Computers and Mathematics with Applications 60 (2010) 2601–2608

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Constructing nonlinear discrete integrable Hamiltonian couplings

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ARTICLE INFO

Article history:
Received 20 February 2010
Received in revised form 26 August 2010
Accepted 26 August 2010

Keywords:
Integrable couplings
Hamiltonian structures
Variational identities

ABSTRACT

Beginning with Lax pairs from special non-semisimple matrix Lie algebras, we establish a scheme for constructing nonlinear discrete integrable couplings. Discrete variational identities over the associated loop algebras are used to build Hamiltonian structures for the resulting integrable couplings. We illustrate the application of the scheme by means of an enlarged Volterra spectral problem and present an example of nonlinear discrete integrable Hamiltonian couplings for the Volterra lattice equations.

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1. Introduction

Let E be a shift operator and define

$$f^{(m)}(n)=(E^m f)(n)=f(n+m),\quad m\in\mathbb{Z}. \tag{1.1}$$

We consider a pair of matrix discrete spectral problems

$$\begin{cases} E\phi=U\phi=U(u,\lambda)\phi,\\ \phi_t=V\phi=V(u,Eu,E^{-1}u,\ldots;\lambda)\phi,\end{cases} \tag{1.2}$$

where $u=u(n,t)$ is the potential, ϕ_t denotes the derivative with respect to t, U and V, called a Lax pair, belong to a given matrix Lie algebra g, and λ is a spectral parameter. Assume that the compatibility condition of (1.2) (i.e., the discrete zero-curvature equation)

$$U_t=(EV)U-VU \tag{1.3}$$

determines a discrete soliton equation (see, e.g., [1–3]):

$$u_t=K=K(n,t,u,Eu,E^{-1}u,\ldots). \tag{1.4}$$

That is to say, a triple (U,V,K) satisfies

$$U'[K]=(EV)U-VU,$$

where $U'[K]$ denotes the Gateaux derivative of U with respect to u in a direction K:

$$U'[K]=\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}U(u+\varepsilon K).$$

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doi:10.1016/j.camwa.2010.08.076

The Lie algebraic structure for such triples was analyzed in [4] and can be applied to the study of non-isospectral flows in both 1 + 1 dimensions [5] and 2 + 1 dimensions [6].

To generate integrable couplings [7,8] of Eq. (1.4), we have to take semi-direct sums of g with other matrix loop algebras g_c (see [9,10]):

$$\bar{g} = g \in g_c.$$

The notion of semi-direct sums implies that g and g_c satisfy $[g, g_c] \subseteq g_c$, where $[g, g_c] = \{[A, B] \mid A \in g,\ B \in g_c\}$. Variational identities on non-semisimple Lie algebras provide tools for generating Hamiltonian structures of the resulting integrable couplings [11,12]. It is known that most of the integrable couplings presented in the literature are linear with respect to the supplementary variables (see, e.g., [7,8,13–20]). For example, the spectral matrices of the perturbation form

$$\bar{U} = \begin{bmatrix} U(u) & U'[v] \\ 0 & U(u) \end{bmatrix} \tag{1.5}$$

lead to special dark equations—the perturbation equations [7,21,22]. In such integrable couplings, the equation for the supplementary variable v is linear with respect to v.

Definition 1.1. If the second equation of an integrable coupling

$$u_t = K(u), \qquad v_t = S(u, v),$$

defines a nonlinear equation for v, then the whole system is called a nonlinear integrable coupling of $u_t = K(u)$.

Linear integrable couplings contain extensions of symmetry equations [7,13] and are important in classifying integrable lattice equations. But naturally, nonlinear ones have much richer structures. There are a few systematic approaches for constructing linear discrete integrable couplings, starting from the perturbed spectral matrices [8,13], defined as before, or the amended spectral matrices [14,17]:

$$\bar{U} = \begin{bmatrix} U(u) & U_a(v) \\ 0 & 0 \end{bmatrix}, \tag{1.6}$$

where U_a might not be square. However, there is no feasible way which allows us to construct nonlinear discrete integrable couplings as yet.

