + a_{ik} a_{jl} + a_{il} a_{jk}). \tag{3.15}$$

Thus, if (3.11) solves the Hirota bilinear equation (2.5), i.e., the reduced Hirota bilinear equation (3.12), then we have

$$8 \sum_{i,j,k,l=1}^{M} p_{ijkl} (a_{ij} a_{kl} + a_{ik} a_{jl} + a_{il} a_{jk}) + 4d \sum_{i,j=1}^{M} p_{ij} a_{ij}$$

$$+ y^T \Big[\sum_{i,j=1}^{M} p_{ij} (a_{ij} A - A_i A_j^T - A_j A_i^T) \Big] y = 0. \tag{3.16}$$

Note $x \in \mathbb{R}^M$ is arbitrary, and so is $y = x - \alpha$. Therefore, we obtain the following result.

Theorem 3.4. *Let $A = (a_{ij})_{M \times M} \in \mathbb{R}^{M \times M}$ be symmetric and $d \in \mathbb{R}$ be arbitrary. A quadratic function f defined by (3.11) solves the Hirota bilinear equation (2.5) if and only if*

$$2 \sum_{i,j,k,l=1}^{M} p_{ijkl} (a_{ij} a_{kl} + a_{ik} a_{jl} + a_{il} a_{jk}) + d \sum_{i,j=1}^{M} p_{ij} a_{ij} = 0 \tag{3.17}$$

and

$$\sum_{i,j=1}^{M} p_{ij} (a_{ij} A - A_i A_j^T - A_j A_i^T) = 0, \tag{3.18}$$

where A_i denotes the ith column vector of the symmetric matrix A for $1 \le i \le M$.

Corollary 3.5. *If $f(x) = x^T A x + d$ solves the Hirota bilinear equation (2.5), then for any $\alpha \in \mathbb{R}^M$, $f(x - \alpha)$ solves the Hirota bilinear equation (2.5), too.*

Proof. This is because (3.17) and (3.18) only depend on the matrix A and the constant d, but do not depend on the shift vector α. □

We denote the coefficient matrix of the second order Hirota bilinear derivative terms by

$$P^{(2)} = (p_{ij})_{M\times M} \in \mathbb{R}^{M\times M}, \tag{3.19}$$

in the Hirota bilinear equation (2.5). When $P^{(2)} = 0$, the matrix equation (3.18) is automatically satisfied and the scalar equation (3.17) reduces to

$$\sum_{i,j,k,l=1}^{M} p_{ijkl}(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}) = 0. \tag{3.20}$$

If $M \geq 2$, for a fixed matrix A, obviously there exists infinitely many non-zero solutions of p_{ijkl}, $1 \leq i, j, k, l \leq M$, to the equation (3.20).

Let us now consider quadratic function solutions with $|A| \neq 0$.

If $M = 1$, then $a_{11} \neq 0$. Therefore, (3.17) and (3.18) equivalently yield

$$p_{11} = p_{1111} = 0.$$

This means that a bilinear ordinary differential equation defined by (2.5) has a quadratic function solution if and only if the least degree of a polynomial P must be greater than 5.

If $M = 2$, we have the following example in (1 + 1)-dimensions. Consider the function $f(x,t) = 3x^2 - 2xt + t^2 + \frac{27}{2}$, where $A = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$ with $|A| = 2 > 0$. Obviously, this quadratic polynomial is positive, and solves the following (1 + 1)-dimensional Hirota bilinear equation:

$$(D_x^4 - D_x^2 - 2D_tD_x - 3D_t^2) f \cdot f = 0,$$

where the symmetric coefficient matrix $P^{(2)} = \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix}$ is not zero. This function f leads to lump solutions to the corresponding nonlinear equations under $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$.

When $M \geq 3$, there is a totally different situation. What kind of Hirota bilinear equations (2.5) can possess a quadratic function solution defined by (3.11) with $|A| \neq 0$? The following theorem provides a complete answer to this question.

Theorem 3.6. *Let $M \geq 3$. Assume that a quadratic function f defined by (3.11) solves the Hirota bilinear equation (2.5) with P defined by (2.6). If $|A| \neq 0$, i.e., A is non-singular, then*

$$p_{ij} + p_{ji} = 0,\ 1 \leq i, j \leq M, \tag{3.21}$$

which means that the Hirota bilinear equation (2.5) doesn't contain any second-order Hirota bilinear derivative term.

Proof. First, assume that $P^{(2)T} = P^{(2)}$. Then, (3.18) becomes

$$\tilde{a}A - 2AP^{(2)}A = 0,\ \text{where } \tilde{a} = \sum_{i,j=1}^{M} p_{ij}a_{ij}. \tag{3.22}$$

Since A is symmetric, there exists an orthogonal matrix $U \in \mathbb{R}^{M\times M}$ such that

$$\hat{A} = U^T A U = \mathrm{diag}(\hat{a}_1, \cdots, \hat{a}_M).$$

Set $\hat{P}^{(2)} = U^T P^{(2)} U$, and by (3.22), we have

$$\tilde{a}\hat{A} - 2\hat{A}\hat{P}^{(2)}\hat{A} = 0. \tag{3.23}$$

Since $|A| \neq 0$, we have $|\hat{A}| \neq 0$. Thus, (3.23) tells that $\hat{P}^{(2)} = \frac{\tilde{a}}{2}\hat{A}^{-1}$ and further $\hat{P}^{(2)}$ is diagonal. Therefore, we can express

$$\hat{P}^{(2)} = \mathrm{diag}(\hat{p}_1, \cdots, \hat{p}_M).$$

Plugging the two diagonal matrices $\hat{A}$ and $\hat{P}^{(2)}$ into (3.23) engenders

$$\tilde{a} = 2\hat{a}_k\hat{p}_k,\ 1 \le k \le M. \tag{3.24}$$

On the other hand, a direct calculation can show that $\tilde{a} = \sum_{i,j=1}^{M} a_{ij}p_{ij}$ is an invariant under an orthogonal similarity transformation, and thus, from $\hat{A} = U^T A U$ and $\hat{P}^{(2)} = U^T P^{(2)} U$, we have

$$\tilde{a} = \sum_{k=1}^{M} \hat{a}_k\hat{p}_k. \tag{3.25}$$

Now a combination of (3.24) and (3.25) tells that $M\tilde{a} = 2\tilde{a}$. Since $M \ge 3$, we see $\tilde{a} = 0$, and so, $\hat{P}^{(2)} = 0$, which implies that $P^{(2)} = 0$.

