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A 4 × 4 Matrix Spectral Problem Involving Four Potentials and Its Combined Integrable Hierarchy

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Abstract

This paper introduces a specific matrix spectral problem involving four potentials and derives an associated soliton hierarchy using the zero-curvature formulation. The bi-Hamiltonian formulation is derived via the trace identity, thereby establishing the hierarchy's Liouville integrability. This is exemplified through two systems: generalized combined NLS-type equations and modified KdV-type equations. Owing to Liouville integrability, each member of the hierarchy admits a bi-Hamiltonian structure and, consequently, possesses infinitely many symmetries and conservation laws.

Keywords: Lax pair; soliton hierarchy; NLS equations; mKdV equations

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1. Introduction

Matrix spectral problems serve as a fundamental starting point for the formulation of integrable models [1,2]. These problems offer a robust framework for revealing the intricate mathematical structures of integrable models, such as the bi-Hamiltonian formulation and the presence of infinitely many symmetries and conserved quantities [3]. Owing to their rich mathematical structure and solvability, integrable models have become instrumental in diverse applications, ranging from nonlinear optics and water wave dynamics to plasma physics and quantum mechanics.

A fundamental example of an integrable hierarchy is the soliton hierarchy presented by Ablowitz, Kaup, Newell and Segur (AKNS) [4], which has inspired the development of numerous soliton hierarchies of integrable couplings. Matrix Lie algebras provide a powerful framework for constructing Lax pairs associated with matrix spectral problems [5,6], enabling the identification and classification of integrable models into distinct categories. Although many integrable models involving one or two potentials have been extensively studied, investigations into systems with multiple potentials remain comparatively scarce. We aim to construct a soliton hierarchy of integrable flows involving four distinct potentials, each admitting a bi-Hamiltonian formulation. This hierarchy is derived from a matrix spectral problem associated with a specific matrix Lie subalgebra of the general linear algebra, which we will develop in this work.

The Lax pair formulation serves as a fundamental framework for constructing soliton hierarchies of integrable bi-Hamiltonian models from matrix spectral problems (see [6] for details). In our discussion, we denote the spectral parameter by λ and represent the potential as a q-dimensional column vector $v = (v_1, \cdots, v_q)^T$. The initial step involves selecting a matrix Lie algebra $\tilde{g}$ parameterized by the spectral parameter λ, and formulating the spatial spectral matrix accordingly:

$$\mathcal{M} = \mathcal{M}(v, \lambda) = v_1 E_1(\lambda) + \cdots + v_q E_q(\lambda) + E_0(\lambda), \tag{1}$$

where $E_1, \cdots, E_q$ form a linearly independent set in the matrix Lie algebra under consideration. We assume that the element E_0 satisfies the pseudo-regular conditions:

(a) The direct sum of the kernel and the image of ad_{E_0} spans the entire algebra $\tilde{g}$;
(b) The kernel of ad_{E_0} forms an abelian (commuting) subalgebra.

Here, $\mathrm{ad}E_0$ refers to the adjoint action on $\tilde{g}$, given by $\mathrm{ad}E_0(X) = [E_0, X]$. With these conditions in place, it is possible to construct a particular series solution of the form $Y = \sum_{n \geq 0} Y^{[n]} \lambda^{-n}$ satisfying the stationary zero-curvature equation

$$Y_x = [\mathcal{M}, Y] \tag{2}$$

where $Y \in \tilde{g}$, the chosen loop algebra, i.e., the matrix Lie algebra parameterized by the spectral parameter λ.

Following this, the next step is to propose an infinite sequence of appropriate Lax matrices:

$$\mathcal{N}^{[m]} = \mathcal{N}^{[m]}(v, \lambda) = (\lambda^m Y)_+ + \Delta_m, \ m \geq 0, \tag{3}$$

where

$$(\lambda^m Y)_+ = \sum_{n=0}^{m} \lambda^n Y^{[m-n]}, \tag{4}$$

and $\Delta_m \in \tilde{g}$ are modification terms for $m \geq 0$, ensuring closure within the loop algebra. These temporal Lax matrices, together with the spatial spectral matrix, constitute the Lax pairs that generate the integrable hierarchy via the zero-curvature condition. Serving as the other elements, these matrices form part of the Lax pair sequence, enabling the construction of an integrable hierarchy:

$$v_{t_m} = P^{[m]} = P^{[m]}(v), \ m \geq 0, \tag{5}$$

which is obtained from the corresponding zero-curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}^{[m]}_x - [\mathcal{N}^{[m]}, \mathcal{M}] = 0, \ m \geq 0. \tag{6}$$

These zero-curvature equations ensure the compatibility of the spatial and temporal matrix spectral problems:

$$\varphi_x = \mathcal{M}(v, \lambda)\varphi, \ \varphi_{t_m} = \mathcal{N}^{[m]}(v, \lambda)\varphi, \ m \geq 0. \tag{7}$$

