Reduced D-Kaup–Newell soliton hierarchies from $\text{sl}(2,\mathbb{R})$ and $\text{so}(3,\mathbb{R})$

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Received 31 December 2014
Accepted 16 May 2016
Published 7 July 2016

Two reduced D-Kaup–Newell spectral problems from $\text{sl}(2,\mathbb{R})$ and $\text{so}(3,\mathbb{R})$ are considered, and the corresponding soliton hierarchies are generated by using the zero curvature formulation. The resulting systems are shown to be bi-Hamiltonian and their hereditary recursion operators are explicitly computed.

Keywords: Soliton hierarchies; spectral problems; bi-Hamiltonian systems; Liouville integrable.

Mathematics Subject Classification 2010: 37K05, 37K10, 35Q53, 35C08, 35P30
PACS: 02.30.Ik

1. Introduction

There exist many interesting soliton hierarchies, which consist of commuting evolution equations. Typical examples of soliton hierarchies, which fit into the zero curvature formulation, include the Korteweg–de Vries hierarchy [1], the Ablowitz–Kaup–Newell–Segur hierarchy [2], the Kaup–Newell hierarchy [3, 4], the Wadati–Konno–Ichikawa hierarchy [5], the Boiti–Pempinelli–Tu hierarchy [6], the Dirac hierarchy [7], and the coupled AKNS–Kaup–Newell hierarchy [8]. Those hierarchies only contain dependent variables less than or equal to three, and soliton hierarchies with more dependent variables are highly complicated, which require considerable efforts in computations [9–11]. Integrable couplings associated with nonsemisimple loop algebras present such examples of soliton hierarchies, which can possess many dependent variables [12–15]. Spectral problems formulated using matrix Lie algebras are a starting point in generating soliton hierarchies (see, e.g. [16–22] for more details).
Very recently, the three-dimensional special orthogonal Lie algebra so(3, \mathbb{R}) has been used in constructing soliton hierarchies. The first few examples of soliton hierarchies associated with so(3, \mathbb{R}) are the Ablowitz–Kaup–Newell–Segur type soliton hierarchy [23], the Kaup–Newell type soliton hierarchy [24], the Wadati–Konno–Ichikawa type soliton hierarchy [25], and the Heisenberg type soliton hierarchy [26]. The Lie algebra so(3, \mathbb{R}) is the counterpart of the Lie algebra sl(2, \mathbb{R}), and they are only the two three-dimensional real Lie algebras with a three-dimensional derived algebra.

In this paper, we would like to consider two reduced D-Kaup–Newell spectral problems possessing two dependent variables associated with sl(2, \mathbb{R}) and so(3, \mathbb{R}). The two D-Kaup–Newell spectral problems associated with sl(2, \mathbb{R}) and so(3, \mathbb{R}) are presented respectively in [19, 27], and both of them possess three dependent variables.

A standard procedure for generating soliton hierarchies [29, 28] is stated as follows. Let \tilde{g} be the matrix loop algebra associated with a given matrix Lie algebra g. We first introduce a spatial spectral problem

\[ \phi_x = U\phi, \quad U = U(u, \lambda) \in \tilde{g}, \]  

where \( u \) denotes a column vector of dependent variables and \( \lambda \) is a spectral parameter. Then, we take a solution of the form

\[ W = W(u, \lambda) = \sum_{i \geq 0} W_{0,i} \lambda^{-i}, \quad W_{0,i} \in g, \quad i \geq 0 \]  

to the stationary zero curvature equation

\[ W_x = [U, W]. \]  

Next, we try to determine the Lax matrices

\[ V^{[m]} = V^{[m]}(u, \lambda) = (\lambda f^{(m)} W)_+ + \Delta_m \in \tilde{g}, \quad m \geq 0, \]  

\((P)_+\) denoting the polynomial part of \( P \) in \( \lambda \) and \( f \) being an appropriate function from \( \mathbb{N} \) to \( \mathbb{N} \), to formulate the temporal spectral problems

\[ \phi_{tm} = V^{[m]} \phi, \quad m \geq 0. \]  

The modification terms \( \Delta_m \in \tilde{g} \) should be selected, such that the corresponding zero curvature equations

\[ U_{tm} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \]  

will produce a hierarchy of soliton equations:

\[ u_{tm} = K_m(u), \quad m \geq 0. \]  