Second, if $P^{(2)}$ is not symmetric, noting that

$$\sum_{i,j=1}^{N} p_{ij}x_ix_j = \sum_{i,j=1}^{N} \bar{p}_{ij}x_ix_j,\ \bar{p}_{ij} = \frac{p_{ij}+p_{ji}}{2},\ 1 \le i, j \le M.$$

we can begin with a symmetric coefficient matrix of second order Hirota bilinear derivative terms, $\bar{P}^{(2)} = (\bar{p}_{ij})_{M\times M}$, to analyze quadratic function solutions. Thus, as we just showed, $\bar{P}^{(2)} = 0$. This is exactly what we need to get. The proof is finished. □

Theorem 3.6 tells us about the case of $|A| \neq 0$, which says that if a Hirota bilinear equation admits a quadratic function solution determined by (3.11) with $|A| \neq 0$, then it cannot contain any second-order Hirota bilinear derivative term.

For the KPI and KPII equations, since the corresponding symmetric coefficient matrix $P^{(2)}$ is not zero, Theorem 3.6 tells that any quadratic function solution f cannot be expressed as a sum of squares of three linear functions and a constant: $f = g_1^2 + g_2^2 + g_3^2 + d$, where

$$g_i = c_{i1}x + c_{i2}y + c_{i3}t + \alpha_i,\ 1 \le i \le 3,$$

with $(c_{ij})_{3\times 3}$ being non-singular, which will also be showed clearly later.

The other case is $|A| = 0$, for which there is no requirement on inclusion of second-order Hirota bilinear derivative terms. Obviously, when $A = \mathrm{diag}(a_1, \cdots, a_{M-1}, 0) \neq 0$, (3.22) has a non-zero symmetric matrix solution $P^{(2)} = \mathrm{diag}(\underbrace{0, \cdots, 0}_{M-1}, 1) \neq 0$ with $\tilde{a} = 0$, and (3.17) has infinitely many non-zero solutions for $\{p_{ijkl} | 1 \leq i, j, k, l, \leq M\}$. Therefore, we can have both second- and fourth-order Hirota bilinear derivative terms in the Hirota bilinear equation (2.5).

3.3. Solutions as sums of squares of linear functions

We will explore relations between quadratic function solutions and sums of squares of linear functions, and discuss quadratic function solutions which can be written as sums of squares of linear functions.

Theorem 3.7. *Let a quadratic function f be defined by (3.11). Suppose $r = \mathrm{rank}(A)$. Then there exist $b^{(j)} \in \mathbb{R}^M$, $c_j \in \mathbb{R}$, $1 \leq j \leq r$, such that*

$$f(x) = \sum_{j=1}^{r} (b^{(j)T} x + c_j)^2 + d. \tag{3.26}$$

Proof. We assume that the symmetric matrix A has the singular value decomposition:

$$A = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{3.27}$$

where $V \in \mathbb{R}^{M \times M}$ is orthogonal and

$$\Sigma = \mathrm{diag}(d_1, \cdots, d_r), \ d_1 \geq \cdots \geq d_r > 0.$$

Upon denoting $V = (v^{(1)}, v^{(2)}, \cdots, v^{(M)})$ and setting

$$b^{(j)} = \sqrt{d_j}\, v^{(j)}, \ c_j = -\alpha^T b^{(j)}, \ 1 \leq j \leq r, \tag{3.28}$$

we have

$$A = \sum_{j=1}^{r} d_j v^{(j)} v^{(j)T} = \sum_{j=1}^{r} (\sqrt{d_j}\, v^{(j)})(\sqrt{d_j}\, v^{(j)})^T = \sum_{j=1}^{r} b^{(j)} b^{(j)T},$$

and thus

$$\begin{aligned} f(x) &= \sum_{j=1}^{r} (x-\alpha)^T b^{(j)} b^{(j)T} (x-\alpha) + d \\ &= \sum_{j=1}^{r} [(x-\alpha)^T b^{(j)}][(x-\alpha)^T b^{(j)}]^T + d \end{aligned}$$

$$= \sum_{j=1}^{r} (b^{(j)T} x + c_j)^2 + d.$$

The proof is finished. □

Based on Theorem 3.2, and noting that constant functions are particular linear functions, the following result is a direct consequence of Theorem 3.7.

Corollary 3.8. *Any non-negative quadratic function can be written as a sum of squares of linear functions.*

This corollary guarantees that completing squares can transform non-negative quadratic functions into sums of squares of linear functions. It also proves Hilbert's 17th problem for quadratic functions.

Lemma 3.9. *Let N be a natural number, and $b^{(j)} \in \mathbb{R}^M$, $c_j \in \mathbb{R}$, $1 \le j \le N$, be arbitrary. Then the linear system*

$$\Big(\sum_{j=1}^{N} b^{(j)} b^{(j)T}\Big)\alpha = -\sum_{j=1}^{N} c_j b^{(j)}, \tag{3.29}$$

is consistent, where $\alpha \in \mathbb{R}^M$ an unknown vector.

Proof. Note that the columns of the coefficient matrix $\sum_{j=1}^{N} b^{(j)} b^{(j)T}$ read

$$\sum_{j=1}^{N} b_1^{(j)} b^{(j)}, \cdots, \sum_{j=1}^{N} b_M^{(j)} b^{(j)},$$

where $b_i^{(j)}$ is the ith component of $b^{(j)}$. It follows that the dimension of the column space of the coefficient matrix is equal to the rank of the $M \times N$ matrix $(b_i^{(j)})_{1 \le i \le M, 1 \le j \le N}$. This implies that the column space of the coefficient matrix is just the space spanned by $b^{(j)}$, $1 \le j \le N$. On the other hand, the given vector $-\sum_{j=1}^{N} c_j b^{(j)}$ belongs to the space spanned by $b^{(j)}$, $1 \le j \le N$. Therefore, the linear system is consistent. □

Theorem 3.10. *Let N be a natural number, and $b^{(j)} \in \mathbb{R}^M$, $c_j \in \mathbb{R}$, $1 \le j \le N$, $h \in \mathbb{R}$ be arbitrary. Suppose that a quadratic function f is given by*

$$f(x) = \sum_{j=1}^{N} (b^{(j)T} x + c_j)^2 + h. \tag{3.30}$$

Then (a) we have

$$f(x) = x^T A x - 2b^T x + c = (x - A^+ b)^T A (x - A^+ b) + d, \tag{3.31}$$

where

$$A=\sum_{j=1}^{N} b^{(j)}b^{(j)T},\ b=-\sum_{j=1}^{N} c_j b^{(j)},\ c=\sum_{j=1}^{N} c_j^2+h,\ d=c-b^TA^+b; \tag{3.32}$$

(b) f is non-negative (or positive) if and only if $d\ge 0$ (or $d>0$); and (c) f solves the Hirota bilinear equation (2.5), or equivalently (3.12), if and only if (3.17) and (3.18) are true for the matrix A and the constant d defined in (3.32).