In this paper, we would like to present a kind of Lie algebras which can generate nonlinear discrete integrable couplings. More specifically, we would like to show that the following choice of spectral matrices:

$$\bar{U} = \begin{bmatrix} U(u) & U_a(v) \\ 0 & U(u) + U_a(v) \end{bmatrix}, \tag{1.7}$$

can engender nonlinear discrete integrable couplings. The set of all matrices above is closed under the matrix product, and thus it constitutes a matrix Lie algebra under the matrix commutator. The resulting Lie algebras are non-semisimple, since they have a non-trivial ideal Lie sub-algebra consisting of matrices of the form

$$\begin{bmatrix} 0 & U_a \\ 0 & U_a \end{bmatrix}. \tag{1.8}$$

The variational identities over this kind of Lie algebras can furnish Hamiltonian structures for the associated discrete integrable couplings. We will illustrate such an idea for generating nonlinear discrete integrable Hamiltonian couplings by means of the Volterra lattice hierarchy.

2. A scheme for constructing nonlinear integrable couplings

2.1. The general scheme

Assume that an integrable equation (1.4) has a discrete zero-curvature representation (1.3), where two Lax matrices U and V usually belong to a semisimple matrix Lie algebra g. Let us introduce an enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}) = \begin{bmatrix} U(u) & U_a(v) \\ 0 & U(u) + U_a(v) \end{bmatrix}, \tag{2.1}$$

where the new dependent variable $\bar{u}$ consists of the original one u and the supplementary one v. Now, upon choosing

$$\bar{V} = \bar{V}(\bar{u}) = \begin{bmatrix} V(u) & V_a(\bar{u}) \\ 0 & V(u) + V_a(\bar{u}) \end{bmatrix}, \tag{2.2}$$

an enlarged discrete zero-curvature equation

$$\bar{U}_t = (E\bar{V})\bar{U} - \bar{U}\bar{V} \tag{2.3}$$

yields

$$\begin{cases} U_t = (EV)U - UV, \\ U_{a,t} = (EV)U_a - U_aV + (EV_a)U - UV_a + (EV_a)U_a - U_aV_a. \end{cases} \tag{2.4}$$

This is an integrable coupling of the Eq. (1.4) due to the assumption (1.3), and it is normally a nonlinear integrable coupling because the matrix $(EV_a)U_a - U_aV_a$ often produces nonlinear terms.

Let us further take a solution $\bar{W}$ to the enlarged stationary discrete zero-curvature equation

$$(E\bar{W})(E\bar{U}) - \bar{U}\bar{W} = 0. \tag{2.5}$$

Then, we use the corresponding discrete variational identity [23]:

$$\frac{\delta}{\delta \bar{u}} \sum_{n\in\mathbb{Z}} \langle \bar{W}, \bar{U}_\lambda \rangle = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{W}, \bar{U}_{\bar{u}} \rangle, \quad \gamma = \text{const.}, \tag{2.6}$$

to furnish Hamiltonian structures for the discrete integrable couplings described above. In the variational identity (2.6), $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric and invariant bilinear form (see [11,23,19] for detailed discussion), over the non-semisimple Lie algebra:

$$\bar{g} = \left\{ \begin{bmatrix} A & B \\ 0 & A+B \end{bmatrix} \middle| A, B \in g \right\}. \tag{2.7}$$

In what follows, we will make an application of this general scheme to the Volterra lattice hierarchy.

2.2. An application to the Volterra lattice hierarchy

2.2.1. The Volterra hierarchy

The Volterra lattice hierarchy can be associated with the following discrete spectral problem (see, e.g., [4]):

$$E\phi = U\phi, \quad U = U(u,\lambda) = \begin{bmatrix} 1 & u \\ \lambda^{-1} & 0 \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{2.8}$$

Upon setting

$$\Gamma = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\ge 0} \Gamma_i \lambda^{-i}, \quad \Gamma_i = \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix}, \ i \ge 0, \tag{2.9}$$

the stationary discrete zero-curvature equation

$$(E\Gamma)U - U\Gamma = 0 \tag{2.10}$$

equivalently generates

$$\begin{cases} b = u(a^{(1)} + a), \qquad c = \lambda^{-1}(a + a^{(-1)}), \\ a^{(1)} - a = -\lambda^{-1}[u^{(1)}(a^{(2)} + a^{(1)}) - u(a + a^{(-1)})]. \end{cases} \tag{2.11}$$

