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Reduced D-Kaup–Newell soliton hierarchies from sl(2,ℝ) and so(3,ℝ)

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Two reduced D-Kaup–Newell spectral problems from sl(2,ℝ) and so(3,ℝ) are considered, and the corresponding soliton hierarchies are generated by using the zero curvature formulation. The resulting systems are shown to be bi-Hamiltonian and their hereditary recursion operators are explicitly computed.

Keywords: Soliton hierarchies; spectral problems; bi-Hamiltonian systems; Liouville integrable.

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1. Introduction

There exist many interesting soliton hierarchies, which consist of commuting evolution equations. Typical examples of soliton hierarchies, which fit into the zero curvature formulation, include the Korteweg–de Vries hierarchy [1], the Ablowitz–Kaup–Newell–Segur hierarchy [2], the Kaup–Newell hierarchy [3, 4], the Wadati–Konno–Ichikawa hierarchy [5], the Boiti–Pempinelli–Tu hierarchy [6], the Dirac hierarchy [7], and the coupled AKNS–Kaup–Newell hierarchy [8]. Those hierarchies only contain dependent variables less than or equal to three, and soliton hierarchies with more dependent variables are highly complicated, which require considerable efforts in computations [9–11]. Integrable couplings associated with nonsemisimple loop algebras present such examples of soliton hierarchies, which can possess many dependent variables [12–15]. Spectral problems formulated using matrix Lie algebras are a starting point in generating soliton hierarchies (see, e.g. [16–22] for more details).

Very recently, the three-dimensional special orthogonal Lie algebra so(3,ℝ) has been used in constructing soliton hierarchies. The first few examples of soliton hierarchies associated with so(3,ℝ) are the Ablowitz–Kaup–Newell–Segur type soliton hieiarchy [23], the Kaup–Newell type soliton hieiarchy [24], the Wadati–Konno–Ichikawa type soliton hieiarchy [25], and the Heisenberg type soliton hieiarchy [26]. The Lie algebra so(3,ℝ) is the counterpart of the Lie algebra sl(2,ℝ), and they are only the two three-dimensional real Lie algebras with a three-dimensional derived algebra.

In this paper, we would like to consider two reduced D-Kaup–Newell spectral problems possessing two dependent variables associated with sl(2,ℝ) and so(3,ℝ). The two D-Kaup–Newell spectral problems associated with sl(2,ℝ) and so(3,ℝ) are presented respectively in [19, 27], and both of them possess three dependent variables.

A standard procedure for generating soliton hierarchies [29, 28] is stated as follows. Let $\tilde{\mathfrak{g}}$ be the matrix loop algebra associated with a given matrix Lie algebra $\mathfrak{g}$. We first introduce a spatial spectral problem

$$\phi_x = U\phi, \quad U = U(u,\lambda) \in \tilde{\mathfrak{g}}, \tag{1.1}$$

where u denotes a column vector of dependent variables and λ is a spectral parameter. Then, we take a solution of the form

$$W = W(u,\lambda) = \sum_{i\geq 0} W_{0,i}\lambda^{-i}, \quad W_{0,i} \in \mathfrak{g}, \ \ i \geq 0 \tag{1.2}$$

to the stationary zero curvature equation

$$W_x = [U, W]. \tag{1.3}$$

Next, we try to determine the Lax matrices

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^{f(m)}W)_+ + \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \tag{1.4}$$

$(P)_+$ denoting the polynomial part of P in λ and f being an appropriate function from $\mathbb{N}$ to $\mathbb{N}$, to formulate the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi, \quad m \geq 0. \tag{1.5}$$