The final step involves establishing the required Liouville integrability of the hierarchy by constructing a bi-Hamiltonian formulation equipped with a hereditary recursion operator, which generates two infinite sequences of commuting symmetries and conserved quantities. To achieve this, we employ the trace identity to establish Hamiltonian formulations:

$$\frac{\delta}{\delta v} \int \mathrm{tr}(Y \frac{\partial \mathcal{M}}{\partial \lambda}) \, dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \mathrm{tr}(Y \frac{\partial \mathcal{M}}{\partial v}). \tag{8}$$

In this expression, $\frac{\delta}{\delta v}$ indicates the variational derivative relative to the function v, and κ is a constant that does not depend on the spectral parameter λ. Applying this identity to the integrable hierarchy given in (5), and utilizing a recursion operator Φ that connects consecutive flows through computing $P^{[m+1]}$ via $\Phi P^{[m]}$, one can construct a bi-Hamiltonian structure. This bi-Hamiltonian structure thus confirms the required Liouville integrability of the hierarchy under consideration (see, e.g., [6,7]). As a result, each system in the hierarchy admits infinitely many symmetries and conservation laws.

A broad class of Liouville integrable hierarchies has been systematically constructed within the Lax pair framework, which serves as a foundational approach for revealing integrable structures in nonlinear systems (see, e.g., [4–15]). In the two-component scenario ($q = 2$), the following four integrable hierarchies are well established: AKNS [4], Heisenberg [16], Kaup–Newell [17], and Wadati–Konno–Ichikawa [18]. The corresponding spectral matrices $\mathcal{M}$ for these hierarchies are given, respectively, by

$$\mathcal{M} = \begin{bmatrix} \lambda & v_1 \\ v_2 & -\lambda \end{bmatrix}, \mathcal{M} = \begin{bmatrix} \lambda v_3 & \lambda v_1 \\ \lambda v_2 & -\lambda v_3 \end{bmatrix}, \mathcal{M} = \begin{bmatrix} \lambda^2 & \lambda v_1 \\ \lambda v_2 & -\lambda^2 \end{bmatrix}, \mathcal{M} = \begin{bmatrix} \lambda & \lambda v_1 \\ \lambda v_2 & -\lambda \end{bmatrix}, \tag{9}$$

where in the second case, $v_1 v_2 + v_3^2 = 1$ holds.

The primary aim of this work is to propose a soliton hierarchy that incorporates four-component Liouville integrable bi-Hamiltonian models, employing the Lax pair framework as the fundamental tool. The central novelty of this paper lies in formulating a specific 4×4 matrix spectral problem involving four potentials, constructed from a particular matrix Lie subalgebra. By applying the trace identity to the matrix spectral problem, we construct a bi-Hamiltonian structure, thereby establishing the Liouville integrability of the resulting hierarchy. Two concrete examples are presented to illustrate the theory: a set of generalized combined integrable nonlinear Schrödinger (NLS) equations and a set of modified Korteweg–de Vries (mKdV) equations. The paper concludes with a summary of the principal outcomes along with a few final comments.

2. A 4 × 4 Matrix Spectral Problem and Its Integrable Hierarchy

Let σ be an arbitrary constant, and let S be an $r \times r$ square matrix satisfying $S^{-1} = S$ (i.e., S is an involutive matrix). Consider the set $\tilde{g}$ composed of block matrices structured as follows:

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}_{2r \times 2r} \text{with } M_4 = SM_1S^{-1},\ M_3 = \sigma SM_2S^{-1}. \tag{10}$$

The set $\tilde{g}$ is shown to form a matrix Lie algebra under the matrix commutator, where $[A, B] = AB - BA$. In particular, when $r = 2$, we consider

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

both of which satisfy $S^{-1} = S$. By leveraging this Lie algebra structure, a specific spectral matrix $\mathcal{M}$ is constructed, from which an integrable hierarchy follows.

Consider four arbitrary constants $\alpha_1, \alpha_2, \sigma_1$ and σ_2, and $v = v(x, t) = (v_1, v_2, v_3, v_4)^T$ a four-component potential vector with real variables x and t. For the subsequent discussion, we require the non-degeneracy condition,

$$(\alpha_1 - \alpha_2)\sigma_1\sigma_2 \neq 0, \tag{11}$$

ensuring that the parameters are suitably distinct and non-zero.

Recent investigations into matrix spectral problems involving four potentials have revealed rich structures and led to the discovery of novel integrable models (see, e.g., [19] by the present first author, and [20,21] by other researchers). Building on existing studies, we propose the following formulation of a 4×4 matrix spectral problem:

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(v,\lambda)\varphi, \tag{12}$$

where the spectral matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{bmatrix} 0 & \sigma_1 v_1 & v_2 & \alpha_1\lambda \\ \sigma_1 v_3 & 0 & \alpha_2\lambda & v_4 \\ \sigma_1\sigma_2 v_4 & \sigma_1\sigma_2\alpha_2\lambda & 0 & \sigma_1 v_3 \\ \sigma_1\sigma_2\alpha_1\lambda & \sigma_1\sigma_2 v_2 & \sigma_1 v_1 & 0 \end{bmatrix}, \tag{13}$$

where $v = (v_1, v_2, v_3, v_4)^T$ and λ is the spectral parameter. Although this matrix spectral problem is not a reduction of the AKNS system (cf. [22]), it nonetheless generates an integrable hierarchy. Each member of this hierarchy admits a bi-Hamiltonian formulation and displays a combined structural feature.