The last equation above uniquely determines all functions a_i, $i \ge 1$, if we take

$$a_0 = \frac{1}{2}, \qquad a_i|_{u=0} = 0, \quad i \ge 1. \tag{2.12}$$

The first two equations automatically give all functions of b_i and c_i, $i \ge 0$. In particular, the first two sets of a_i, b_i and c_i are

$$a_0 = \frac{1}{2}, \qquad b_0 = u, \qquad c_0 = 0;$$
$$a_1 = -u, \qquad b_1 = -u(u^{(1)} + u), \qquad c_1 = 1.$$

The compatibility conditions of the matrix discrete spectral problems

$$E\phi = U\phi, \qquad \phi_t = V^{[m]}\phi, \qquad V^{[m]} = (\lambda^{m+1}\Gamma)_+ + \Delta_m, \quad \Delta_m = \begin{bmatrix} 0 & -b_{m+1} \\ 0 & a_{m+1} + a_{m+1}^{(-1)} \end{bmatrix}, \ m \ge 0, \tag{2.13}$$

where $(P)_+$ denotes the polynomial part of P in λ, determine (see, e.g., [4]) the Volterra lattice hierarchy of soliton equations

$$u_{t_m} = K_m = \Phi^m K_0 = u(a^{(1)}_{m+1} - a^{(-1)}_{m+1}), \quad m \geq 0, \tag{2.14}$$

where the hereditary recursion operator Φ reads

$$\Phi = \Phi(u) = u(1 + E^{-1})(-u^{(1)}E^2 + u)(E - 1)^{-1}u^{-1}. \tag{2.15}$$

The first equation in (2.14) is the Volterra lattice equation

$$u_{t_0} = K_0 = u(u^{(-1)} - u^{(1)}). \tag{2.16}$$

Because of

$$\langle V, U_\lambda \rangle = \mathrm{tr}(VU_\lambda) = \lambda^{-1}a^{(1)}, \qquad \langle V, U_u \rangle = \mathrm{tr}(VU_u) = -\frac{a}{u},$$

where $V = \Gamma U^{-1}$, an application of the trace identity with $\gamma = 0$ in [1] yields the Hamiltonian structures of the Volterra lattice hierarchy:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad J = u(E^{-1} - E)u, \ \mathcal{H}_m = \sum_{n\in\mathbb{Z}} \left(-\frac{a_{m+1}}{m+1}\right), \ m \geq 0. \tag{2.17}$$

2.2.2. Integrable couplings

Let us now begin with an enlarged spectral matrix:

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_a \\ 0 & U + U_a \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.18}$$

where U is defined as in (2.8) and the supplementary matrix U_a is taken as

$$U_a = U_a(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}. \tag{2.19}$$

For the enlarged stationary discrete zero-curvature equation

$$(E\bar{\Gamma})\bar{U} - \bar{U}\bar{\Gamma} = 0, \tag{2.20}$$

we look for a solution

$$\bar{\Gamma} = \begin{bmatrix} \Gamma & \Gamma_a \\ 0 & \Gamma + \Gamma_a \end{bmatrix}, \quad \Gamma_a = \Gamma_a(\bar{u}, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \tag{2.21}$$

where Γ, defined by (2.9), solves the stationary discrete zero-curvature equation (2.10). Then, Eq. (2.20) requires

$$(E\Gamma)U_a - U_a\Gamma + (E\Gamma_a)U - U\Gamma_a + (E\Gamma_a)U_a - U_a\Gamma_a = 0, \tag{2.22}$$

which implies

$$\begin{cases} e^{(1)} - e + \lambda^{-1}f^{(1)} - ug - v(c + g) = 0, \\ u(e^{(1)} + e) - f + v(a^{(1)} + a + e^{(1)} + e) = 0, \\ g^{(1)} - \lambda^{-1}(e^{(1)} + e) = 0, \\ ug^{(1)} - \lambda^{-1}f + v(c^{(1)} + g^{(1)}) = 0. \end{cases} \tag{2.23}$$

Noting the system (2.11) for defining a, b and c, the above system equivalently gives

$$\begin{cases} f = u(e^{(1)} + e) + v(a^{(1)} + a + e^{(1)} + e), \qquad g = \lambda^{-1}(e + e^{(-1)}), \\ \lambda(e^{(1)} - e) = -u^{(1)}(e^{(2)} + e^{(1)}) + u(e + e^{(-1)}) - v^{(1)}(a^{(2)} + a^{(1)} + e^{(2)} + e^{(1)}) + v(a + a^{(-1)} + e + e^{(-1)}). \end{cases} \tag{2.24}$$