The modification terms $\Delta_m \in \tilde{\mathfrak{g}}$ should be selected, such that the corresponding zero curvature equations

$$U_{t_m} - V^{[m]}_x + [U, V^{[m]}] = 0, \quad m \geq 0, \tag{1.6}$$

will produce a hierarchy of soliton equations:

$$u_{t_m} = K_m(u), \quad m \geq 0. \tag{1.7}$$

A soliton hierarchy usually possesses Hamiltonian structures

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{1.8}$$

where $\frac{\delta}{\delta u}$ is the variational derivative, J is a Hamiltonian operator, and $\mathcal{H}_m, \ m \geq 0$, are common conserved functionals. Such Hamiltonian structures can often be generated by applying the trace identity, where $\mathfrak{g}$ is semisimple [28, 29]:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(\frac{\partial U}{\partial\lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\operatorname{tr}\left(\frac{\partial U}{\partial u}W\right), \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\operatorname{tr}(W^2)| \tag{1.9}$$

or the variational identity, where $\mathfrak{g}$ is nonsemisimple [30, 31]:

$$\frac{\delta}{\delta u}\int \left\langle\frac{\partial U}{\partial\lambda}, W\right\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle\frac{\partial U}{\partial u}, W\right\rangle, \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{1.10}$$

where $\langle\cdot,\cdot\rangle$ is a nondegenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$.

The three-dimensional real special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{1.11}$$

whose commutators are

$$[e_1, e_2] = 2e_2, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = 2e_3, \tag{1.12}$$

whereas the special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.13}$$

whose commutators are

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \tag{1.14}$$

We will adopt the following two matrix loop algebras associated with $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$:

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \left\{\sum_{i\geq 0} M_i\lambda^{n-i} \mid M_i \in \mathrm{sl}(2,\mathbb{R}),\ i \geq 0 \text{ and } n \in \mathbb{Z}\right\}, \tag{1.15}$$

and

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \left\{\sum_{i\geq 0} M_i\lambda^{n-i} \mid M_i \in \mathrm{so}(3,\mathbb{R}),\ i \geq 0 \text{ and } n \in \mathbb{Z}\right\}. \tag{1.16}$$

Those are spaces of all Laurent series in λ with coefficients in $\mathrm{sl}(2,\mathbb{R})$ or $\mathrm{so}(3,\mathbb{R})$ and with a finite regular part.

The rest of the paper is structured as follows. In Secs. 2 and 3, we will introduce two reduced D-Kaup–Newell spectral problems associated with $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$, and then generate the corresponding soliton hierarchies of bi-Hamiltonian equations, respectively. In Sec. 4, we will give a conclusion and a few remarks.

2. A Reduced D-Kaup–Newell Soliton Hierarchy from sl(2,$\mathbb{R}$)

In this section, we will derive a soliton hierarchy from matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$. We begin with a spectral problem and let e_1, e_2, e_3 be defined by (1.11).

2.1. *Spectral problem*

Let α be an arbitrary real constant. Let us introduce a spectral matrix

$$U = U(u,\lambda) = (\lambda^2+\alpha)e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} \lambda^2+\alpha & \lambda p \\ \lambda q & -\lambda^2-\alpha \end{bmatrix} \tag{2.1}$$

and consider the following isospectral problem

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2+\alpha & \lambda p \\ \lambda q & -\lambda^2-\alpha \end{bmatrix}\phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.2}$$

associated with sl(2,$\mathbb{R}$).

It is known [19] that, the D-Kaup–Newell spectral problem associated with sl(2,$\mathbb{R}$) reads

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2+r & \lambda p \\ \lambda q & -\lambda^2-r \end{bmatrix}\phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

which possess three potentials: p, q and r. The new spectral problem (2.2) is just a reduced case of the above D-Kaup–Newell spectral problem under $r = \alpha$. It is, actually, also a generalization of the standard Kaup–Newell spectral problem [3], which corresponds to the case of $\alpha = 0$. There is another interesting reduction $r = \alpha pq$ of the D-Kaup–Newell spectral problem associated with sl(2,$\mathbb{R}$), which generates an integrable hierarchy (see [19, 32] for details).