The construction of an integrable hierarchy through the Lax pair approach commences with the determination of a particular series solution to the stationary zero-curvature Equation (2). We consider a solution Y of the specific form

$$Y = \begin{bmatrix} \sigma_1 a & \sigma_1 b & e & f \\ \sigma_1 c & -\sigma_1 a & -f & g \\ \sigma_1\sigma_2 g & -\sigma_1\sigma_2 f & -\sigma_1 a & \sigma_1 c \\ \sigma_1\sigma_2 f & \sigma_1\sigma_2 e & \sigma_1 b & \sigma_1 a \end{bmatrix} = \sum_{n\geq 0} \lambda^{-n} Y^{[n]}. \tag{14}$$

This matrix form is chosen because the commutator between the spectral matrix $\mathcal{M}$ and any element of the matrix Lie algebra $\tilde{g}$ must also lie in the algebraic structure. Upon substitution into the stationary zero-curvature Equation (2), we obtain the following recursive system governing the coefficients of the Laurent series solution:

$$\begin{cases} a_x = \sigma_1 c v_1 + \sigma_2 g v_2 - \sigma_1 b v_3 - \sigma_2 e v_4, \\ b_x = \alpha\sigma_2\lambda e - 2\sigma_1 a v_1 - 2\sigma_2 f v_2, \\ c_x = -\alpha\sigma_2\lambda g + 2\sigma_1 a v_3 + 2\sigma_2 f v_4, \end{cases} \tag{15}$$

$$\begin{cases} e_x = \alpha\sigma_1\lambda b - 2\sigma_1 f v_1 - 2\sigma_1 a v_2, \\ g_x = -\alpha\sigma_1\lambda c + 2\sigma_1 f v_3 + 2\sigma_1 a v_4, \\ f_x = \sigma_1 g v_1 + \sigma_1 c v_2 - \sigma_1 e v_3 - \sigma_1 b v_4, \end{cases} \tag{16}$$

where $\alpha = \alpha_1 - \alpha_2$. Assuming that the functions a, b, c, e, f, g admit Laurent series expansions of the form

$$\begin{cases} a = \sum_{n\leq 0} a^{[-n]}\lambda^n,\ b = \sum_{n\leq 0} b^{[-n]}\lambda^n,\ c = \sum_{n\leq 0} c^{[-n]}\lambda^n, \\ e = \sum_{n\leq 0} e^{[-n]}\lambda^n,\ f = \sum_{n\leq 0} f^{[-n]}\lambda^n,\ g = \sum_{n\leq 0} g^{[-n]}\lambda^n, \end{cases} \tag{17}$$

we deduce the initial conditions

$$\partial_x a^{[0]} = 0,\ b^{[0]} = c^{[0]} = e^{[0]} = g^{[0]} = 0,\ \partial_x f^{[0]} = 0, \tag{18}$$

and the following recursion relations for $n \geq 0$:

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha\sigma_1}(e_x^{[n]} + 2\sigma_1 f^{[n]} v_1 + 2\sigma_1 a^{[n]} v_2), \\ c^{[n+1]} = -\frac{1}{\alpha\sigma_1}(g_x^{[n]} - 2\sigma_1 f^{[n]} v_3 - 2\sigma_1 a^{[n]} v_4), \end{cases} \tag{19}$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha\sigma_2}(b_x^{[n]} + 2\sigma_1 a^{[n]} v_1 + 2\sigma_2 f^{[n]} v_2), \\ g^{[n+1]} = -\frac{1}{\alpha\sigma_2}(c_x^{[n]} - 2\sigma_1 a^{[n]} v_3 - 2\sigma_2 f^{[n]} v_4), \end{cases} \tag{20}$$

$$\begin{cases} a_x^{[n+1]} = \sigma_1 c^{[n+1]} v_1 + \sigma_2 g^{[n+1]} v_2 - \sigma_1 b^{[n+1]} v_3 - \sigma_2 e^{[n+1]} v_4, \\ f_x^{[n+1]} = \sigma_1 g^{[n+1]} v_1 + \sigma_1 c^{[n+1]} v_2 - \sigma_1 e^{[n+1]} v_3 - \sigma_1 b^{[n+1]} v_4. \end{cases} \tag{21}$$

To determine Y explicitly, we take the initial values,

$$a^{[0]} = \frac{1}{2}\beta, \; f^{[0]} = \frac{1}{2}\gamma. \tag{22}$$

Within this framework, the two constant parameters β and γ are arbitrary but not both zero simultaneously; that is, at least one of them must be non-zero to ensure a nontrivial structure in the system. Additionally, we impose that the constants of integration vanish:

$$a^{[n]}|_{v=0} = 0, \; f^{[n]}|_{v=0} = 0, \; n \geq 1. \tag{23}$$

Under (22) and (23), with symbolic computations, we can systematically derive the first few coefficients of the expansion, explicitly computing up to order $n = 4$, including second and third derivatives, nonlinear products, and coupling between components:

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}(\beta v_2 + \gamma v_1), \; c^{[1]} = \frac{1}{\alpha}(\beta v_4 + \gamma v_3), \\ e^{[1]} = \frac{1}{\alpha\sigma_2}(\sigma_1 \beta v_1 + \sigma_2 \gamma v_2), \; g^{[1]} = \frac{1}{\alpha\sigma_2}(\sigma_1 \beta v_3 + \sigma_2 \gamma v_4), \\ a^{[1]} = 0, \; f^{[1]} = 0; \end{cases}$$

$$\begin{cases} b^{[2]} = \frac{1}{\alpha^2\sigma_1\sigma_2}(\sigma_1 \beta v_{1,x} + \sigma_2 \gamma v_{2,x}), \; c^{[2]} = -\frac{1}{\alpha^2\sigma_1\sigma_2}(\sigma_1 \beta v_{3,x} + \sigma_2 \gamma v_{4,x}), \\ e^{[2]} = \frac{1}{\alpha^2\sigma_2}(\gamma v_{1,x} + \beta v_{2,x}), \; g^{[2]} = -\frac{1}{\alpha^2\sigma_2}(\gamma v_{3,x} + \beta v_{4,x}), \end{cases}$$

$$\begin{cases} a^{[2]} = -\frac{1}{\alpha^2\sigma_2}[(\sigma_1 \beta v_3 + \sigma_2 \gamma v_4) v_1 + \sigma_2(\gamma v_3 + \beta v_4) v_2], \\ f^{[2]} = -\frac{1}{\alpha^2\sigma_2}[\sigma_1(\gamma v_1 + \beta v_2) v_3 + (\sigma_1 \beta v_1 + \sigma_2 \gamma v_2) v_4]; \end{cases}$$

$$\begin{cases} b^{[3]} = \frac{1}{\alpha^3\sigma_1\sigma_2}[\gamma v_{1,xx} + \beta v_{2,xx} - 2\sigma_1^2(\gamma v_3 + \beta v_4) v_1^2 - 4\sigma_1(\sigma_1 \beta v_3 + \sigma_2 \gamma v_4) v_1 v_2 \\ \qquad -2\sigma_1\sigma_2(\gamma v_3 + \beta v_4) v_2^2], \\ c^{[3]} = \frac{1}{\alpha^3\sigma_1\sigma_2}[\gamma v_{3,xx} + \beta v_{4,xx} - 2\sigma_1^2(\gamma v_1 + \beta v_2) v_3^2 - 4\sigma_1(\sigma_1 \beta v_1 + \sigma_2 \gamma v_2) v_3 v_4 \\ \qquad -2\sigma_1\sigma_2(\gamma v_1 + \beta v_2) v_4^2], \end{cases}$$

$$\begin{cases} e^{[3]} = \frac{1}{\alpha^3\sigma_1\sigma_2^2}[\sigma_1 \beta v_{1,xx} + \sigma_2 \gamma v_{2,xx} - 2\sigma_1^2(\sigma_1 \beta v_3 + \sigma_2 \gamma v_4) v_1^2 - 4\sigma_1^2\sigma_2(\gamma v_3 + \beta v_4) v_1 v_2 \\ \qquad -2\sigma_1\sigma_2(\sigma_1 \beta v_3 + \sigma_2 \gamma v_4) v_2^2], \\ g^{[3]} = \frac{1}{\alpha^3\sigma_1\sigma_2^2}[\sigma_1 \beta v_{3,xx} + \sigma_2 \gamma v_{4,xx} - 2\sigma_1^2(\sigma_1 \beta v_1 + \sigma_2 \gamma v_2) v_3^2 - 4\sigma_1^2\sigma_2(\gamma v_1 + \beta v_2) v_3 v_4 \\ \qquad -2\sigma_1\sigma_2(\sigma_1 \beta v_1 + \sigma_2 \gamma v_2) v_4^2], \end{cases}$$

$$\begin{cases} a^{[3]} = \frac{1}{\alpha^3\sigma_1\sigma_2}[-\sigma_1(\gamma v_3+\beta v_4)v_{1,x} - (\sigma_1\beta v_3+\sigma_2\gamma v_4)v_{2,x} + \sigma_1(\gamma v_1+\beta v_2)v_{3,x} \\ \qquad +(\sigma_1\beta v_1+\sigma_2\gamma v_2)v_{4,x}], \\ f^{[3]} = \frac{1}{\alpha^3\sigma_2^2}[-(\sigma_1\beta v_3+\sigma_2\gamma v_4)v_{1,x} - \sigma_2(\gamma v_3+\beta v_4)v_{2,x} + (\sigma_1\beta v_1+\sigma_2\gamma v_2)v_{3,x} \\ \qquad +\sigma_2(\gamma v_1+\beta v_2)v_{4,x}]; \end{cases}$$