A soliton hierarchy usually possesses Hamiltonian structures

\[ u_{tm} = K_m(u) = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \]  

where \( \frac{\delta}{\delta u} \) is the variational derivative, \( J \) is a Hamiltonian operator, and \( \mathcal{H}_m \), \( m \geq 0 \), are common conserved functionals. Such Hamiltonian structures can often be
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generated by applying the trace identity, where $\mathfrak{g}$ is semisimple [28, 29]:

$$\frac{\delta}{\delta u} \int tr \left( \frac{\partial U}{\partial \lambda} W \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left( \frac{\partial U}{\partial u} W \right), \quad \gamma = -\frac{\lambda}{2 d \lambda} \ln |tr(W^2)|$$  \hspace{1cm} (1.9)

or the variational identity, where $\mathfrak{g}$ is nonsemisimple [30, 31]:

$$\frac{\delta}{\delta u} \int \left( \frac{\partial U}{\partial \lambda} , W \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left( \frac{\partial U}{\partial u} , W \right), \quad \gamma = -\frac{\lambda}{2 d \lambda} \ln |\langle W, W \rangle|$$ \hspace{1cm} (1.10)

where $\langle \cdot, \cdot \rangle$ is a nondegenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$.

The three-dimensional real special linear Lie algebra $sl(2, \mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$ \hspace{1cm} (1.11)

whose commutators are

$$[e_1, e_2] = 2e_2, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = 2e_3$$ \hspace{1cm} (1.12)

whereas the special orthogonal Lie algebra $so(3, \mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ \hspace{1cm} (1.13)

whose commutators are

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$ \hspace{1cm} (1.14)

We will adopt the following two matrix loop algebras associated with $sl(2, \mathbb{R})$ and $so(3, \mathbb{R})$:

$$\tilde{sl}(2, \mathbb{R}) = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} \mid M_i \in sl(2, \mathbb{R}), \ i \geq 0 \text{ and } n \in \mathbb{Z} \right\}$$ \hspace{1cm} (1.15)

and

$$\tilde{so}(3, \mathbb{R}) = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} \mid M_i \in so(3, \mathbb{R}), \ i \geq 0 \text{ and } n \in \mathbb{Z} \right\}$$ \hspace{1cm} (1.16)

Those are spaces of all Laurent series in $\lambda$ with coefficients in $sl(2, \mathbb{R})$ or $so(3, \mathbb{R})$ and with a finite regular part.

The rest of the paper is structured as follows. In Secs. 2 and 3, we will introduce two reduced D-Kaup–Newell spectral problems associated with $sl(2, \mathbb{R})$ and $so(3, \mathbb{R})$, and then generate the corresponding soliton hierarchies of bi-Hamiltonian equations, respectively. In Sec. 4, we will give a conclusion and a few remarks.
2. A Reduced D-Kaup–Newell Soliton Hierarchy from $\mathfrak{sl}(2,\mathbb{R})$

In this section, we will derive a soliton hierarchy from matrix loop algebra $\tilde{\mathfrak{sl}}(2,\mathbb{R})$. We begin with a spectral problem and let $e_1, e_2, e_3$ be defined by (1.11).

2.1. Spectral problem

Let $\alpha$ be an arbitrary real constant. Let us introduce a spectral matrix

$$U(u, \lambda) = \begin{bmatrix} \lambda^2 + \alpha & \lambda p \\ \lambda q & -\lambda^2 - \alpha \end{bmatrix}$$

and consider the following isospectral problem

$$\phi_x = U \phi = \begin{bmatrix} \lambda^2 + \alpha & \lambda p \\ \lambda q & -\lambda^2 - \alpha \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

associated with $\mathfrak{sl}(2,\mathbb{R})$.

It is known [19] that, the D-Kaup–Newell spectral problem associated with $\mathfrak{sl}(2,\mathbb{R})$ reads

$$\phi_x = U \phi = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

which possess three potentials: $p, q$ and $r$. The new spectral problem (2.2) is just a reduced case of the above D-Kaup–Newell spectral problem under $r = \alpha$. It is, actually, also a generalization of the standard Kaup–Newell spectral problem [3], which corresponds to the case of $\alpha = 0$. There is another interesting reduction $r = \alpha pq$ of the D-Kaup–Newell spectral problem associated with $\mathfrak{sl}(2,\mathbb{R})$, which generates an integrable hierarchy (see [19, 32] for details).