Proof. To prove part (a), we begin by computing that

$$\begin{aligned} f(x) &= \sum_{j=1}^{N}(b^{(j)T}x+c_j)^2+h \\ &= \sum_{j=1}^{N}(b^{(j)T}x)^T(b^{(j)T}x)+2\sum_{j=1}^{N}(b^{(j)T}x)c_j+\sum_{j=1}^{N}c_j^2+h \\ &= x^T\Big(\sum_{j=1}^{N}b^{(j)}b^{(j)T}\Big)x+2\Big(\sum_{j=1}^{N}c_jb^{(j)T}\Big)x+\sum_{j=1}^{N}c_j^2+h \\ &= x^TAx-2b^Tx+c, \end{aligned} \tag{3.33}$$

where A, b and c are defined in (3.32). It then follows from Lemma 3.9 and Theorem 3.2 that

$$f(x)=(x-\alpha)^TA(x-\alpha)+d=(x-A^+b)^TA(x-A^+b)+d,$$

where α solves $A\alpha=b$ and d is defined in (3.32). Therefore, part (a) is true.

Now, based on part (a) and noting that A is positive-semidefinite, parts (b) and (c) are just consequences of Theorem 3.2 and Theorem 3.4. The proof is finished. □

This theorem tells us the way of constructing positive quadratic function solutions through taking sums of squares of linear functions. It also leads to the following inequality involving the Moore–Penrose pseudoinverse.

Corollary 3.11. *Let N be a natural number, and $b^{(j)}\in\mathbb{R}^M$, $c_j\in\mathbb{R}$, $1\le j\le N$, be arbitrary. Then*

$$\Big(\sum_{j=1}^{N}c_jb^{(j)T}\Big)A^+\Big(\sum_{j=1}^{N}c_jb^{(j)}\Big)\le\sum_{j=1}^{N}c_j^2, \tag{3.34}$$

where A^+ is the Moore–Penrose pseudoinverse of $A=\sum_{j=1}^{N}b^{(j)}b^{(j)T}$.

Proof. In Theorem 3.10, we assume that $h\ge 0$, and then the quadratic function f defined by (3.30) is non-negative, which means that

$$\sum_{j=1}^{N} c_j^2 + h - b^T A^+ b \geq 0,$$

where $b = -\sum_{j=1}^{N} c_j b^{(j)}$. The required result in the corollary follows immediately from taking a limit of the above inequality as $h \to 0$. □

If the linear system (3.29) has a particular solution $\alpha \in \mathbb{R}^M$ determined by

$$b^{(j)T}\alpha = -c_j,\ 1 \leq j \leq N,$$

then we have

$$\Big(\sum_{j=1}^{N} c_j b^{(j)T}\Big) A^+ \Big(\sum_{j=1}^{N} c_j b^{(j)}\Big) = \sum_{j=1}^{N} c_j^2.$$

This is because by (3.6), we can compute that

$$\Big(\sum_{j=1}^{N} c_j b^{(j)T}\Big) A^+ \Big(\sum_{j=1}^{N} c_j b^{(j)}\Big) = \alpha^T A\alpha$$
$$= \alpha^T \Big(\sum_{j=1}^{N} b^{(j)} b^{(j)T}\Big)\alpha = -\alpha^T \sum_{j=1}^{N} b^{(j)} c_j = \sum_{j=1}^{N} c_j^2.$$

Next, we are going to present a basic characteristic of sums of squares of linear functions.

Theorem 3.12. *Let N be a natural number, and $b^{(j)} \in \mathbb{R}^M$, $c_j \in \mathbb{R}$, $1 \leq j \leq N$, $h \in \mathbb{R}$ be arbitrary. Suppose that a quadratic function f is defined by (3.30), i.e.,*

$$f(x) = \sum_{j=1}^{N} (b^{(j)T} x + c_j)^2 + h,$$

and set $A = \sum_{j=1}^{N} b^{(j)} b^{(j)T}$ and $r = \mathrm{rank}(A)$. Then (a) there exist $\tilde{b}^{(j)} \in \mathbb{R}^M$, $\tilde{c}_j \in \mathbb{R}$, $1 \leq j \leq r$, such that

$$f(x) = \sum_{j=1}^{r} (\tilde{b}^{(j)T} x + \tilde{c}_j)^2 + \sum_{j=1}^{N} c_j^2 + h - b^T A^+ b, \tag{3.35}$$

where $b = -\sum_{j=1}^{N} c_j b^{(j)}$; (b) if $f(x) = \sum_{j=1}^{s} (\hat{b}^{(j)T} x + \hat{c}_j)^2 + \hat{h}$, where $\hat{b}^{(j)} \in \mathbb{R}^M$, $\hat{c}_j \in \mathbb{R}$, $1 \leq j \leq s$, $\hat{h} \in \mathbb{R}$, then $s \geq r$.

Proof. (a) A combination of Theorem 3.7 and Theorem 3.10 leads to part (a).

(b) Note that we can rewrite

$$f(x)=\sum_{j=1}^{s}(\hat{b}^{(j)T}x+\hat{c}_j)^2+\hat{h}=x^T\hat{A}x-2\hat{b}^Tx+\hat{c},$$

where

$$\hat{A}=\sum_{j=1}^{s}\hat{b}^{(j)}\hat{b}^{(j)T},\ \hat{b}=-\sum_{j=1}^{s}\hat{c}_j\hat{b}^{(j)},\ \hat{c}=\sum_{j=1}^{s}\hat{c}_j^2+\hat{h}.$$

Compared with (3.30), (3.31) and (3.32), we see $\hat{A}=A$. Set $\hat{B}=(\hat{b}^{(1)},\hat{b}^{(2)},\cdots,\hat{b}^{(s)})$. Then $\hat{A}=\hat{B}\hat{B}^T$ and so

$$r=\text{rank}(A)=\text{rank}(\hat{A})=\text{rank}\hat{B}\le s.$$

This completes the proof. □

The result (b) of Theorem 3.12 tells the largest number of squares of linearly independent non-constant linear functions in a sum for a non-negative quadratic function.

When $x=(x_1,\cdots,x_{M-1},t)$, where t denotes time and x_i, $1\le i\le M-1$, are spatial variables, positive quadratic function solutions determined by (3.11) with a non-zero (M,M) minor of A lead to lump solutions, and otherwise, lump-type solutions to the corresponding nonlinear equations under either of the two transformations in (2.10).