2.2. *Soliton hierarchy*

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2,\mathbb{R}) \tag{2.3}$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda(pc - qb), \\ b_x = 2(\lambda^2+\alpha)b - 2\lambda pa, \\ c_x = 2\lambda qa - 2(\lambda^2+\alpha)c. \end{cases} \tag{2.4}$$

We further, assume that

$$a = \sum_{i\geq 0} a_i\lambda^{-2i}, \quad b = \sum_{i\geq 0} b_i\lambda^{-2i-1}, \quad c = \sum_{i\geq 0} c_i\lambda^{-2i-1} \tag{2.5}$$

and take the initial values

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q \tag{2.6}$$

which are required by

$$a_{0,x} = pc_0 - qb_0, \quad b_0 - pa_0 = 0, \quad qa_0 - c_0 = 0. \tag{2.7}$$

Now based on (2.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = 2\alpha b_i + 2b_{i+1} - 2pa_{i+1}, \qquad i \geq 0. \\ c_{i,x} = 2qa_{i+1} - 2\alpha c_i - 2c_{i+1}. \end{cases} \tag{2.8}$$

From this, we can derive the recursion relations

$$\begin{cases} a_{i+1,x} = \alpha q b_i - \alpha p c_i - \frac{q}{2} b_{i,x} - \frac{p}{2} c_{i,x}, \\ b_{i+1} = \frac{1}{2} b_{i,x} - \alpha b_i + pa_{i+1}, \qquad i \geq 0, \\ c_{i+1} = qa_{i+1} - \frac{1}{2} c_{i,x} - \alpha c_i, \end{cases} \tag{2.9}$$

since (2.8) tells

$$\begin{aligned} a_{i+1,x} &= pc_{i+1} - qb_{i+1} \\ &= p\left(qa_{i+1} - \frac{1}{2}c_{i,x} - \alpha c_i\right) - q\left(\frac{1}{2}b_{i,x} - \alpha b_i + pa_{i+1}\right) \\ &= \alpha q b_i - \alpha p c_i - \frac{q}{2} b_{i,x} - \frac{p}{2} c_{i,x}, \quad i \geq 0. \end{aligned}$$

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1 \tag{2.10}$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. We list the first two sets as follows:

$$\begin{aligned} a_1 &= -\frac{1}{2}pq, \\ b_1 &= \frac{1}{2}p_x - \alpha p - \frac{1}{2}p^2 q, \\ c_1 &= -\frac{1}{2}q_x - \alpha q - \frac{1}{2}pq^2; \\ a_2 &= \alpha pq + \frac{1}{4}(pq_x - p_x q) + \frac{3}{8}p^2 q^2, \\ b_2 &= \alpha^2 p - \alpha p_x + \frac{1}{4}p_{xx} - \frac{3}{4}pp_x q + \frac{3}{2}\alpha p^2 q + \frac{3}{8}p^3 q^2, \\ c_2 &= \alpha^2 q + \alpha q_x + \frac{1}{4}q_{xx} + \frac{3}{4}pqq_x + \frac{3}{2}\alpha pq^2 + \frac{3}{8}p^2 q^3. \end{aligned}$$

Based on the recursion relations (2.9), we can have

$$\begin{bmatrix} c_{i+1} \\ b_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} c_i \\ b_i \end{bmatrix}, \quad i \geq 0, \tag{2.11}$$

where

$$\Psi = \begin{bmatrix} -\alpha - \frac{1}{2}\partial - \alpha q\partial^{-1}p - \frac{1}{2}q\partial^{-1}p\partial & \alpha q\partial^{-1}q - \frac{1}{2}q\partial^{-1}q\partial \\ -\alpha p\partial^{-1}p - \frac{1}{2}p\partial^{-1}p\partial & -\alpha + \frac{1}{2}\partial + \alpha p\partial^{-1}q - \frac{1}{2}p\partial^{-1}q\partial \end{bmatrix} \tag{2.12}$$

in which $\partial = \frac{\partial}{\partial x}$. We will see that all vectors $(c_i, b_i)^T$, $i \geq 0$, are gradient, and will generate conserved functionals.