2.2. Soliton hierarchy

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \tilde{\mathfrak{sl}}(2,\mathbb{R})$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda(p c - q b), \\ b_x = 2(\lambda^2 + \alpha)b - 2\lambda p a, \\ c_x = 2\lambda qa - 2(\lambda^2 + \alpha)c. \end{cases}$$

We further, assume that

$$a = \sum_{i \geq 0} a_i \lambda^{-2i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i \geq 0} c_i \lambda^{-2i-1}$$
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and take the initial values

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q$$

(2.6)

which are required by

$$a_{0,x} = pc_0 - qb_0, \quad b_0 - pa_0 = 0, \quad qa_0 - c_0 = 0.$$  

(2.7)

Now based on (2.4), we have

$$\begin{align*}
\begin{cases}
a_{i,x} &= pc_i - qb_i, \\
b_{i,x} &= 2\alpha b_i + 2b_{i+1} - 2pa_{i+1}, \\
c_{i,x} &= 2qa_{i+1} - 2\alpha c_i - 2c_{i+1}.
\end{cases}
\end{align*}$$  

(2.8)

From this, we can derive the recursion relations

$$\begin{align*}
\begin{cases}
a_{i+1,x} &= \alpha qb_i - \alpha pc_i - \frac{q}{2}b_{i,x} - \frac{p}{2}c_{i,x}, \\
b_{i+1} &= \frac{1}{2}b_{i,x} - \alpha b_i + pa_{i+1}, \\
c_{i+1} &= qa_{i+1} - \frac{1}{2}c_{i,x} - \alpha c_i,
\end{cases}
\end{align*}$$  

(2.9)

since (2.8) tells

$$a_{i+1,x} = pc_{i+1} - qb_{i+1}$$

$$= p \left(qa_{i+1} - \frac{1}{2}c_{i,x} - \alpha c_i\right) - q \left(\frac{1}{2}b_{i,x} - \alpha b_i + pa_{i+1}\right)$$

$$= \alpha qb_i - \alpha pc_i - \frac{q}{2}b_{i,x} - \frac{p}{2}c_{i,x}, \quad i \geq 0.$$  

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1$$  

(2.10)

to determine the sequence of \{\(a_i, b_i, c_i\)|\(i \geq 1\)} uniquely. We list the first two sets as follows:

$$a_1 = -\frac{1}{2}pq,$$
$$b_1 = \frac{1}{2}px - \alpha p - \frac{1}{2}p^2q,$$
$$c_1 = -\frac{1}{2}qx - \alpha q - \frac{1}{2}pq^2;$$
$$a_2 = \alpha pq + \frac{1}{4}(pq_x - px q) + \frac{3}{8}p^2q^2,$$
$$b_2 = \alpha^2 p - \alpha px + \frac{1}{4}px x - \frac{3}{4}pp_x q + \frac{3}{2}\alpha p^2 q + \frac{3}{8}p^3 q^2,$$
$$c_2 = \alpha^2 q + \alpha qx + \frac{1}{4}qx x + \frac{3}{4}pq x q + \frac{3}{2}\alpha pq^2 + \frac{3}{8}p^2 q^3.$$  

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Based on the recursion relations (2.9), we can have
\[
\begin{bmatrix}
  c_{i+1} \\
  b_{i+1}
\end{bmatrix} = \Psi \begin{bmatrix}
  c_i \\
  b_i
\end{bmatrix}, \quad i \geq 0,
\] (2.11)
where
\[
\Psi = \begin{bmatrix}
  -\alpha - \frac{1}{2} \partial - \alpha q \partial^{-1} p - \frac{1}{2} q \partial^{-1} p \partial \\
  -\alpha p \partial^{-1} q - \frac{1}{2} p \partial^{-1} p \partial
\end{bmatrix}
\] (2.12)
in which \( \partial = \frac{\partial}{\partial x} \). We will see that all vectors \((c_i, b_i)^T, i \geq 0\), are gradient, and will generate conserved functionals.

Now for each \( m \geq 0 \), we introduce
\[
V^{[m]} = \lambda (\lambda^{2m+1} W^+) + (\lambda^{2m+2} W^+) - a_{m+1} e_1
= \sum_{i=0}^{m} (a_i \lambda^{2(m-i)+2} e_1 + b_i \lambda^{2(m-i)+1} e_2 + c_i \lambda^{2(m-i)+1} e_3)
\] (2.13)
and the corresponding zero curvature equations
\[
U_t - V^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0
\] (2.14)
engender a hierarchy of solution equations
\[
u_t = K_m = \begin{bmatrix}
  b_{m,x} - 2\alpha b_m \\
  c_{m,x} + 2\alpha c_m
\end{bmatrix} = J \begin{bmatrix}
  c_m \\
  b_m
\end{bmatrix}, \quad m \geq 0,
\] (2.15)
where
\[
J = \begin{bmatrix}
  0 & \partial - 2\alpha \\
  \partial + 2\alpha & 0
\end{bmatrix}.
\] (2.16)
It is obvious that \( J \) is a Hamiltonian operator, since it is skew-adjoint and does not depend on the potentials [33].