4. Applications to generalized KP and BKP equations

4.1. Generalized KP equations in $(N+1)$-dimensions

Let us first consider the generalized Kadomtsev–Petviashvili (gKP) equations in $(N+1)$-dimensions:

$$(u_t+6uu_{x_1}+u_{x_1x_1x_1})_{x_1}+\sigma(u_{x_2x_2}+u_{x_3x_3}+\cdots+u_{x_Nx_N})=0, \tag{4.1}$$

where $\sigma=\mp1$ and $N\ge2$. When $\sigma=-1$, it is called the gKPI equation, and when $\sigma=1$, the gKPII equation.

Denote $x=(x_1,x_2,\cdots,x_N,t)^T\in\mathbb{R}^{N+1}$. Take a positive quadratic function:

$$f(x)=x^TAx+d \tag{4.2}$$

with $A=A^T\in\mathbb{R}^{(N+1)\times(N+1)}$, $A\ge0$ and $d>0$. For any $x\in\mathbb{R}^{N+1}$, the rational function

$$u=2(\ln f)_{x_1x_1}=\frac{2(ff_{11}-f_1^2)}{f^2}$$

is analytical in $\mathbb{R}^{N+1}$. Substituting it into (4.1), we have

$$(u_t + 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} + \sigma \sum_{j=2}^{N} u_{x_j x_j}$$
$$= \frac{\partial^2}{\partial x_1^2}\Big[f^{-2}(D_1^4 + D_1 D_{N+1} + \sigma \sum_{j=2}^{N} D_j^2) f \cdot f \Big] = 0,\ \sigma = \mp 1,$$

where D_{N+1} is the Hirota bilinear derivative with respect to time t. Therefore, if f solves the bilinear gKPI or gKPII equation:

$$(D_1^4 + D_1 D_{N+1} + \sigma \sum_{j=2}^{N} D_j^2) f \cdot f = 0,\ \sigma = \mp 1, \tag{4.3}$$

then $u = 2(\ln f)_{x_1x_1}$ solves the gKPI or gKPII equation in (4.1). Such a solution process provides us with lump or lump-type solutions to the gKPI or gKPII equation.

Theorem 4.1. *A positive quadratic function f defined by (4.2) solves the bilinear gKPI or gKPII equation by (4.3) if and only if*

$$6a_{11}^2 + d\tilde{a} = 0, \tag{4.4}$$

and

$$\tilde{a}A - (A_1 A_{N+1}^T + A_{N+1} A_1^T) - 2\sigma \sum_{i=2}^{N} A_i A_i^T = 0, \tag{4.5}$$

where

$$\tilde{a} := a_{1N+1} + \sigma \sum_{i=2}^{N} a_{ii} \le 0. \tag{4.6}$$

Proof. An application of Theorem 3.4 to the bilinear gKPI and gKPII equations in (4.3) tells (4.4) and (4.5). The property $\tilde{a} \le 0$ in (4.6) follows from (4.4) and $d > 0$. The proof is finished. □

If $\tilde{a} = 0$, then we have $a_{11} = 0$ by (4.4). Since $A \ge 0$, we have $a_{1,N+1} = 0$. Further

$$\sigma \sum_{i=2}^{N} a_{ii} = \tilde{a} - a_{1N+1} = 0.$$

However, $\sigma \ne 0$ and $a_{ii} \ge 0$ for $i = 1, \cdots, N+1$. Thus, $a_{22} = \cdots = a_{NN} = 0$, and there exists only a non-zero solution $A = (a_{ij})_{(N+1)\times(N+1)}$ with all $a_{ij} = 0$ except $a_{N+1,N+1}$. The corresponding solution is $u = 2(\ln f)_{x_1x_1} \equiv 0$, a trivial solution.

Now let us introduce

$$B = 2\bar{P}^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2\sigma I_{N-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{4.7}$$

where I_{N-1} is the identity matrix of size $N-1$, and then the algebraic equation (4.5) can be written in a compact form:

$$\tilde{a}A - ABA = 0, \tag{4.8}$$

where $\tilde{a}$ is defined by (4.6).

Corollary 4.2. *If a positive-semidefinite matrix A satisfies the condition (4.8), then $|A| = 0$.*

Proof. If $|A| \neq 0$, then $\tilde{a}I_{N+1} - AB = 0$, and so $A = \tilde{a}B^{-1}$. The matrix B has two eigenvalues ± 1 (and an eigenvalue 2σ of multiplicity $N-1$), and thus B^{-1} also has two eigenvalues ± 1. Therefore, A is not positive-semidefinite unless $\tilde{a} = 0$. In this case, $ABA = 0$, and then $|ABA| = |A|^2|B| = 0$, which leads to $|A| = 0$. A contradiction! □

This corollary is also a consequence of Theorem 3.6. For the $(N+1)$-dimensional KP equations, since the corresponding symmetric coefficient matrix $P^{(2)}$, defined by (3.19), is not zero, their corresponding Hirota bilinear equations in (4.3) do not possess any quadratic function solution which can be written as a sum of squares of $N+1$ linearly independent linear functions.

We remark that it is not easy to find all solutions to the system of quadratic equations in (4.8). The following examples show us that the gKPI equations have lump or lump-type solutions. It is also direct to observe that any lump or lump-type solution to an $(N+1)$-dimensional gKPI equation is a lump-type solution to an $((N+1)+1)$-dimensional gKPI equation of the same type as well.

Example 4.3. Let us consider the simplest case: $N = 2$. This corresponds to the $(2+1)$-dimensional KPI and KPII equations:

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0, \ \sigma = \mp 1, \tag{4.9}$$

where we set $x_1 = x$ and $x_2 = y$. By using Maple, we can have

$$A = \begin{bmatrix} a & b & \sigma(ac-2b^2)/a \\ b & c & -\sigma bc/a \\ \sigma(ac-2b^2)/a & -\sigma bc/a & \sigma^2c^2/a \end{bmatrix} \text{ with } a > 0, \ c > 0, \ ac - b^2 > 0.$$