Now for each $m \geq 0$, we introduce

$$\begin{aligned} V^{[m]} &= \lambda(\lambda^{2m+1}W)_+ \\ &= (\lambda^{2m+2}W)_+ - a_{m+1}e_1 \\ &= \sum_{i=0}^{m}[a_i\lambda^{2(m-i)+2}e_1 + b_i\lambda^{2(m-i)+1}e_2 + c_i\lambda^{2(m-i)+1}e_3] \end{aligned} \tag{2.13}$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0 \tag{2.14}$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} - 2\alpha b_m \\ c_{m,x} + 2\alpha c_m \end{bmatrix} = J \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 0, \tag{2.15}$$

where

$$J = \begin{bmatrix} 0 & \partial - 2\alpha \\ \partial + 2\alpha & 0 \end{bmatrix}. \tag{2.16}$$

It is obvious that J is a Hamiltonian operator, since it is skew-adjoint and does not depend on the potentials [33].

2.3. *Bi-Hamiltonian structures*

It is easy to compute that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda & p \\ q & -2\lambda \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}$$

and so, we have

$$\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = 4\lambda a + qb + pc, \quad \operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) = \lambda c, \quad \operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) = \lambda b.$$

By the trace identity (1.9), we get

$$\frac{\delta}{\delta u}\int(4\lambda a + qb + pc)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda c \\ \lambda b \end{bmatrix}.$$

Balancing coefficients of all power of λ in above equality tells

$$\frac{\delta}{\delta u}\int (4a_{m+1}+qb_m+pc_m)dx = (\gamma-2m)\begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 0.$$

Taking $m=1$, we obtain $\gamma=0$, and further, we arrive at

$$\frac{\delta}{\delta u}\int \left(-\frac{4a_{m+1}+qb_m+pc_m}{2m}\right)dx = \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 1. \tag{2.17}$$

Therefore, we get Hamiltonian structures for the reduced D-Kaup–Newell soliton hierarchy (2.15) associated with sl(2,$\mathbb{R}$):

$$u_{t_m} = K_m = J\begin{bmatrix} c_m \\ b_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{2.18}$$

where the Hamiltonian functionals are given by

$$\begin{aligned} \mathcal{H}_0 &= \int pq dx, \\ \mathcal{H}_m &= \int \left(-\frac{4a_{m+1}+qb_m+pc_m}{2m}\right)dx, \quad m \geq 1. \end{aligned} \tag{2.19}$$

Now, we introduce a second Hamiltonian operator

$$M = J\Psi = \Psi^{\dagger}J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{2.20}$$

where all elements can be explicitly worked out:

$$\begin{cases} M_{11} = 2\alpha^2 p\partial^{-1}p + \alpha p\partial^{-1}p\partial - \alpha\partial p\partial^{-1}p - \frac{1}{2}\partial p\partial^{-1}p\partial, \\ M_{12} = \frac{1}{2}\partial^2 - 2\alpha\partial + 2\alpha^2 - 2\alpha^2 p\partial^{-1}q + \alpha\partial p\partial^{-1}q + \alpha p\partial^{-1}q\partial \\ \qquad\quad - \frac{1}{2}\partial p\partial^{-1}q\partial, \\ M_{21} = -\frac{1}{2}\partial^2 - 2\alpha\partial - 2\alpha^2 - 2\alpha^2 q\partial^{-1}p - \alpha\partial q\partial^{-1}p - \alpha q\partial^{-1}p\partial \\ \qquad\quad - \frac{1}{2}\partial q\partial^{-1}p\partial, \\ M_{22} = 2\alpha^2 q\partial^{-1}q + \alpha\partial q\partial^{-1}q - \alpha q\partial^{-1}q\partial - \frac{1}{2}\partial q\partial^{-1}q\partial. \end{cases} \tag{2.21}$$