### 2.3. Bi-Hamiltonian structures

It is easy to compute that
\[
\frac{\partial U}{\partial \lambda} = \begin{bmatrix}
  2\lambda & p \\
  q & -2\lambda
\end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix}
  0 & \lambda \\
  0 & 0
\end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix}
  0 & 0
\end{bmatrix}
\]
and so, we have
\[
\text{tr} \left( W \frac{\partial U}{\partial \lambda} \right) = 4\lambda a + qb + pc, \quad \text{tr} \left( W \frac{\partial U}{\partial p} \right) = \lambda c, \quad \text{tr} \left( W \frac{\partial U}{\partial q} \right) = \lambda b.
\]
By the trace identity (1.9), we get
\[
\frac{\delta}{\delta u} \int (4\lambda a + qb + pc) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda \begin{bmatrix}
  \lambda c \\
  \lambda b
\end{bmatrix}.
\]
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Balancing coefficients of all power of \( \lambda \) in above equality tells

\[
\frac{\delta}{\delta u} \left( 4a_{m+1} + qb_m + pc_m \right) = \left( \gamma - 2m \right) \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 0.
\]

Taking \( m = 1 \), we obtain \( \gamma = 0 \), and further, we arrive at

\[
\frac{\delta}{\delta u} \left( - \frac{4a_{m+1} + qb_m + pc_m}{2m} \right) = \left( \gamma - 2m \right) \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \quad m \geq 1.
\] (2.17)

Therefore, we get Hamiltonian structures for the reduced D-Kaup–Newell soliton hierarchy (2.15) associated with sl(2, R):

\[
u_t = K_m = J \begin{bmatrix} c_m \\ b_m \end{bmatrix} = J \frac{\delta \epsilon_m}{\delta u}, \quad m \geq 0,
\] (2.18)

where the Hamiltonian functionals are given by

\[
\begin{align*}
\mathcal{K}_0 &= \int pqdx, \\
\mathcal{K}_m &= \int \left( - \frac{4a_{m+1} + qb_m + pc_m}{2m} \right) dx, \quad m \geq 1.
\end{align*}
\] (2.19)

Now, we introduce a second Hamiltonian operator

\[
M = J \Psi = \Psi J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},
\] (2.20)

where all elements can be explicitly worked out:

\[
\begin{align*}
M_{11} &= 2\alpha^2 \partial^{-1} p + \alpha p \partial^{-1} p - \alpha \partial p \partial^{-1} p - \frac{1}{2} \partial p \partial^{-1} p, \\
M_{12} &= \frac{1}{2} \partial^2 - 2\alpha \partial + 2\alpha^2 - 2\alpha^2 \partial^{-1} q + \alpha \partial p \partial^{-1} q + \alpha p \partial^{-1} q \partial \\
&\quad - \frac{1}{2} \partial p \partial^{-1} q \partial, \\
M_{21} &= -\frac{1}{2} \partial^2 - 2\alpha \partial - 2\alpha^2 - 2\alpha^2 \partial^{-1} p - \alpha \partial q \partial^{-1} p - \alpha q \partial^{-1} p \partial \\
&\quad - \frac{1}{2} \partial q \partial^{-1} p \partial, \\
M_{22} &= 2\alpha^2 \partial^{-1} q + \alpha \partial q \partial^{-1} q - \alpha q \partial^{-1} q \partial - \frac{1}{2} \partial q \partial^{-1} q \partial.
\end{align*}
\] (2.21)