This leads to

$$\begin{aligned} f(x,y,t) &= ax^2 + cy^2 + \frac{\sigma^2c^2}{a}t^2 + 2bxy - \frac{2\sigma bc}{a}yt + \frac{2\sigma}{a}(ac-2b^2)xt + d \\ &= a[x + \frac{b}{a}y + \frac{\sigma}{a^2}(ac-2b^2)t]^2 + \frac{ac-b^2}{a}(y - \frac{2\sigma b}{a}t)^2 + d, \end{aligned} \tag{4.10}$$

which reduces to

$$f(x,y,t)=ax^2+cy^2+\frac{\sigma^2c^2}{a}t^2+2\sigma cxt+d=a(x+\frac{\sigma ct}{a})^2+cy^2+d,$$

when $b=0$. The condition (4.4) now reads

$$6a^2+d[\frac{\sigma(ac-2b^2)}{a}+\sigma c]=6a^2+2d\frac{\sigma(ac-b^2)}{a}=0,$$

which yields

$$d=-\frac{3a^3}{\sigma(ac-b^2)}>0. \tag{4.11}$$

By Corollary 3.5, for any constants $\gamma_1,\gamma_1,\gamma_3\in\mathbb{R}$, we have the following quadratic function solutions:

$$\begin{aligned}f(x,y,t)&=a[(x-\gamma_1)+\frac{b}{a}(y-\gamma_2)+\frac{\sigma}{a^2}(ac-2b^2)(t-\gamma_3)]^2\\&\quad+\frac{ac-b^2}{a}[(y-\gamma_2)-\frac{2\sigma b}{a}(t-\gamma_3)]^2+d\\&=a[x+\frac{b}{a}y+\frac{\sigma}{a^2}(ac-2b^2)t-\delta_1]^2\\&\quad+\frac{ac-b^2}{a}(y-\frac{2\sigma b}{a}t-\delta_2)^2+d,\end{aligned} \tag{4.12}$$

with δ_1 and δ_2 being defined by

$$\delta_1=\gamma_1+\frac{b}{a}\gamma_2+\frac{\sigma}{a^2}(ac-2b^2)\gamma_3,\ \delta_2=\gamma_2-\frac{2\sigma b}{a}\gamma_3.$$

Because $\gamma_1,\gamma_2,\gamma_3$ are arbitrary, so are δ_1 and δ_2. Furthermore, the corresponding lump solutions to the $(2+1)$-dimensional KPI equation in (4.9) read

$$\begin{aligned}&u(x,y,t)=2(\ln f)_{xx}\\&=\frac{4\big\{-a^2[x+\frac{b}{a}y+\frac{\sigma}{a^2}(ac-2b^2)t-\delta_1]^2+(ac-b^2)(y-\frac{2\sigma b}{a}t-\delta_2)^2+ad\big\}}{\big\{a[x+\frac{b}{a}y+\frac{\sigma}{a^2}(ac-2b^2)t-\delta_1]^2+\frac{ac-b^2}{a}(y-\frac{2\sigma b}{a}t-\delta_2)^2+d\big\}^2},\end{aligned}$$

where d is defined by (4.11), $a,b,c\in\mathbb{R}$ satisfy $a>0$, $c>0$, $ac-b^2>0$, and δ_1 and δ_2 are arbitrary. When taking

$$a=1,\ b=\sqrt{3}a,\ c=3(a^2+b^2),\ d=\frac{1}{b^2},\ \delta_1=\delta_2=0,\ y\to\frac{1}{\sqrt{3}}y,$$

the resulting lump solutions reduce to the solutions in (1.2).

Remark 4.4. The condition in (4.11) implies that $\sigma = -1$ in order to have lump solutions generated from positive quadratic functions. This shows that the $(2+1)$-dimensional KPI equation $(\sigma = -1)$ possesses the discussed lump solutions whereas the $(2+1)$-dimensional KPII equation $(\sigma = 1)$ does not.

Example 4.5. We consider the $(3+1)$-dimensional gKPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} - u_{zz} = 0. \tag{4.13}$$

By using Maple, we have two classes of lump-type solutions below. Moreover, we will prove that there is no lump solution from quadratic functions.

Case I – Sum of two squares: In this case, by Maple, we can have

$$f(x, y, z, t) = (f_1(x, y, z, t))^2 + (f_2(x, y, z, t))^2 + d, \tag{4.14}$$

with

$$\begin{cases} f_1(x, y, z, t) = x + l_1 y + m_1 z + \omega_1 t - \delta_1, \\ f_2(x, y, z, t) = k_2 x + l_2 y + m_2 z + \omega_2 t - \delta_2, \end{cases} \tag{4.15}$$

where $k_2, l_1, l_2, m_1, m_2, \delta_1, \delta_2 \in \mathbb{R}$ are arbitrary, $l_1 m_2 \neq l_2 m_1$ and

$$\omega_1 = \frac{2k_2(l_1 l_2 + m_1 m_2) + (l_1^2 - l_2^2) + (m_1^2 - m_2^2)}{k_2^2 + 1},$$
$$\omega_2 = -\frac{k_2[(l_1^2 - l_2^2) + (m_1^2 - m_2^2)] - 2(l_1 l_2 + m_1 m_2)}{k_2^2 + 1},$$
$$d = \frac{3(k_2^2 + 1)^3}{(k_2 l_1 - l_2)^2 + (k_2 m_1 - m_2)^2}.$$

The corresponding lump-type solutions read

$$u(x, y, z, t) = \frac{4[(1 + k_2^2)d + (k_2^2 - 1)(f_1^2 - f_2^2) - 4k_2 f_1 f_2]}{(f_1^2 + f_2^2 + d)^2},$$

where f_1 and f_2 are defined by (4.15).

Case II – Sum of three squares: In this case, by Maple, we can have

$$f(x, y, z, t) = (f_1(x, y, z, t))^2 + (f_2(x, y, z, t))^2 + (f_3(x, y, z, t))^2 + d, \tag{4.16}$$

with

$$\begin{cases} f_1(x, y, z, t) = x + l_1 y + m_1 z + \omega_1 t - \delta_1, \\ f_2(x, y, z, t) = k_2 x + l_2 y + m_2 z + \omega_2 t - \delta_2, \\ f_3(x, y, z, t) = l_3 y + m_3 z + \omega_3 t - \delta_3, \end{cases} \tag{4.17}$$

where $k_2, l_1, l_2, l_3 \neq 0, m_1, m_3, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$ are arbitrary, and

$$\omega_1 = -\frac{\rho_1}{(k_2^2+1)l_3^2},\ \omega_2 = \frac{\rho_2}{(k_2^2+1)l_3^2},\ \omega_3 = \frac{2\rho_3}{(k_2^2+1)l_3},$$
$$m_2 = \frac{-k_2 l_1 m_3 + k_2 l_3 m_1 + l_2 m_3}{l_3},\ d = \frac{3(k_2^2+1)^3 l_3^2}{(l_3^2+m_3^2)[(k_2 l_1 - l_2)^2 + (k_2^2+1)l_3^2]},$$

with

$$\begin{aligned}
\rho_1 &= k_2^2(l_1^2 m_3^2 - l_3^2 m_1^2) - l_3^2(l_1^2 + m_1^2) + (l_2^2 - 2k_2 l_1 l_2 + l_3^2)(l_3^2 + m_3^2),\\
\rho_2 &= k_2^3(l_1 m_3 - l_3 m_1)^2 - 2k_2^2 l_2 (l_1 m_3 - l_3 m_1) m_3\\
&\quad - k_2(l_1^2 l_3^2 - l_2^2 l_3^2 + 2 l_1 l_3 m_1 m_3 - l_2^2 m_3^2 - l_3^2 m_1^2 + l_3^2 m_3^2 + l_3^4)\\
&\quad + 2 l_2 l_3 (l_1 l_3 + m_1 m_2),\\
\rho_3 &= -k_2^2(l_1 m_3 - l_3 m_1) m_3 + k_2 l_2 (l_3^2 + m_3^2) + l_3(l_1 l_3 + m_1 m_3).
\end{aligned}$$

The corresponding lump-type solutions read

$$u(x,y,z,t) = \frac{4[(1+k_2^2)(f_3^2+d) + (k_2^2-1)(f_1^2 - f_2^2) - 4k_2 f_1 f_2]}{(f_1^2+f_2^2+f_3^2+d)^2},$$

where f_1, f_2 and f_3 are defined by (4.17).