It is easy to check that M is skew-adjoint. By a long and tedious computation with Maple, we can verify that J and M constitute a Hamiltonian pair [34, 35], i.e. any linear combination N of J and M is skew-symmetry and satisfies the Jacobi identity:

$$\int K^T N'(u)[NS]T dx + \text{cycle}(K,S,T) = 0$$

for all vector fields K, S and T. Thus, the operator

$$\Phi = \Psi^\dagger = \begin{bmatrix} -\alpha + \frac{1}{2}\partial + \alpha p \partial^{-1} q - \frac{1}{2}\partial p \partial^{-1} q & \alpha p \partial^{-1} p - \frac{1}{2}\partial p \partial^{-1} p \\ -\alpha q \partial^{-1} q - \frac{1}{2}\partial q \partial^{-1} q & -\alpha - \frac{1}{2}\partial - \alpha q \partial^{-1} p - \frac{1}{2}\partial q \partial^{-1} p \end{bmatrix} \tag{2.22}$$

is hereditary, that is, it satisfies that

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K$$

for all vector fields K and S (see [36] for definition of hereditary operators). Here, Φ' denotes the Gateaux derivative of Φ as usual.

The above condition for the hereditary operators is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K \Phi \tag{2.23}$$

for any vector field K. Here $L_K\Phi$ is the Lie derivative defined by

$$(L_K\Phi)S := \Phi[K, S] - [K, \Phi S],$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields. Note that, for any autonomous operator $\Psi = \Psi(u, u_x, \ldots)$ is a recursion operator of a given evolution equation $u_t = K(u)$, if and only if Φ satisfies

$$L_K\Phi = \Phi'[K] - [K', \Phi] = 0,$$

S' denoting the Gateaux derivative operator of a vector field S (see [33] for more details).

For the reduced D-Kaup–Newell hierarchy (2.15), it is direct to show that

$$L_{K_0}\Phi = \Phi'[K_0] - [K_0', \Phi] = 0. \tag{2.24}$$

Thus, it follows now that the hierarchy (2.15) is bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1 \tag{2.25}$$

and Φ is a common hereditary recursion operator for the whole hierarchy (2.15). All this implies that the reduced D-Kaup–Newell hierarchy (2.15) is Liouville integrable [26]. We point out that no bi-Hamiltonian structure was presented for the D-Kaup–Newell soliton hierarchy in [19], though the hierarchy was shown to have infinitely many symmetries.

When $m = 0$, we get a linear system:

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x - 2\alpha p \\ q_x + 2\alpha q \end{bmatrix} = J\frac{\delta \mathcal{H}_0}{\delta u}. \tag{2.26}$$

When $m = 1$, we have a nonlinear system of bi-Hamiltonian equations:

$$\begin{aligned} u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\ &= \begin{bmatrix} 2\alpha^2 p + \alpha p^2 q + \frac{1}{2}p_{xx} - 2\alpha p_x - \frac{1}{2}p^2 q_x - pqp_x \\ -2\alpha^2 q - \alpha pq^2 - 2\alpha q_x - \frac{1}{2}q_{xx} - pqq_x - \frac{1}{2}p_x q^2 \end{bmatrix} \\ &= J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}, \end{aligned} \tag{2.27}$$

where $\mathcal{H}_1$ can also be explicitly given by

$$\mathcal{H}_1 = \int \left[-\alpha pq - \frac{1}{4}p^2q^2 - \frac{1}{4}(pq_x - p_x q) \right] dx. \tag{2.28}$$

3. A Reduced D-Kaup–Newell Soliton Hierarchy from so$(3,\mathbb{R})$

In this section, we will construct the second soliton hierarchy based on the three-dimensional orthogonal Lie algebra so$(3,\mathbb{R})$. The basis of so$(3,\mathbb{R})$ consisting of e_1, e_2, e_3 is defined by (1.13).