It is easy to check that \( M \) is skew-adjoint. By a long and tedious computation with Maple, we can verify that \( J \) and \( M \) constitute a Hamiltonian pair [34, 35], i.e. any linear combination \( N \) of \( J \) and \( M \) is skew-symmetry and satisfies the Jacobi identity:

\[
\int K^T N'(u)[NS]Tdx + \text{cycle}(K, S, T) = 0.
\]
for all vector fields $K, S$ and $T$. Thus, the operator

$$
\Phi = \Psi^\dagger = \begin{bmatrix}
-\alpha + \frac{1}{2} \partial + \alpha p \partial^{-1} q - \frac{1}{2} \partial q \partial^{-1} p & \alpha p \partial^{-1} p - \frac{1}{2} \partial p \partial^{-1} p \\
-\alpha q \partial^{-1} q - \frac{1}{2} \partial q \partial^{-1} q & -\alpha - \frac{1}{2} \partial - \alpha q \partial^{-1} p - \frac{1}{2} \partial q \partial^{-1} p
\end{bmatrix}
$$

(2.22)

is hereditary, that is, it satisfies that

$$
\Phi(\Phi_K S - \Phi \Phi'_K) = \Phi(\Phi_S K - \Phi \Phi'_S) K
$$

for all vector fields $K$ and $S$ (see [36] for definition of hereditary operators). Here, $\Phi'$ denotes the Gateaux derivative of $\Phi$ as usual.

The above condition for the hereditary operators is equivalent to

$$
L_{\Phi K} \Phi = \Phi L_K \Phi
$$

(2.23)

for any vector field $K$. Here $L_K \Phi$ is the Lie derivative defined by

$$(L_K \Phi) S := \Phi[K, S] - [K, \Phi S],$$

where $[,]$ is the Lie bracket of vector fields. Note that, for any autonomous operator $\Psi = \Psi(u, u_x, \ldots)$ is a recursion operator of a given evolution equation $u_t = K(u)$, if and only if $\Phi$ satisfies

$$
L_K \Phi = \Phi'[K] - [K', \Phi] = 0,
$$

$S'$ denoting the Gateaux derivative operator of a vector field $S$ (see [33] for more details).

For the reduced D-Kaup–Newell hierarchy (2.15), it is direct to show that

$$
L_{K_0} \Phi = \Phi'[K] = [K', \Phi] = 0.
$$

(2.24)

Thus, it follows now that the hierarchy (2.15) is bi-Hamiltonian:

$$
u_{t_m} = K_m = J \frac{\delta H_m}{\delta u} = M \frac{\delta H_{m-1}}{\delta u}, \quad m \geq 1
$$

(2.25)

and $\Phi$ is a common hereditary recursion operator for the whole hierarchy (2.15). All this implies that the reduced D-Kaup–Newell hierarchy (2.15) is Liouville integrable [26]. We point out that no bi-Hamiltonian structure was presented for the D-Kaup–Newell soliton hierarchy in [19], though the hierarchy was shown to have infinitely many symmetries.

When $m = 0$, we get a linear system:

$$
u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p - 2 \alpha p \\ p + 2 \alpha q \end{bmatrix} = J \frac{\delta H_0}{\delta u}.
$$

(2.26)
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When \(m = 1\), we have a nonlinear system of bi-Hamiltonian equations:

\[
\begin{align*}
    u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\
    &= \begin{bmatrix} 2\alpha^2 p + \alpha p^2 q + \frac{1}{2} p_{xx} - 2\alpha p_x - \frac{1}{2} p^2 q_x - pq_x \\
    &\quad -2\alpha^2 q - \alpha p q^2 - 2\alpha q_x - \frac{1}{2} q_{xx} - p q_x - \frac{1}{2} p_x q^2 \\
    &= \frac{\delta \mathcal{H}_1}{\delta u} = M \frac{\delta \mathcal{H}_0}{\delta u},
\end{align*}
\]

(2.27)

where \(\mathcal{H}_1\) can also be explicitly given by

\[
\mathcal{H}_1 = \int \left[ -\alpha p q - \frac{1}{4} p^2 q^2 - \frac{1}{4} (pq_x - p_x q) \right] dx.
\]

(2.28)

3. A Reduced D-Kaup–Newell Soliton Hierarchy from \(so(3, \mathbb{R})\)

In this section, we will construct the second soliton hierarchy based on the three-dimensional orthogonal Lie algebra \(so(3, \mathbb{R})\). The basis of \(so(3, \mathbb{R})\) consisting of \(e_1, e_2, e_3\) is defined by (1.13).

3.1. Spectral problem

Let us introduce the second spectral matrix with an arbitrary real constant \(\alpha\):

\[
U = U(u, \lambda) = (\lambda^2 + \alpha) e_1 + \lambda p e_2 + \lambda q e_3
\]

(3.1)

\[
= \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - \alpha \\
\lambda q & 0 & -\lambda p \\
\lambda^2 + \alpha & \lambda p & 0 \end{bmatrix}
\]

and consider the following isospectral problem

\[
\phi_x = U \phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - \alpha \\
\lambda q & 0 & -\lambda p \\
\lambda^2 + \alpha & \lambda p & 0 \end{bmatrix} \phi,
\]

(3.2)

\[
u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},
\]

associated with \(so(3, \mathbb{R})\).