The formula for m_2 in the above example means the corresponding first minor M_{44} is zero, and so, the presented solution is not a lump solution. Generally, when $N \geq 3$, there is no solution to the matrix equation (4.8) with a non-zero first minor $M_{N+1,N+1}$, indeed. Therefore, the above gKP equations in $(N+1)$-dimensions with $N \geq 3$ have no lump solutions generated from quadratic functions. We prove a more general result as follows.

Theorem 4.6. *Let $N \geq 3$. Then there is no symmetric matrix solution $A \in \mathbb{R}^{(N+1)\times(N+1)}$ to the matrix equation (4.8) with* $\mathrm{rank}(A) = N$, *which implies that the $(N+1)$-dimensional gKP equations (4.1) have no lump solution generated from quadratic functions under the transformation $u = 2(\ln f)_{xx}$.*

Proof. Suppose that there is a symmetric matrix $A \in \mathbb{R}^{(N+1)\times(N+1)}$ which solves the equation (4.8) and whose rank is N. Then, since A is symmetric and $\mathrm{rank}(A) = N$, there exists an orthogonal matrix $U \in \mathbb{R}^{(N+1)\times(N+1)}$ such that

$$\hat{A} = U^T A U = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & 0 \end{bmatrix},\ \hat{A}_1 = \mathrm{diag}(\lambda_1, \cdots, \lambda_N),$$

where $\lambda_i \neq 0,\ 1 \leq i \leq N$. Set

$$\hat{B} = U^T B U = \begin{bmatrix} \hat{B}_1 & \hat{B}_2 \\ \hat{B}_3 & \hat{B}_4 \end{bmatrix},\ \hat{B}_1 = (\hat{b}_{ij})_{N\times N} \in \mathbb{R}^{N\times N}.$$

Upon noting that $\tilde{a}$ is an invariant under an orthogonal similarity transformation, it follows from (4.8) that

$$\tilde{a}\hat{A}_1 - \hat{A}_1\hat{B}_1\hat{A}_1 = 0,\ \tilde{a} = \sum_{i,j=1}^{N+1} a_{ij}p_{ij} = \frac{1}{2}\sum_{k=1}^{N}\lambda_k \hat{b}_{kk}.$$

Then, based on this sub-matrix equation, using the same idea in the proof of Theorem 3.6 shows that $N\tilde{a} = 2\tilde{a}$, which leads to $\tilde{a} = 0$ since $N \ge 3$. Further, we have $\hat{B}_1 = 0$, and thus, $\text{rank}(\hat{B}) \le 2$, which is a contradiction to $\text{rank}(\hat{B}) = \text{rank}(B) = N + 1$. Therefore, there is no symmetric matrix solution A to the equation (4.8) with $\text{rank}(A) = N$.

Finally, note that the existence of a non-zero $(N+1, N+1)$ minor of A implies that $\text{rank}(A) \ge N$, and thus, by Theorem 3.6, we have $\text{rank}(A) = N$. Now, it follows that there is no symmetric matrix solution A to the equation (4.8) with a non-zero $(N+1, N+1)$ minor. This means that the gKP equations, defined by (4.1), in $(N+1)$-dimensions with $N \ge 3$ have no lump solution, which are generated from quadratic functions under the transformation $u = 2(\ln f)_{xx}$. The proof is finished. □

4.2. Generalized KP and BKP equations with general 2nd-order derivatives

Let us next consider generalized KP and BKP equations with a general sum of second-order Hirota derivative terms. We will present lump solutions to those two generalized KP and BKP equations.

Example 4.7. We consider the following generalized KP (gKP) equation:

$$K_{gKP1}(u) = (6uu_x + u_{xxx})_x + c_1u_{xx} + 2c_2u_{xy} + 2c_3u_{xt} + c_4u_{yy} + 2c_5u_{yt} + c_6u_{tt} = 0, \quad (4.18)$$

with arbitrary constant coefficients c_i, $1 \le i \le 6$. Under the typical transformation $u = 2(\ln f)_{xx}$, this general equation itself has a Hirota bilinear form:

$$B_{gKP1}(f) = (D_x^4 + c_1D_x^2 + 2c_2D_xD_y + 2c_3D_xD_t + c_4D_y^2 + 2c_5D_yD_t + c_6D_t^2)f \cdot f = 0, \quad (4.19)$$

since we have

$$K_{gKP1}(u) = \left(\frac{B_{gKP1}(f)}{f^2}\right)_{xx}.$$

The equation (4.18) reduces to the (2 + 1)-dimensional KPI and KPII equations in (4.9), upon taking

$$c_1 = c_2 = 0,\ c_3 = \frac{1}{2},\ c_4 = \sigma,\ c_5 = c_6 = 0.$$

To search for quadratic function solutions to the (2 + 1)-dimensional bilinear gKP equation (4.19), we start with

$$f = g_1^2 + g_2^2 + a_9,\ g_1 = a_1x + a_2y + a_3t + a_4,\ g_2 = a_5x + a_6y + a_7t + a_8, \quad (4.20)$$

where a_i, $1 \le i \le 9$, are real parameters to be determined. A direct Maple symbolic computation with this function f generates the set of three constraining equations for the parameters and the coefficients:

$$\begin{cases} a_9 = -\dfrac{3(a_1^2+a_5^2)^3}{\mu_1 c_4 + \mu_2 c_5 + \mu_3 c_6}, \\ c_1 = \dfrac{\nu_{1,1}c_3 + \nu_{1,2}c_4 + \nu_{1,3}c_5 + \nu_{1,4}c_6}{(a_1^2+a_5^2)(a_1a_6 - a_2a_5)}, \\ c_2 = -\dfrac{\nu_{2,1}c_3 + \nu_{2,2}c_4 + \nu_{2,3}c_5 + \nu_{2,4}c_6}{(a_1^2+a_5^2)(a_1a_6 - a_2a_5)}, \end{cases} \tag{4.21}$$