3.1. *Spectral problem*

Let us introduce the second spectral matrix with an arbitrary real constant α:

$$\begin{aligned} U = U(u,\lambda) &= (\lambda^2 + \alpha)e_1 + \lambda p e_2 + \lambda q e_3 \\ &= \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - \alpha \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + \alpha & \lambda p & 0 \end{bmatrix} \end{aligned} \tag{3.1}$$

and consider the following isospectral problem

$$\begin{aligned} \phi_x = U\phi &= \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - \alpha \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + \alpha & \lambda p & 0 \end{bmatrix} \phi, \\ u &= \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \end{aligned} \tag{3.2}$$

associated with so$(3,\mathbb{R})$.

The D-Kaup–Newell spectral problem associated with so(3,$\mathbb{R}$) was studied recently in [27]:

$$\phi_x = U\phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - r \\ \lambda q & 0 & -\lambda p \\ \lambda^2 + r & \lambda p & 0 \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},$$

which possesses three potentials: p, q and r. The new spectral problem (3.2) is a reduced case of the above D-Kaup–Newell spectral problem under $r = \alpha$. In fact, it is also a generalization of the standard Kaup–Newell spectral problem in [24], which corresponds to the case of $\alpha = 0$. Another interesting reduction $r = \alpha(p^2 + q^2)$ of the D-Kaup–Newell type spectral problem associated with so(3,$\mathbb{R}$), generating an integrable hierarchy, has been proposed and studied in [37].

3.2. *Soliton hierarchy*

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3, \mathbb{R}) \tag{3.3}$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda(pc - qb), \\ b_x = \lambda qa - (\lambda^2 + \alpha)c, \\ c_x = (\lambda^2 + \alpha)b - \lambda pa. \end{cases} \tag{3.4}$$

We further, assume that

$$a = \sum_{i\geq 0} a_i \lambda^{-2i}, \quad b = \sum_{i\geq 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i\geq 0} c_i \lambda^{-2i-1} \tag{3.5}$$

and take the initial values

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q \tag{3.6}$$

which are required by

$$a_{0,x} = pc_0 - qb_0, \quad -c_0 + qa_0 = 0, \quad -pa_0 + b_0 = 0. \tag{3.7}$$

Now based on (3.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = qa_{i+1} - c_{i+1} - \alpha c_i, \quad i \geq 0. \\ c_{i,x} = b_{i+1} + \alpha b_i - pa_{i+1}. \end{cases} \tag{3.8}$$

From this, we can derive the recursion relations

$$\begin{cases} a_{i+1,x} = -pb_{i,x} + \alpha qb_i - qc_{i,x} - \alpha pc_i, \\ b_{i+1} = c_{i,x} - \alpha b_i + pa_{i+1}, \qquad i \geq 0, \\ c_{i+1} = -b_{i,x} - \alpha c_i + qa_{i+1}, \end{cases} \tag{3.9}$$

since (3.8) tells

$$\begin{aligned} a_{i+1,x} &= pc_{i+1} - qb_{i+1} \\ &= p(-b_{i,x} - \alpha c_i + qa_{i+1}) - q(c_{i,x} - \alpha b_i + pa_{i+1}) \\ &= -pb_{i,x} + \alpha qb_i - qc_{i,x} - \alpha pc_i, \quad i \geq 0. \end{aligned}$$

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1 \tag{3.10}$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. We list the first two sets as follows:

$$\begin{aligned}
a_1 &= -\frac{1}{2}(p^2+q^2),\\
b_1 &= q_x - \alpha p - \frac{1}{2}p(p^2+q^2),\\
c_1 &= -p_x - \alpha q - \frac{1}{2}q(p^2+q^2);\\
a_2 &= \alpha(p^2+q^2) + p_x q - p q_x + \frac{3}{8}(p^2+q^2)^2,\\
b_2 &= -p_{xx} - 2\alpha q_x + \alpha^2 p + \frac{3}{2}(\alpha p - q_x)(p^2+q^2) + \frac{3}{8}p(p^2+q^2)^2,\\
c_2 &= -q_{xx} + 2\alpha p_x + \alpha^2 q + \frac{3}{2}(\alpha q + p_x)(p^2+q^2) + \frac{3}{8}q(p^2+q^2)^2.
\end{aligned}$$