Trying a formal series solution

$$e = \sum_{i=0}^{\infty} e_i\lambda^{-i}, \qquad f = \sum_{i=0}^{\infty} f_i\lambda^{-i}, \qquad g = \sum_{i=0}^{\infty} g_i\lambda^{-i}, \tag{2.25}$$

we obtain a recursion relation for e_i:

$$\begin{aligned} e_{i+1} = {} & -(E - 1)^{-1}[u^{(1)}(e_i^{(2)} + e_i^{(1)}) - u(e_i + e_i^{(-1)}) \\ & + v^{(1)}(a_i^{(2)} + a_i^{(1)} + e_i^{(2)} + e_i^{(1)}) - v(a_i + a_i^{(-1)} + e_i + e_i^{(-1)})], \quad i \geq 0. \end{aligned} \tag{2.26}$$

Upon taking

$$e_0 = \frac{1}{2}, \qquad e_i|_{\bar{u}=0} = 0, \quad i \geq 1, \tag{2.27}$$

the recursion relation (2.26) uniquely determines the sequence of $e_i,\ i \geq 1$, and subsequently, the system (2.24) defines the sequences of f_i and $g_i,\ i \geq 0$. The first two sets are listed as follows:

$$e_0 = \frac{1}{2}, \qquad f_0 = u + 2v, \qquad g_0 = 0;$$

$$e_1 = -u - 2v, \qquad f_1 = -u(u^{(1)} + 2v^{(1)} + u + 2v) - 2v(u^{(1)} + v^{(1)} + u + v), \qquad g_1 = 1.$$

For each integer $m \geq 0$, let us introduce

$$\bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} + V_a^{[m]} \end{bmatrix}, \tag{2.28}$$

where $V^{[m]}$ is defined as in (2.13), and

$$V_a^{[m]} = (\lambda^{m+1}\Gamma_a)_+ + \Delta_{m,a}, \quad \Delta_{m,a} = \begin{bmatrix} 0 & -f_{m+1} \\ 0 & e_{m+1} + e_{m+1}^{(-1)} \end{bmatrix}. \tag{2.29}$$

Then, the enlarged discrete zero-curvature equation

$$\bar{U}_{t_m} = (E\bar{V}^{[m]})\bar{U} - \bar{U}\bar{V}^{[m]} \tag{2.30}$$

yields

$$U_{a,t_m} = (EV^{[m]})U_a - U_aV^{[m]} + (EV_a^{[m]})U - UV_a^{[m]} + (EV_a^{[m]})U_a - U_aV_a^{[m]},$$

together with the m-th Volterra lattice equation in (2.14). This equation leads to

$$v_{t_m} = S_m = S_m(u, v) = u(e_{m+1}^{(1)} - e_{m+1}^{(-1)}) + v(a_{m+1}^{(1)} - a_{m+1}^{(-1)} + e_{m+1}^{(1)} - e_{m+1}^{(-1)}), \quad m \geq 0.$$

Therefore, the hierarchy of enlarged discrete zero-curvature equations generates a hierarchy of discrete integrable couplings

$$\begin{aligned} \bar{u}_{t_m} &= \begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_m(u, v) \end{bmatrix} \\ &= \begin{bmatrix} u(a_{m+1}^{(1)} - a_{m+1}^{(-1)}) \\ u(e_{m+1}^{(1)} - e_{m+1}^{(-1)}) + v(a_{m+1}^{(1)} - a_{m+1}^{(-1)} + e_{m+1}^{(1)} - e_{m+1}^{(-1)}) \end{bmatrix}, \quad m \geq 0, \end{aligned} \tag{2.31}$$

for the Volterra lattice hierarchy (2.14). All discrete integrable couplings above are nonlinear due to the term $v(e_{m+1}^{(1)} - e_{m+1}^{(-1)})$. The first one reads

$$\begin{cases} u_{t_0} = u(u^{(-1)} - u^{(1)}), \\ v_{t_0} = u(u^{(-1)} + 2v^{(-1)} - u^{(1)} - 2v^{(1)}) + 2v(u^{(-1)} - u^{(1)} + v^{(-1)} - v^{(1)}). \end{cases} \tag{2.32}$$

Obviously, the supplementary evolution equation of v is nonlinear with respect to v itself, and so (2.32) gives a nonlinear discrete integrable coupling of the Volterra lattice equation (2.16).