and

$$\begin{cases} b^{[4]} = \frac{1}{\alpha^4\sigma_1^2\sigma_2^2}\{\sigma_1\beta v_{1,xxx} + \sigma_2\gamma v_{2,xxx} - 6\sigma_1^2[(\sigma_1\beta v_3+\sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3+\beta v_4)v_2]v_{1,x} \\ \qquad -6\sigma_1\sigma_2[\sigma_1(\gamma v_3+\beta v_4)v_1 + (\sigma_1\beta v_3+\sigma_2\gamma v_4)v_2]v_{2,x}\}, \\ c^{[4]} = \frac{1}{\alpha^4\sigma_1^2\sigma_2^2}\{-\sigma_1\beta v_{3,xxx} - \sigma_2\gamma v_{4,xxx} + 6\sigma_1^2[(\sigma_1\beta v_3+\sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3+\beta v_4)v_2]v_{3,x} \\ \qquad +6\sigma_1\sigma_2[\sigma_1(\gamma v_3+\beta v_4)v_1 + (\sigma_1\beta v_3+\sigma_2\gamma v_4)v_2]v_{4,x}\}, \end{cases}$$

$$\begin{cases} e^{[4]} = \frac{1}{\alpha^4\sigma_1\sigma_2^2}\{\gamma v_{1,xxx} + \beta v_{2,xxx} - 6\sigma_1[\sigma_1(\gamma v_3+\beta v_4)v_1 + (\sigma_1\beta v_3+\sigma_2\gamma v_4)v_2]v_{1,x} \\ \qquad -6\sigma_1[(\sigma_1\beta v_3+\sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3+\beta v_4)v_2]v_{2,x}\}, \\ g^{[4]} = \frac{1}{\alpha^4\sigma_1\sigma_2^2}\{-\gamma v_{3,xxx} - \beta v_{4,xxx} + 6\sigma_1[\sigma_1(\gamma v_3+\beta v_4)v_1 + (\sigma_1\beta v_3+\sigma_2\gamma v_4)v_2]v_{3,x} \\ \qquad +6\sigma_1[(\sigma_1\beta v_3+\sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3+\beta v_4)v_2]v_{4,x}\}, \end{cases}$$

$$\begin{aligned} a^{[4]} = \frac{1}{\alpha^4\sigma_1\sigma_2^2}[&-(\sigma_1\beta v_3+\sigma_2\gamma v_4)v_{1,xx} - \sigma_2(\gamma v_3+\beta v_4)v_{2,xx} - (\sigma_1\beta v_1+\sigma_2\gamma v_2)v_{3,xx} \\ &-\sigma_2(\gamma v_1+\beta v_2)v_{4,xx} + (\sigma_1\beta v_{3,x}+\sigma_2\gamma v_{4,x})v_{1,x} + \sigma_2(\gamma v_{3,x}+\beta v_{4,x})v_{2,x} \\ &+3\sigma_1^2(\sigma_1\beta v_3^2 + 2\sigma_2\gamma v_3v_4 + \sigma_2\beta v_4^2)v_1^2 + 6\sigma_1\sigma_2(\sigma_1\gamma v_3^2 + 2\sigma_1\beta v_3v_4 + \sigma_2\gamma v_4^2)v_1v_2 \\ &+3\sigma_1\sigma_2(\sigma_1\beta v_3^2 + 2\sigma_2\gamma v_3v_4 + \sigma_2\beta v_4^2)v_2^2], \\ f^{[4]} = \frac{1}{\alpha^4\sigma_1\sigma_2^2}[&-\sigma_1(\gamma v_3+\beta v_4)v_{1,xx} - (\sigma_1\beta v_3+\sigma_2\gamma v_4)v_{2,xx} - \sigma_1(\gamma v_1+\beta v_2)v_{3,xx} \\ &-(\sigma_1\beta v_1+\sigma_2\gamma v_2)v_{4,xx} + \sigma_1(\gamma v_{3,x}+\beta v_{4,x})v_{1,x} + (\sigma_1\beta v_{3,x}+\sigma_2\gamma v_{4,x})v_{2,x} \\ &+3\sigma_1^2(\sigma_1\gamma v_3^2 + 2\sigma_1\beta v_3v_4 + \sigma_2\gamma v_4^2)v_1^2 + 6\sigma_1^2(\sigma_1\beta v_3^2 + 2\sigma_2\gamma v_3v_4 + \sigma_2\beta v_4^2)v_1v_2 \\ &+3\sigma_1\sigma_2(\sigma_1\gamma v_3^2 + 2\sigma_1\beta v_3v_4 + \sigma_2\gamma v_4^2)v_2^2]. \end{aligned}$$

By setting $\Delta_m = 0$ for all $m \geq 0$, based on the above symbolic computations which guarantee closure within the loop algebra, we take the temporal part of the matrix spectral problems to be