with

$$\begin{cases} \mu_1 = (a_1a_6 - a_2a_5)^2, \\ \mu_2 = 2(a_1a_6 - a_2a_5)(a_1a_7 - a_3a_5), \\ \mu_3 = (a_1a_7 - a_3a_5)^2, \end{cases} \tag{4.22}$$

$$\begin{cases} \nu_{1,1} = 2(a_1^2+a_5^2)(a_2a_7 - a_3a_6), \\ \nu_{1,2} = (a_2^2+a_6^2)(a_1a_6 - a_2a_5), \\ \nu_{1,3} = 2(a_2^2+a_6^2)(a_1a_7 - a_3a_5), \\ \nu_{1,4} = 2a_3a_7(a_1a_2 - a_5a_6) - (a_3^2 - a_7^2)(a_1a_6 + a_2a_5) \end{cases} \tag{4.23}$$

and

$$\begin{cases} \nu_{2,1} = (a_1^2+a_5^2)(a_1a_7 - a_3a_5), \\ \nu_{2,2} = (a_1a_2 + a_5a_6)(a_1a_6 - a_2a_5), \\ \nu_{2,3} = (a_1^2 - a_5^2)(a_2a_7 + a_3a_6) - 2\,a_1a_5(a_2a_3 - a_6a_7), \\ \nu_{2,4} = (a_1a_3 + a_5a_7)(a_1a_7 - a_3a_5), \end{cases} \tag{4.24}$$

where all involved other parameters and coefficients are arbitrary provided that the expressions make sense.

When a determinant condition

$$a_1a_6 - a_2a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \ne 0, \tag{4.25}$$

is satisfied, the above quadratic function f will be positive if and only if $a_9 > 0$, i.e.,

$$\mu_1 c_4 + \mu_2 c_5 + \mu_3 c_6 < 0, \tag{4.26}$$

and the resulting class of positive quadratic function solutions generates lump solutions to the $(2+1)$-dimensional gKPI equation (4.18) through the transformation $u = 2(\ln f)_{xx}$:

$$u = \frac{4(a_1^2 + a_5^2)f - 8(a_1g_1 + a_5g_2)^2}{f^2}, \tag{4.27}$$

where the functions f, g_1, g_2 are determined above.

In this class of lump solutions, all involved eight parameters a_i, $1 \le i \le 8$, and four coefficients c_i, $3 \le i \le 6$, are arbitrary provided that the two conditions, (4.25) and (4.26), are satisfied. The determinant condition (4.25) precisely means that two directions (a_1, a_2) and (a_5, a_6) in the (x, y)-plane are not parallel, which guarantees, together with (4.26), that the resulting solutions in (4.27) are lump solutions.

For the standard KPI and KPII equations in (4.9), we have

$$a_9 = -\frac{3\sigma(a_1^2 + a_5^2)^3}{(a_1a_6 - a_2a_5)^2},$$

and obtain

$$a_3 = -\frac{\sigma(a_1{a_2}^2 - a_1{a_6}^2 + 2\,a_2a_5a_6)}{{a_1}^2 + {a_5}^2},\ a_7 = -\frac{\sigma(2\,a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2)}{{a_1}^2 + {a_5}^2},$$

upon solving the system

$$c_1 = \frac{1}{2}v_{1,1} + \sigma v_{1,2} = 0,\ c_2 = \frac{1}{2}v_{2,1} + \sigma v_{2,2} = 0,$$

for a_3 and a_7. This exactly produces to the lump solution presented in [16] for the KPI equation, but the resulting solution to the KPII equation has pole singularity in the (x, y)-plane at any time, due to $a_9 < 0$.

Example 4.8. We consider the following generalized BKP (gBKP) equation:

$$\begin{aligned} K_{gBKP}(u) = (15u_x^3 + 15u_xu_{3x} + u_{5x})_x + c_1[u_{3x,y} + 3(u_xu_y)_x] \\ + c_2u_{xx} + 2c_3u_{xy} + 2c_4u_{xt} + c_5u_{yy} + 2c_6u_{yt} + c_7u_{tt} = 0, \end{aligned} \tag{4.28}$$

with arbitrary constant coefficients c_i, $1 \le i \le 7$. Under the other typical transformation $u = 2(\ln f)_x$, this general equation itself has a Hirota bilinear form:

$$\begin{aligned} B_{gBKP}(f) = (D_x^6 + c_1D_x^3D_y + c_2D_x^2 + 2c_3D_xD_y \\ + 2c_4D_xD_t + c_5D_y^2 + 2c_6D_yD_t + c_7D_t^2)f \cdot f = 0, \end{aligned} \tag{4.29}$$

since we have

$$K_{gBKP}(u) = \Big(\frac{B_{gBKP}(f)}{f^2}\Big)_x.$$

The equation (4.28) reduces to the $(2+1)$-dimensional BKP equations:

$$(u_t + 15u_x^3 + 15u_xu_{3x} - 15u_xu_y + u_{5x})_x - 5u_{3x,y} + 5\sigma u_{yy} = 0,\ \sigma = \mp 1, \tag{4.30}$$

upon taking

$$c_1 = -5, \; c_4 = \frac{1}{2}, \; c_5 = 5\sigma, \; c_2 = c_3 = c_6 = c_7 = 0. \tag{4.31}$$

To search for quadratic function solutions to the $(2+1)$-dimensional bilinear gBKP equation (4.29), we begin with the same class of quadratic functions defined by (4.20). A similar direct Maple symbolic computation with f leads to the set of three constraining equations for the parameters and the coefficients:

$$\begin{cases} a_9 = -\dfrac{3(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)^2 c_1}{\mu_1 c_5 + \mu_2 c_6 + \mu_3 c_7}, \\ c_2 = \dfrac{\nu_{1,1}c_4 + \nu_{1,2}c_5 + \nu_{1,3}c_6 + \nu_{1,4}c_7}{(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)}, \\ c_3 = -\dfrac{\nu_{2,1}c_4 + \nu_{2,2}c_5 + \nu_{2,3}c_6 + \nu_{2,4}c_7}{(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)}, \end{cases} \tag{4.32}$$

with μ_i, $1 \le i \le 3$, $\nu_{1,i}$, $1 \le i \le 4$, and $\nu_{2,i}$, $1 \le i \le 4$, being defined by (4.22), (4.23) and (4.24). All involved parameters and coefficients are arbitrary provided that the expressions make sense.