It follows directly from the recursion relations (3.9), that, we have

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \quad i \geq 0, \tag{3.11}$$

where

$$\Psi = \begin{bmatrix} -\alpha - p\partial^{-1}p\partial + \alpha p\partial^{-1}q & \partial - \alpha p\partial^{-1}p - p\partial^{-1}q\partial \\ -\partial - q\partial^{-1}p\partial + \alpha q\partial^{-1}q & -\alpha - \alpha q\partial^{-1}p - q\partial^{-1}q\partial \end{bmatrix}. \tag{3.12}$$

We will see that all vectors $(b_i, c_i)^T$, $i \geq 0$, above are gradient, and will generate conserved functionals.

Now for each $m \geq 0$, we introduce

$$\begin{aligned}
V^{[m]} &= \lambda(\lambda^{2m+1}W)_+\\
&= (\lambda^{2m+2}W)_+ - a_{m+1}e_1\\
&= \sum_{i=0}^{m}[a_i\lambda^{2(m-i)+2}e_1 + b_i\lambda^{2(m-i)+1}e_2 + c_i\lambda^{2(m-i)+1}e_3]
\end{aligned} \tag{3.13}$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0 \tag{3.14}$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} + \alpha c_m \\ c_{m,x} - \alpha b_m \end{bmatrix} = J \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 0, \tag{3.15}$$

with J being defined by

$$J = \begin{bmatrix} \partial & \alpha \\ -\alpha & \partial \end{bmatrix}. \tag{3.16}$$

It is direct to check that J is Hamiltonian.

3.3. *Bi-Hamiltonian structures*

It is easy to see that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q & -2\lambda \\ q & 0 & -p \\ 2\lambda & p & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so, we obtain

$$\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -4\lambda a - 2pb - 2qc, \quad \operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) = -2\lambda b, \quad \operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) = -2\lambda c.$$

By the trace identity (1.9), we get

$$\frac{\delta}{\delta u}\int (2\lambda a + pb + qc)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda b \\ \lambda c \end{bmatrix}.$$

Balancing coefficients of all power of λ in above equality tells

$$\frac{\delta}{\delta u}\int (2a_{m+1} + pb_m + qc_m)dx = (\gamma - 2m)\begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 0.$$

Taking $m = 1$, we obtain $\gamma = 0$, and further, we have

$$\frac{\delta}{\delta u}\int \left(-\frac{2a_{m+1} + pb_m + qc_m}{2m}\right) dx = \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 1. \tag{3.17}$$

This way, we obtain Hamiltonian structures for the reduced D-Kaup–Newell hierarchy (3.15) associated with so(3,$\mathbb{R}$):

$$u_{t_m} = K_m = J\begin{bmatrix} b_m \\ c_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{3.18}$$

where the Hamiltonian functionals are given by

$$\begin{aligned} \mathcal{H}_0 &= \int \frac{1}{2}(p^2 + q^2)dx, \\ \mathcal{H}_m &= \int \left(-\frac{2a_{m+1} + pb_m + qc_m}{2m}\right) dx, \quad m \geq 1. \end{aligned} \tag{3.19}$$

Introduce a second Hamiltonian operator

$$M = J\Psi = \Psi^{\dagger}J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{3.20}$$

where all elements can be explicitly worked out:

$$\begin{cases} M_{11} = -2\alpha\partial + \alpha^2 q\partial^{-1}q + \alpha\partial p\partial^{-1}q - \alpha q\partial^{-1}p\partial - \partial p\partial^{-1}p\partial, \\ M_{12} = -\alpha^2 + \partial^2 - \alpha^2 q\partial^{-1}p - \alpha q\partial^{-1}q\partial - \alpha\partial p\partial^{-1}p - \partial p\partial^{-1}q\partial, \\ M_{21} = \alpha^2 - \partial^2 - \alpha^2 p\partial^{-1}q + \alpha\partial q\partial^{-1}q + \alpha p\partial^{-1}p\partial - \partial q\partial^{-1}p\partial, \\ M_{22} = -2\alpha\partial + \alpha^2 p\partial^{-1}p - \alpha\partial q\partial^{-1}p + \alpha p\partial^{-1}q\partial - \partial q\partial^{-1}q\partial. \end{cases} \tag{3.21}$$