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(v,\lambda)\varphi,\ m \geq 0, \tag{24}$$

where the Lax matrices are defined by

$$\mathcal{N}^{[m]} = (\lambda^m Y)_+ = Y^{[m]} + \lambda Y^{[m-1]} + \cdots + \lambda^m Y^{[0]}, \tag{25}$$

where $m \geq 0$. This construction establishes the temporal matrix spectral problems as an integral part of the zero-curvature framework. Compatibility between the spatial and temporal problems (12) and (24) is ensured by the zero-curvature Equation (6), giving rise to an integrable hierarchy involving four dependent potentials:

$$v_{t_m} = P^{[m]} = P^{[m]}(v) = (P_1^{[m]}, P_2^{[m]}, P_3^{[m]}, P_4^{[m]})^T,\ m \geq 0. \tag{26}$$

where

$$P^{[m]} = (\alpha\sigma_2 e^{[m+1]}, \alpha\sigma_1 b^{[m+1]}, -\alpha\sigma_2 g^{[m+1]}, -\alpha\sigma_1 c^{[m+1]})^T,\ m \geq 0. \tag{27}$$

More explicitly, the hierarchy splits into the subsystems

$$\begin{cases} v_{1,t_m} = P_1^{[m]} = \alpha\sigma_2 e^{[m+1]}, \\ v_{2,t_m} = P_2^{[m]} = \alpha\sigma_1 b^{[m+1]}, \end{cases} m \geq 0, \tag{28}$$

$$\begin{cases} v_{3,t_m} = P_3^{[m]} = -\alpha\sigma_2 g^{[m+1]}, \\ v_{4,t_m} = P_4^{[m]} = -\alpha\sigma_1 c^{[m+1]}, \end{cases} m \geq 0, \tag{29}$$

thus providing an explicit description of the integrable hierarchy flows.

We can illustrate the above integrable hierarchy by examining some specific examples. The first nonlinear example corresponds to a combined integrable system of NLS-type equations:

$$\begin{cases} v_{1,t_2} = \frac{1}{\alpha^2\sigma_1\sigma_2}[\sigma_1\beta v_{1,xx} + \sigma_2\gamma v_{2,xx} - 2\sigma_1(\sigma_1\beta v_3 + \sigma_2\gamma v_4)(\sigma_1 v_1^2 + \sigma_2 v_2^2) \\ \qquad -4\sigma_1^2\sigma_2(\gamma v_3 + \beta v_4)v_1 v_2], \\ v_{2,t_2} = \frac{1}{\alpha^2\sigma_2}[\gamma v_{1,xx} + \beta v_{2,xx} - 2\sigma_1(\gamma v_3 + \beta v_4)(\sigma_1 v_1^2 + \sigma_2 v_2^2) \\ \qquad -4\sigma_1(\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_1 v_2], \end{cases} \tag{30}$$

$$\begin{cases} v_{3,t_2} = -\frac{1}{\alpha^2\sigma_1\sigma_2}[\sigma_1\beta v_{3,xx} + \sigma_2\gamma v_{4,xx} - 2\sigma_1(\sigma_1\beta v_1 + \sigma_2\gamma v_2)(\sigma_1 v_3^2 + \sigma_2 v_4^2) \\ \qquad -4\sigma_1^2\sigma_2(\gamma v_1 + \beta v_2)v_3 v_4], \\ v_{4,t_2} = -\frac{1}{\alpha^2\sigma_2}[\gamma v_{3,xx} + \beta v_{4,xx} - 2\sigma_1(\gamma v_1 + \beta v_2)(\sigma_1 v_3^2 + \sigma_2 v_4^2) \\ \qquad -4\sigma_1(\sigma_1\beta v_1 + \sigma_2\gamma v_2)v_3 v_4]. \end{cases} \tag{31}$$

The second example corresponds to a combined integrable system of mKdV-type equations:

$$\begin{cases} v_{1,t_3} = \frac{1}{\alpha^3\sigma_1\sigma_2}\{\gamma v_{1,xxx} + \beta v_{2,xxx} - 6\sigma_1[\sigma_1(\gamma v_3 + \beta v_4)v_1 + (\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_2]v_{1,x} \\ \qquad -6\sigma_1[(\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3 + \beta v_4)v_2]v_{2,x}\}, \\ v_{2,t_3} = \frac{1}{\alpha^3\sigma_1\sigma_2^2}\{\sigma_1\beta v_{1,xxx} + \sigma_2\gamma v_{2,xxx} - 6\sigma_1^2[(\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3 + \beta v_4)v_2]v_{1,x} \\ \qquad -6\sigma_1\sigma_2[\sigma_1(\gamma v_3 + \beta v_4)v_1 + (\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_2]v_{2,x}\}, \end{cases} \tag{32}$$

$$\begin{cases} v_{3,t_3} = -\frac{1}{\alpha^3\sigma_1\sigma_2}\{-\gamma v_{3,xxx} - \beta v_{4,xxx} + 6\sigma_1[\sigma_1(\gamma v_3 + \beta v_4)v_1 + (\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_2]v_{3,x} \\ \qquad +6\sigma_1[(\sigma_1\beta v_3 + \delta_2\gamma v_4)v_1 + \sigma_2(\gamma v_3 + \beta v_4)v_2]v_{4,x}\}, \\ v_{4,t_3} = -\frac{1}{\alpha^3\sigma_1\sigma_2^2}\{-\sigma_1\beta v_{3,xxx} - \sigma_2\gamma v_{4,xxx} + 6\sigma_1^2[(\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_1 + \sigma_2(\gamma v_3 + \beta v_4)v_2]v_{3,x} \\ \qquad +6\sigma_1\sigma_2[\sigma_1(\gamma v_3 + \beta v_4)v_1 + (\sigma_1\beta v_3 + \sigma_2\gamma v_4)v_2]v_{4,x}\}. \end{cases} \tag{33}$$