2.2.3. Invariant bilinear forms

To construct Hamiltonian structures of the integrable couplings obtained, we need to compute non-degenerate, symmetric and invariant bilinear forms on the following Lie algebra:

$$\bar{g} = \left\{ \begin{bmatrix} A & B \\ 0 & A + B \end{bmatrix} \middle| A, B \in \mathrm{gl}(2) \right\}. \tag{2.33}$$

For brevity, we transform this Lie algebra $\bar{g}$ into a vector form through the mapping

$$\delta : \bar{g} \to \mathbb{R}^8, \qquad A \mapsto (a_1, a_2, \ldots, a_8)^T, \qquad A = \begin{bmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_7 & a_8 \\ 0 & 0 & a_1 + a_5 & a_2 + a_6 \\ 0 & 0 & a_3 + a_7 & a_4 + a_8 \end{bmatrix} \in \bar{g}. \tag{2.34}$$

The mapping δ induces a Lie algebraic structure on $\mathbb{R}^8$, isomorphic to the matrix Lie algebra $\bar{g}$ above. It is easy to see that the corresponding commutator $[\cdot, \cdot]$ on $\mathbb{R}^8$ is given by

$$[a, b]^T = a^T R(b), \quad a = (a_1, a_2, \ldots, a_8)^T,\ b = (b_1, b_2, \ldots, b_8)^T \in \mathbb{R}^8, \tag{2.35}$$

where

$$R(b)=\begin{bmatrix} 0 & b_2 & -b_3 & 0 & 0 & b_6 & -b_7 & 0\\ b_3 & -b_1+b_4 & 0 & -b_3 & b_7 & -b_5+b_8 & 0 & -b_7\\ -b_2 & 0 & b_1-b_4 & b_2 & -b_6 & 0 & b_5-b_8 & b_6\\ 0 & -b_2 & b_3 & 0 & 0 & -b_6 & b_7 & 0\\ 0 & 0 & 0 & 0 & 0 & b_2+b_6 & -b_3-b_7 & 0\\ 0 & 0 & 0 & 0 & b_3+b_7 & b_8-b_1-b_5+b_4 & 0 & -b_3-b_7\\ 0 & 0 & 0 & 0 & -b_2-b_6 & 0 & -b_8+b_1+b_5-b_4 & b_2+b_6\\ 0 & 0 & 0 & 0 & 0 & -b_2-b_6 & b_3+b_7 & 0 \end{bmatrix}.$$

Define a bilinear form on $\mathbb{R}^8$ as follows:

$$\langle a,b\rangle = a^T F b, \tag{2.36}$$

where F is a constant matrix. Then, the symmetric property $\langle a,b\rangle=\langle b,a\rangle$ and the invariance property under the Lie product $\langle a,[b,c]\rangle=\langle[a,b],c\rangle$ requires that $F^T=F$ and

$$(R(b)F)^T=-R(b)F \quad \text{for all } b\in\mathbb{R}^6.$$

This matrix equation leads to a system of linear equations in the elements of F. Solving the resulting system yields

$$F=\begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 & \eta_3 & 0 & 0 & \eta_3-\eta_4+\eta_5\\ 0 & 0 & \eta_1-\eta_2 & 0 & 0 & 0 & \eta_4-\eta_5 & 0\\ 0 & \eta_1-\eta_2 & 0 & 0 & 0 & \eta_4-\eta_5 & 0 & 0\\ \eta_2 & 0 & 0 & \eta_1 & \eta_3-\eta_4+\eta_5 & 0 & 0 & \eta_3\\ \eta_3 & 0 & 0 & \eta_3-\eta_4+\eta_5 & \eta_4 & 0 & 0 & \eta_5\\ 0 & 0 & \eta_4-\eta_5 & 0 & 0 & 0 & \eta_4-\eta_5 & 0\\ 0 & \eta_4-\eta_5 & 0 & 0 & 0 & \eta_4-\eta_5 & 0 & 0\\ \eta_3-\eta_4+\eta_5 & 0 & 0 & \eta_3 & \eta_5 & 0 & 0 & \eta_4 \end{bmatrix},$$

where η_i, $1\le i\le 5$, are arbitrary constants.