A direct computation by Maple can show that J and M constitute a Hamiltonian pair. Thus once proving $L_{K_0}\Phi = 0$, we can show by the same argument as in the previous section that the operator

$$\Phi = \Psi^\dagger = \begin{bmatrix} -\alpha - \partial p\partial^{-1}p - \alpha q\partial^{-1}p & \partial - \partial p\partial^{-1}q - \alpha q\partial^{-1}q \\ -\partial + \alpha p\partial^{-1}p - \partial q\partial^{-1}p & -\alpha + \alpha p\partial^{-1}q - \partial q\partial^{-1}q \end{bmatrix} \tag{3.22}$$

is a common hereditary recursion operator for the whole hierarchy (3.15), and the hierarchy (3.15) is bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta\mathcal{H}_m}{\delta u} = M\frac{\delta\mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1. \tag{3.23}$$

All this shows that the reduced D-Kaup–Newell hierarchy (3.15) associated with so(3,$\mathbb{R}$) is Liouville integrable [26].

When $m = 0$, we get a linear system

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x + \alpha q \\ -\alpha p + q_x \end{bmatrix} = J\frac{\delta\mathcal{H}_0}{\delta u}. \tag{3.24}$$

When $m = 1$, we obtain a nonlinear system of bi-Hamiltonian equations:

$$\begin{aligned} u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\ &= \begin{bmatrix} -\alpha^2 q - 2\alpha p_x + q_{xx} - \frac{1}{2}(\alpha q + p_x)(p^2+q^2) - p(pp_x + qq_x) \\ \alpha^2 p - 2\alpha q_x - p_{xx} + \frac{1}{2}(\alpha p + q_x)(p^2+q^2) - q(pp_x + qq_x) \end{bmatrix} \\ &= J\frac{\delta\mathcal{H}_1}{\delta u} = M\frac{\delta\mathcal{H}_0}{\delta u}, \end{aligned} \tag{3.25}$$

where $\mathcal{H}_1$ can also be explicitly given by

$$\mathcal{H}_1 = \int \left[\frac{1}{2}(-\alpha p^2 - \alpha q^2 - p_x q + pq_x) - \frac{1}{8}(p^2+q^2)^2\right] dx. \tag{3.26}$$

We point out that, no bi-Hamiltonian structure presented for the second non-reduced D-Kaup–Newell soliton hierarchy associated with so(3,$\mathbb{R}$) [27]. Only one of the two differential operators in the recursive structure is Hamiltonian, and the other is quasi-Hamiltonian for the non-reduced D-Kaup–Newell soliton hierarchy in [27].

4. Concluding Remarks

In this paper, we have introduced two reduced D-Kaup–Newell spectral problems associated with sl(2,ℝ) and so(3,ℝ) and generated their hierarchies of bi-Hamiltonian equations via the zero curvature formulation. The Liouville integrability of the resulting soliton hierarchies has been shown upon establishing bi-Hamiltonian structures through the trace identity.

Compared to the D-Kaup–Newell soliton hierarchies [19, 27], our soliton hierarchies present different features such as bi-Hamiltonian structures, though our spectral problems involve less potentials. Moreover, the newly presented Hamiltonian pairs display different recursion operator structures from the known Kaup–Newell recursion operators associated with sl(2,ℝ) and so(3,ℝ) [3, 24].

Different choices of spectral matrices from matrix loop algebras bring us various soliton hierarchies with hereditary recursion operators, and any study on integrable couplings will help in classifying multicomponent integrable systems associated with nonsemisimple Lie algebras.

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