The presented systems extend coupled NLS- and mKdV-type integrable models (cf. [23]) by featuring two distinct highest-order derivative terms in each equation, which motivates referring to them as "combined models."

Two distinct reduced hierarchies arise by specializing the parameters to $\beta = 1, \gamma = 0$ and $\beta = 0, \gamma = 1$, respectively. These particular choices yield reduced hierarchies consisting of uncombined integrable models, simplifying the structure and separating the coupled dynamics present in the general combined hierarchy. Each choice leads to a separate integrable subsystem that highlights different nonlinear wave interactions within the four-component framework. These reductions not only simplify the general model but also preserve its integrability properties, allowing for explicit construction of bi-Hamiltonian structures and soliton solutions.

In particular, by choosing $\alpha = \sigma_1 = \sigma_2 = 1$, and setting $\beta = 1$ and $\gamma = 0$, the system of (30) and (31) simplifies to a reduced hierarchy corresponding to a coupled NLS-type integrable model:

$$\begin{cases} v_{1,t_2} = v_{1,xx} - 2v_3(v_1^2 + v_2^2) - 4v_1v_2v_4, \\ v_{2,t_2} = v_{2,xx} - 2v_4(v_1^2 + v_2^2) - 4v_1v_2v_3, \end{cases} \tag{34}$$

$$\begin{cases} v_{3,t_2} = -v_{3,xx} + 2v_1(v_3^2 + v_4^2) + 4v_2v_3v_4, \\ v_{4,t_2} = -v_{4,xx} + 2v_2(v_3^2 + v_4^2) + 4v_1v_3v_4. \end{cases} \tag{35}$$

Likewise, taking $\alpha = \sigma_1 = \sigma_2 = 1$, $\beta = 0$ and $\gamma = 1$, we arrive at another coupled NLS-type integrable model:

$$\begin{cases} v_{1,t_2} = v_{2,xx} - 2v_4(v_1^2 + v_2^2) - 4v_1v_2v_3, \\ v_{2,t_2} = v_{1,xx} - 2u_3(v_1^2 + v_2^2) - 4v_1v_2v_4, \end{cases} \tag{36}$$

$$\begin{cases} v_{3,t_2} = -v_{4,xx} + 2v_2(v_3^2 + v_4^2) + 4v_1v_3v_4, \\ v_{4,t_2} = -v_{3,xx} + 2v_1(v_3^2 + v_4^2) + 4v_2v_3v_4. \end{cases} \tag{37}$$

Turning to the mKdV-type systems, setting $\alpha = \sigma_1 = \sigma_2 = 1$, $\beta = 1$ and $\gamma = 0$ in (32) and (33), we obtain

$$\begin{cases} v_{1,t_3} = v_{2,xxx} - 6(v_1v_4 + v_2v_3)v_{1,x} - 6(v_1v_3 + v_2v_4)v_{2,x}, \\ v_{2,t_3} = v_{1,xxx} - 6(v_1v_3 + v_2v_4)v_{1,x} - 6(v_1v_4 + v_2v_3)v_{2,x}, \end{cases} \tag{38}$$

$$\begin{cases} v_{3,t_3} = v_{4,xxx} - 6(v_1v_4 + v_2v_3)v_{3,x} - 6(v_1v_3 + v_2v_4)v_{4,x}, \\ v_{4,t_3} = v_{3,xxx} - 6(v_1v_3 + v_2v_4)v_{3,x} - 6(v_1v_4 + v_2v_3)v_{4,x}. \end{cases} \tag{39}$$

Similarly, setting $\alpha = \sigma_1 = \sigma_2 = 1$, $\beta = 0$ and $\gamma = 1$ in the model of (32) and (33), we derive the alternative coupled mKdV-type integrable model:

$$\begin{cases} v_{1,t_3} = v_{1,xxx} - 6(v_1v_3 + v_2v_4)v_{1,x} - 6(v_1v_4 + v_2v_3)v_{2,x}, \\ v_{2,t_3} = v_{2,xxx} - 6(v_1v_4 + v_2v_3)v_{1,x} - 6(v_1v_3 + v_2v_4)v_{2,x}, \end{cases} \tag{40}$$

$$\begin{cases} v_{3,t_3} = v_{3,xxx} - 6(v_1v_3 + v_2v_4)v_{3,x} - 6(v_1v_4 + v_2v_3)v_{4,x}, \\ v_{4,t_3} = v_{4,xxx} - 6(v_1v_4 + v_2v_3)v_{3,x} - 6(v_1v_3 + v_2v_4)v_{4,x}. \end{cases} \tag{41}$$

These reduced systems differ from the vector AKNS-type integrable models discussed in [22,24,25]. Each pair of reduced models is connected by a component-swapping symmetry, whereby the vector fields on the right-hand sides are transformed by interchanging the first and second components as well as the third and fourth components. Despite this structural variation, all four integrable systems remain mutually commuting, preserving the integrability of the hierarchy under these reductions. These systems are anticipated to have potential applications in engineering problems, for example, as second- and third-order perturbed models of nonlinear motion, highlighting one of the key advantages of the proposed theoretical framework.