Therefore, a bilinear form on the underlying Lie algebra $\bar{g}$ in (2.33) is given by

$$\begin{aligned}\langle A,B\rangle_{\bar{g}} &= \langle \delta^{-1}(A),\delta^{-1}(B)\rangle_{\mathbb{R}^8}=(a_1,a_2,\ldots,a_8)F(b_1,b_2,\ldots,b_8)^T\\ &= \eta_1(a_1b_1+a_2b_3+a_3b_2+a_4b_4)+\eta_2(a_1b_4-a_2b_3-a_3b_2+a_4b_1)\\ &\quad +\eta_3[(a_1+a_4)(b_5+b_8)+(a_5+a_8)(b_1+b_4)]\\ &\quad +\eta_4[-a_1b_8+a_2b_7+a_3b_6-a_4b_5-a_5(b_4-b_5)+a_6(b_3+b_7)\\ &\quad +a_7(b_2+b_6)-a_8(b_1-b_8)]+\eta_5[a_1b_8-a_2b_7-a_3b_6+a_4b_5\\ &\quad +a_5(b_4+b_8)-a_6(b_3+b_7)-a_7(b_2+b_6)+a_8(b_1+b_5)],\end{aligned} \tag{2.37}$$

where

$$A=\begin{bmatrix} a_1 & a_2 & a_5 & a_6\\ a_3 & a_4 & a_7 & a_8\\ 0 & 0 & a_1+a_5 & a_2+a_6\\ 0 & 0 & a_3+a_7 & a_4+a_8 \end{bmatrix},\qquad B=\begin{bmatrix} b_1 & b_2 & b_5 & b_6\\ b_3 & b_4 & b_7 & b_8\\ 0 & 0 & b_1+b_5 & b_2+b_6\\ 0 & 0 & b_3+b_7 & b_4+b_8 \end{bmatrix}.$$

This bilinear form (2.37) is symmetric and invariant under the Lie product:

$$\langle A,B\rangle=\langle B,A\rangle,\qquad \langle A,[B,C]\rangle=\langle[A,B],C\rangle,\quad A,B,C\in\bar{g}. \tag{2.38}$$

The bilinear form is invariant under the matrix product:

$$\langle A,BC\rangle=\langle AB,C\rangle,\quad A,B,C\in\bar{g}, \tag{2.39}$$

if and only if

$$\eta_2=\eta_5=0,\qquad \eta_3=\eta_4, \tag{2.40}$$

which can be obtained by checking the coefficients of a_1b_1 and a_1b_5 in $\langle A,BC\rangle-\langle AB,C\rangle$. It is non-degenerate if and only if $\det F\neq 0$, i.e.,

$$\begin{cases}\eta_4-\eta_5\neq 0,\qquad \eta_1-\eta_2-\eta_4+\eta_5\neq 0,\\ \eta_4^2-\eta_1\eta_4-\eta_2\eta_4-2\eta_4\eta_5-4\eta_3\eta_4+4\eta_3\eta_5+4\eta_3^2+\eta_5^2-\eta_1\eta_5-\eta_2\eta_5\neq 0.\end{cases} \tag{2.41}$$

2.2.4. Hamiltonian structures

Let us now choose that

$$\eta_1 = 0, \qquad \eta_2 = 0, \qquad \eta_3 = 1, \qquad \eta_4 = 1, \qquad \eta_5 = 0, \tag{2.42}$$

to guarantee that the corresponding bilinear form (2.37) is non-degenerate and invariant under the matrix product. Then, it is directly computed that

$$\langle \bar{W}, \bar{U}_\lambda \rangle = -\frac{u^2 e + uve + vb - uf}{\lambda u(u+v)} = \lambda^{-1} e^{(1)},$$
$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle = \left(\frac{va - ue}{u(u+v)}, -\frac{a+e}{u+v} \right)^T,$$

where $\bar{W} = \bar{\Gamma}\bar{U}^{-1}$, $\bar{\Gamma}$ being defined by (2.21). Thus, the discrete variational identity (2.6) with $\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln |\langle \bar{\Gamma}, \bar{\Gamma} \rangle| = 0$ generates

$$\frac{\delta}{\delta \bar{u}} \sum_{n\in\mathbb{Z}} \left(-\frac{e_{m+1}}{m+1} \right) = \left(\frac{va_{m+1} - ue_{m+1}}{u(u+v)}, -\frac{a_{m+1} + e_{m+1}}{u+v} \right)^T, \quad m \geq 0. \tag{2.43}$$