The D-Kaup–Newell spectral problem associated with \(so(3, \mathbb{R})\) was studied recently in [27]:

\[
\phi_x = U \phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 - r \\
\lambda q & 0 & -\lambda p \\
\lambda^2 + r & \lambda p & 0 \end{bmatrix} \phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},
\]

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which possesses three potentials: \( p, q \) and \( r \). The new spectral problem (3.2) is a reduced case of the above D-Kaup–Newell spectral problem under \( r = \alpha \). In fact, it is also a generalization of the standard Kaup–Newell spectral problem in [24], which corresponds to the case of \( \alpha = 0 \). Another interesting reduction \( r = \alpha(p^2 + q^2) \) of the D-Kaup–Newell type spectral problem associated with \( \text{so}(3, \mathbb{R}) \), generating an integrable hierarchy, has been proposed and studied in [37].

3.2. Soliton hierarchy

Define

\[
W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} \in \text{so}(3, \mathbb{R})
\] (3.3)

and then, the stationary zero curvature equation \( W_x = [U, W] \) becomes

\[
\begin{align*}
& a_x = \lambda(pc - qb), \\
& b_x = \lambda qa - (\lambda^2 + \alpha)c, \\
& c_x = (\lambda^2 + \alpha)b - \lambda pa.
\end{align*}
\] (3.4)

We further, assume that

\[
a = \sum_{i \geq 0} a_i \lambda^{-2i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i \geq 0} c_i \lambda^{-2i-1} \tag{3.5}
\]

and take the initial values

\[
a_0 = 1, \quad b_0 = p, \quad c_0 = q \tag{3.6}
\]

which are required by

\[
a_{0,x} = pc_0 - qb_0, \quad -c_0 + qa_0 = 0, \quad -pa_0 + b_0 = 0. \tag{3.7}
\]

Now based on (3.4), we have

\[
\begin{align*}
& a_{i,x} = pc_i - qb_i, \\
& b_{i,x} = qa_{i+1} - c_{i+1} - \alpha c_i, \quad i \geq 0, \\
& c_{i,x} = b_{i+1} + ab_i - pa_{i+1}.
\end{align*}
\] (3.8)

From this, we can derive the recursion relations

\[
\begin{align*}
& a_{i+1,x} = -pb_{i,x} + qb_i - qc_{i,x} - \alpha pc_i, \\
& b_{i+1} = c_{i,x} - \alpha b_i + pa_{i+1}, \quad i \geq 0, \\
& c_{i+1} = -b_{i,x} - \alpha c_i + qa_{i+1},
\end{align*}
\] (3.9)

since (3.8) tells

\[
a_{i+1,x} = pc_{i+1} - qb_{i+1}
\]

\[
= p(-b_{i,x} - \alpha c_i + qa_{i+1}) - q(c_{i,x} - \alpha b_i + pa_{i+1})
\]

\[
= -pb_{i,x} + qb_i - qc_{i,x} - \alpha pc_i, \quad i \geq 0.
\]
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We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1$$

(3.10)

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. We list the first two sets as follows:

$$a_1 = -\frac{1}{2}(p^2 + q^2),$$

$$b_1 = q_x - \alpha p - \frac{1}{2}p(p^2 + q^2),$$

$$c_1 = -p_x - \alpha q - \frac{1}{2}q(p^2 + q^2);$$

$$a_2 = \alpha(p^2 + q^2) + pxq - pq_x + \frac{3}{8}(p^2 + q^2)^2,$$

$$b_2 = -pxx - \frac{1}{2}p\partial^{-1}p\partial + \alpha p\partial^{-1}q\partial - \frac{1}{2}q\partial^{-1}q\partial,
\quad c_2 = -qx + \alpha p\partial^{-1}p\partial + \frac{3}{2}(\alpha p + qx)(p^2 + q^2) + \frac{3}{8}p(p^2 + q^2)^2.$$  

It follows directly from the recursion relations (3.9), that, we have

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \quad i \geq 0,$$

(3.11)

where

$$\Psi = \begin{bmatrix} -\alpha - p\partial^{-1}p\partial + \alpha p\partial^{-1}q\partial & \partial - \alpha p\partial^{-1}p\partial - p\partial^{-1}q\partial \\ -\partial - q\partial^{-1}p\partial + \alpha q\partial^{-1}q\partial & -\alpha - \alpha q\partial^{-1}p\partial - q\partial^{-1}q\partial \end{bmatrix}.$$  

(3.12)

We will see that all vectors $(b_i, c_i)^T, i \geq 0$, above are gradient, and will generate conserved functionals.