When we require a determinant condition in (4.25), the presented quadratic function f will be positive if and only if $a_9 > 0$, which means

$$\frac{(a_1a_2 + a_5a_6)c_1}{\mu_1 c_5 + \mu_2 c_6 + \mu_3 c_7} < 0, \tag{4.33}$$

and the resulting class of positive quadratic function solutions yields lump solutions to the $(2+1)$-dimensional gBKP equation (4.28) through the transformation $u = 2(\ln f)_x$:

$$u = \frac{4(a_1g_1 + a_5g_2)}{f}, \tag{4.34}$$

where the functions f, g_1, g_2 are defined above.

In this presented class of lump solutions, all involved eight parameters a_i, $1 \le i \le 8$, and five coefficients c_1, c_i, $4 \le i \le 7$, are arbitrary provided that the two conditions, (4.25) and (4.33), are satisfied. The determinant condition (4.25) exactly requires that two directions (a_1, a_2) and (a_5, a_6) in the (x, y)-plane are not parallel, which similarly guarantees, together with (4.33), that the presented solutions in (4.34) are lump solutions.

The coefficient constraints (4.31) engender the standard BKP equations in (4.30). In this case, similarly we have

$$a_9 = \frac{3\sigma(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)^2}{(a_1a_6 - a_2a_5)^2},$$

and obtain

$$a_3 = -\frac{5\sigma(a_1{a_2}^2 - a_1{a_6}^2 + 2\,a_2a_5a_6)}{{a_1}^2 + {a_5}^2}, \; a_7 = -\frac{5\sigma(2\,a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2)}{{a_1}^2 + {a_5}^2},$$

upon solving the system

$$c_2 = \frac{1}{2}v_{1,1} + 5\sigma v_{1,2} = 0,\ c_3 = \frac{1}{2}v_{2,1} + 5\sigma v_{2,2} = 0,$$

for a_3 and a_7. This generates lump solutions to the BKP equations in (4.30), when $(a_1a_2 + a_5a_6) < 0$ with minus sign ($\sigma = -1$) or $(a_1a_2 + a_5a_6) > 0$ with plus sign ($\sigma = 1$). This solution phenomenon is pretty different from what we presented for the standard KP equations in the previous example. The solution case with minus sign also covers the lump solution presented in [7]. Actually, if we take

$$a_1 = 1,\ a_2 = 3(\alpha^2 - \beta^2),\ a_5 = 0,\ a_6 = 6\alpha\beta,$$

where α and β are arbitrary but $\beta^2 < \alpha^2$, which leads to

$$a_3 = 45(\alpha^4 - 6\alpha^2\beta^2 + \beta^2),\ a_7 = 180\alpha\beta(\alpha^2 - \beta^2),\ a_9 = \frac{\beta^2 - \alpha^2}{4\alpha^2\beta^2},$$

then the resulting lump solution is exactly the one in [7].

5. Concluding remarks

In this paper, we studied positive quadratic function solutions to Hirota bilinear equations. Sufficient and necessary conditions for the existence of such polynomial solutions were given. In turn, positive quadratic function solutions generate lump or lump-type solutions to nonlinear partial differential equations possessing Hirota bilinear forms. Applications were made for a few generalized KP and BKP equations.

We remark that putting Theorem 3.2 and Theorem 3.7 together proves Hilbert's 17th problem for quadratic functions, but the conjecture is not true for higher-order polynomial functions. Moreover, Theorem 3.2 provides a criterion for the positivity of quadratic functions. It, however, still remains open how to determine the positivity of higher-order multivariate polynomials, which is a further problem of Hilbert's 17th problem. It should be also interesting to look for positive polynomial solutions to generalized bilinear equations [13], which generate exact rational function solutions to novel types of nonlinear differential equations [26,28,29]. The first example of such solutions one can try could be positive quartic function solutions.

It is evident that if A is positive-definite in the quadratic function f defined by (3.11), then $f \to \infty$ when $|x| \to \infty$ in any direction in $\mathbb{R}^M$. This guarantees that $u = 2(\ln f)_{x_1} \to 0$ and $u = 2(\ln f)_{x_1x_1} \to 0$ as $|x| \to \infty$ in any direction in $\mathbb{R}^M$, and so they yield lump solutions, rationally localized solutions in all directions in space and time. If A is positive-semidefinite, then $u = 2(\ln f)_{x_1}$ and $u = 2(\ln f)_{x_1x_1}$ do not go to zero in all directions in $\mathbb{R}^M$ but may go to zero in all directions in a subspace of $\mathbb{R}^M$. Therefore, they usually lead to lump-type solutions, and lump solutions if the subspace is the actual space which the spatial variables belong to. Three of such examples about the generalized KP and BKP equations were just discussed.

Through Theorem 3.2, we can obtain a by-product, which partially answers an open question in [14,15,20]: how to determine a real multivariate polynomial which has only one zero? By Theorem 3.2, a quadratic function f has only one zero at $\alpha \in \mathbb{R}^M$ if and only if

$$f(x) = (x - \alpha)^T A(x - \alpha),\ x \in \mathbb{R}^M,$$

where $A \in \mathbb{R}^{M\times M}$ is positive-definite or negative-definite, since a multivariate polynomial is either non-negative or non-positive if it has one zero. If A above is only positive-semidefinite, then there must exist infinitely many zeros.

Moreover, Theorem 3.2 tells that if a quadratic function f is positive on $\mathbb{R}^M$, i.e., $f(x) > 0$ for all $x \in \mathbb{R}^M$, then there is a positive constant d such that $f(x) \ge d$ for all $x \in \mathbb{R}^M$. But this is not the case for higher order multivariate polynomials. There are counterexamples:

$$f_{mn}(x, y) = x^{2m} + (x^n y^n - 1)^2, \quad m, n \in \mathbb{N},$$

for which $f_{mn}(x, y) > 0$ since $f_{mn}(0, y) = 1$ and $f_{mn}(x, y) \ge x^{2m} > 0$ for $x \ne 0$. It is apparent that $\lim_{k\to\infty} f_{mn}(k^{-1}, k) = 0$. This clearly implies that f_{mn} cannot be bounded from below by any positive constant.

Acknowledgments

The work was supported in part by NSFC under the grants 11371326, 11271008, 11301331 and 11371086, NSF under the grant DMS-1664561, a university grant XKY2016112 from Xuzhou Institute of Technology, Natural Science Fund for Colleges and Universities of Jiangsu Province under the grant 17KJB110020, and the Distinguished Professorships by Shanghai University of Electric Power and Shanghai Second Polytechnic University. The authors would also like to thank X. Gu, X. Lü, H.C. Ma, S. Manukure, M. McAnally, B. Shekhtman, F.D. Wang, X.L. Yang, Y.Q. Yao and Y.J. Zhang for their valuable discussions about lump solutions.

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