It follows from this that the Volterra integrable couplings (2.31) possess the following Hamiltonian structures:

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}) = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \quad m \geq 0, \tag{2.44}$$

where the Hamiltonian operator is given by

$$\bar{J} = \begin{bmatrix} -u(E^{-1} - E)u & u(E^{-1} - E)u \\ u(E^{-1} - E)u & (u+v)(E^{-1} - E)(u+v) - u(E^{-1} - E)u \end{bmatrix}, \tag{2.45}$$

and the Hamiltonian functionals, by

$$\bar{\mathcal{H}}_m = \sum_{n\in\mathbb{Z}} \left(-\frac{e_{m+1}}{m+1} \right), \quad m \geq 0. \tag{2.46}$$

Noting that the recursion relation for $a_i + e_i$ has the same format as for a_i, we obtain

$$\bar{K}_{m+1} = \bar{\Phi}\bar{K}_m, \quad \bar{\Phi} = \begin{bmatrix} \Phi(u) & 0 \\ \Phi_a(\bar{u}) & \Phi(u) + \Phi_a(\bar{u}) \end{bmatrix}, \ m \geq 0,$$

where Φ is defined as in (2.15) and

$$\Phi_a(\bar{u}) = (u+v)(1+E^{-1})[-(u^{(1)} + v^{(1)})E^2 + u + v](E-1)^{-1}(u+v)^{-1} - \Phi(u). \tag{2.47}$$

It is straightforward to check $\bar{J}\bar{\Phi}^\dagger = \bar{\Phi}\bar{J}$, where $\bar{\Phi}^\dagger$ is the adjoint operator of $\bar{\Phi}$. Then, it follows immediately that all discrete integrable couplings (2.31) commute with each other, and so do all conserved functionals (2.46), that is to say,

$$[\bar{K}_k, \bar{K}_l] = 0, \qquad \{\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_l\} = 0, \quad k, l \geq 0. \tag{2.48}$$

It is also not difficult to verify that $\bar{J}$ and $\bar{\Phi}\bar{J}$ constitute a Hamiltonian pair and $\bar{\Phi}$ is a hereditary recursion operator for the hierarchy of discrete integrable couplings (2.31).

3. Concluding remarks

We introduced a class of specific non-semisimple Lie algebras which generate nonlinear discrete integrable couplings. The variational identities on the Lie algebras considered were used to furnish Hamiltonian structures for the resulting discrete integrable couplings. An application to the Volterra lattice hierarchy yielded a hierarchy of nonlinear discrete integrable Hamiltonian couplings. The results obtained provide a supplement to the existing theories on the perturbation equations and linear integrable couplings [7,11,14].

We remark that vertex operator representations of polynomial Lie algebras are used to study coupled integrable systems and their soliton solutions and Bäcklund transformations [24,25], and component-trace identities are applied to Hamiltonian structures of multi-component integrable couplings [26]. We also point out that using the block type matrix algebras will lead to other classes of integrable couplings with more supplementary equations. Furthermore, combining different forms of spectral matrices will generate more diverse discrete integrable couplings, both linear and nonlinear (see, e.g., [17,27,28] for other forms). For example,

$$\begin{bmatrix} U & 0 & U_a \\ 0 & U & U_b \\ 0 & 0 & U + U_a \end{bmatrix}$$

will yield a new kind of integrable couplings—integrable bi-couplings [28]. The discrete integrable couplings presented can also have other integrable properties, for instance, Hirota bilinear forms [29]. Such studies will enrich the theory of multi-component integrable equations (see [30–32] for examples of higher-order matrix spectral problems), and we hope that our results will help explore diverse algebraic and geometric structures of integrable couplings, particularly integrable multi-couplings.

Acknowledgements

The first author was supported in part by the Established Researcher Grant, the CAS faculty development grant and the CAS Dean research grant of the University of South Florida, Chunhui Plan of the Ministry of Education of China, and the State Administration of Foreign Experts Affairs of China. The second author was supported in part by the National Natural Science Foundation of China under grant 10971136, and the Ministry of Education and Science of Spain under contract MTM2009-12670. The first author would also like to thank Shanghai Jiao Tong University, where the work was initiated, for warm hospitality, and thank H.T. Chen, S. Kouchekian, V. Vanaja, Y.F. Zhang and H.Q. Zhao for valuable discussions.

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