Now for each $m \geq 0$, we introduce

$$V^{[m]} = \lambda^{2m+1}W^+ = \lambda^{2m+2}W^+ - a_{m+1}e_1$$

$$= \sum_{i=0}^{m} [a_i \lambda^{2(m-i)+2}e_1 + b_i \lambda^{2(m-i)+1}e_2 + c_i \lambda^{2(m-i)+1}e_3]$$

(3.13)

and the corresponding zero curvature equations

$$U_{tm} - V^{[m]}_x + [U, V^{[m]}] = 0, \quad m \geq 0$$

(3.14)

engender a hierarchy of solution equations

$$u_{tm} = K_m = \begin{bmatrix} b_{m,x} + \alpha c_m \\ c_{m,x} - \alpha b_m \end{bmatrix} = J \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 0,$$

(3.15)
with $J$ being defined by

$$J = \begin{bmatrix} \partial & \alpha \\ -\alpha & \partial \end{bmatrix}.$$  \hfill (3.16)

It is direct to check that $J$ is Hamiltonian.

### 3.3. Bi-Hamiltonian structures

It is easy to see that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q & -2\lambda \\ q & 0 & -p \\ 2\lambda & p & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and so, we obtain

$$\text{tr} \left( W \frac{\partial U}{\partial \lambda} \right) = -4\lambda a - 2pb - 2qc, \quad \text{tr} \left( W \frac{\partial U}{\partial p} \right) = -2\lambda b, \quad \text{tr} \left( W \frac{\partial U}{\partial q} \right) = -2\lambda c.$$

By the trace identity (1.9), we get

$$\frac{\delta}{\delta u} \int (2\lambda a + pb + qc) dx = -\gamma \frac{\partial}{\partial \gamma} \lambda^\gamma \begin{bmatrix} \lambda b \\ \lambda c \end{bmatrix}.$$

Balancing coefficients of all power of $\lambda$ in above equality tells

$$\frac{\delta}{\delta u} \int (2a_{m+1} + pb_m + qc_m) dx = (\gamma - 2m) \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 0.$$

Taking $m = 1$, we obtain $\gamma = 0$, and further, we have

$$\frac{\delta}{\delta u} \int \left( -\frac{2a_{m+1} + pb_m + qc_m}{2m} \right) dx = \begin{bmatrix} b_m \\ c_m \end{bmatrix}, \quad m \geq 1. \hfill (3.17)$$

This way, we obtain Hamiltonian structures for the reduced D-Kaup–Newell hierarchy (3.15) associated with so(3, $\mathbb{R}$):

$$u_{tm} = K_m = J \begin{bmatrix} b_m \\ c_m \end{bmatrix} = J \frac{\delta H_m}{\delta u}, \quad m \geq 0, \hfill (3.18)$$

where the Hamiltonian functionals are given by

$$H_0 = \frac{1}{2} (p^2 + q^2) dx,$$

$$H_m = \int \left( -\frac{2a_{m+1} + pb_m + qc_m}{2m} \right) dx, \quad m \geq 1. \hfill (3.19)$$

Introduce a second Hamiltonian operator

$$M = J \Psi = \Psi^\dagger J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

$$M_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
where all elements can be explicitly worked out:

\[
\begin{align*}
M_{11} &= -2\alpha \partial + \alpha^2 q \partial^{-1} q + \alpha \partial p \partial^{-1} q - \alpha q \partial^{-1} p \partial - \partial p \partial^{-1} p \partial, \\
M_{12} &= -\alpha^2 + \partial^2 - \alpha \partial q \partial^{-1} p - \alpha q \partial^{-1} q \partial - \alpha \partial p \partial^{-1} p \partial - \partial p \partial^{-1} q \partial, \\
M_{21} &= \alpha^2 - \partial^2 - \alpha^2 p \partial^{-1} q + \alpha \partial q \partial^{-1} q + \alpha \partial^{-1} p \partial - \partial q \partial^{-1} p \partial, \\
M_{22} &= -2\alpha \partial + \alpha^2 p \partial^{-1} p - \alpha q \partial^{-1} p + \alpha \partial^{-1} q \partial - \partial q \partial^{-1} q \partial.
\end{align*}
\]  

(3.21)

A direct computation by Maple can show that \( L_K, \Phi = 0 \), thus once proving \( L_K, \Psi = 0 \), we can show by the same argument as in the previous section that the operator

\[
\Phi = \Psi^1 = \begin{bmatrix}
-\alpha - \partial q \partial^{-1} p - \alpha q \partial^{-1} q - \alpha q \partial^{-1} q \\
-\partial + \partial q \partial^{-1} p - \partial q \partial^{-1} q - \alpha + \alpha p \partial^{-1} q - \partial q \partial^{-1} q
\end{bmatrix}
\]  

(3.22)

is a common hereditary recursion operator for the whole hierarchy (3.15), and the hierarchy (3.15) is bi-Hamiltonian:

\[
u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1. \tag{3.23}
\]

All this shows that the reduced D-Kaup–Newell hierarchy (3.15) associated with \( \text{so}(3, \mathbb{R}) \) is Liouville integrable [26].

When \( m = 0 \), we get a linear system

\[
u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x + \alpha q \\ -\alpha p + q_x \end{bmatrix} = J \frac{\delta \mathcal{H}_0}{\delta u}. \tag{3.24}
\]

When \( m = 1 \), we obtain a nonlinear system of bi-Hamiltonian equations:

\[
u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1
\]

\[
= \begin{bmatrix}
-\alpha^2 q - 2\alpha p_x + q_{xx} - \frac{1}{2} \alpha q(p_x + p_{xx}) \\
\alpha^2 q - 2\alpha q_x - p_{xx} + \frac{1}{2} \alpha q(p_x + p_{xx})
\end{bmatrix}
\]

\[
= J \frac{\delta \mathcal{H}_1}{\delta u} = M \frac{\delta \mathcal{H}_0}{\delta u}, \tag{3.25}
\]

where \( \mathcal{H}_1 \) can also be explicitly given by

\[
\mathcal{H}_1 = \int \left( \frac{1}{2} (-\alpha p^2 - \alpha q^2 - p_x q + pq_x) - \frac{1}{8} (p^2 + q^2)^2 \right) dx. \tag{3.26}
\]

We point out that, no bi-Hamiltonian structure presented for the second non-reduced D-Kaup–Newell soliton hierarchy associated with \( \text{so}(3, \mathbb{R}) \) [27]. Only one of the two differential operators in the recursive structure is Hamiltonian, and the other is quasi-Hamiltonian for the non-reduced D-Kaup–Newell soliton hierarchy in [27].
4. Concluding Remarks

In this paper, we have introduced two reduced D-Kaup–Newell spectral problems associated with \(\text{sl}(2,\mathbb{R})\) and \(\text{so}(3,\mathbb{R})\) and generated their hierarchies of bi-Hamiltonian equations via the zero curvature formulation. The Liouville integrability of the resulting soliton hierarchies has been shown upon establishing bi-Hamiltonian structures through the trace identity.

Compared to the D-Kaup–Newell soliton hierarchies [19, 27], our soliton hierarchies present different features such as bi-Hamiltonian structures, though our spectral problems involve less potentials. Moreover, the newly presented Hamiltonian pairs display different recursion operator structures from the known Kaup–Newell recursion operators associated with \(\text{sl}(2,\mathbb{R})\) and \(\text{so}(3,\mathbb{R})\) [3, 24].

Different choices of spectral matrices from matrix loop algebras bring us various soliton hierarchies with hereditary recursion operators, and any study on integrable couplings will help in classifying multicomponent integrable systems associated with nonsemisimple Lie algebras.

Acknowledgments

The work was supported in part by NSF under the grant DMS-1301675, NSFC under the grants 11371326, 11271008, 61072147, 11271266 and 11371323, Natural Science Foundation of Shanghai (Grant No. 11ZR1414100), Zhejiang Innovation Project of China (Grant No. T200905), and the First-class Discipline of Universities in Shanghai and the Shanghai University Leading Academic Discipline Project (No. A.13-0101-12-004). The author would also like to thank E. A. Appiah, X. Gu, C. X. Li, H. C. Ma, S. Manukure, M. Mcanally, S. F. Shen, Y. Q. Yao, S. M. Yu and W. Zhang for their valuable discussions about soliton hierarchies.

References

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