3. Hereditary Recursion Operator and Bi-Hamiltonian Structure

To construct a bi-Hamiltonian structure [26] and establish the Liouville integrability of the soliton hierarchy given by (28) and (29), we employ the trace identity (8), which is linked to the spatial part of the Lax pair defined in (12) and (13).

Substituting $\mathcal{M}$ from (12) together with the particular series solution Y from (14) into the trace identity yields the relation

$$\frac{\delta}{\delta v} \int \lambda^{-(n+1)} \alpha \sigma_2 f^{[n+1]}\, dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa - n} (\sigma_1 c^{[n]}, \sigma_2 g^{[n]}, \sigma_1 b^{[n]}, \sigma_2 e^{[n]})^T,\ n \geq 0. \tag{42}$$

This identity relies on the evaluations

$$\begin{cases} \operatorname{tr}\big(Y\dfrac{\partial \mathcal{M}}{\partial v}\big) = (2\sigma_1^2 c, 2\sigma_1\sigma_2 g, 2\sigma_1^2 b, 2\sigma_1\sigma_2 e)^T, \\ \operatorname{tr}\big(Y\dfrac{\partial \mathcal{M}}{\partial \lambda}\big) = 2\alpha\sigma_1\sigma_2 f. \end{cases} \tag{43}$$

A check with $n = 2$ determines that $\kappa = 0$, and thus we arrive at the variational identities

$$\frac{\delta}{\delta v_1}\mathcal{H}^{[n]} = \sigma_1 c^{[n+1]},\ \frac{\delta}{\delta v_2}\mathcal{H}^{[n]} = \sigma_2 g^{[n+1]},\ \frac{\delta}{\delta v_3}\mathcal{H}^{[n]} = \sigma_1 b^{[n+1]},\ \frac{\delta}{\delta v_4}\mathcal{H}^{[n]} = \sigma_2 e^{[n+1]},\ n \geq 0, \tag{44}$$

where the Hamiltonian functionals are defined by

$$\mathcal{H}^{[n]} = -\int \frac{\alpha\sigma_2}{n+1} f^{[n+2]}\, dx,\ n \geq 0. \tag{45}$$

The identities presented in (44) yield a Hamiltonian representation of the integrable hierarchy, highlighting its underlying geometric structure:

$$v_{t_m} = P^{[m]}(v) = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta v},\ m \geq 0, \tag{46}$$

where the Hamiltonian operator J_1 is given by

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{bmatrix}, \tag{47}$$

and $\mathcal{H}^{[m]}$ are the functionals determined by (45). This structure implies a fundamental relation between symmetries and conserved quantities in the hierarchy: any symmetry P of the system satisfies $P = J_1 \frac{\delta \mathcal{H}}{\delta v}$ for some conserved functional $\mathcal{H}$, consistent with the general theory of Hamiltonian systems.

The vector fields $P^{[n]}$ give rise to commuting flows, as they span an abelian Lie algebra:

$$[\![P^{[n_1]}, P^{[n_2]}]\!] = 0,\ n_1, n_2 \geq 0. \tag{48}$$

Here, the commutator bracket is defined by

$$[\![Q^{[1]}, Q^{[2]}]\!] = Q^{[1]\prime}(v)[Q^{[2]}] - Q^{[2]\prime}(v)[Q^{[1]}], \tag{49}$$

where the prime denotes the Gateaux derivative. This commutativity of flows generated by $P^{[n]}$ is a consequence of the abelian structure of the associated temporal Lax matrices $\mathcal{N}^{[n]}$, which satisfy

$$[\![\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}]\!] = 0,\ n_1, n_2 \geq 0, \tag{50}$$

with the bracket defined as

$$[\![\mathcal{M}^{[1]}, \mathcal{M}^{[2]}]\!] = \mathcal{M}^{[1]\prime}(v)[M^{[2]}] - \mathcal{M}^{[2]\prime}(v)[M^{[1]}] + [\mathcal{M}^{[1]}, \mathcal{M}^{[2]}], \tag{51}$$

where again the prime denotes the Gateaux derivative and the last term represents the usual matrix commutator. This reflects the integrability structure inherent in the isospectral zero-curvature representation, where the compatibility conditions ensure that all flows in the hierarchy commute. For further discussion and a detailed algebraic treatment of such zero-curvature hierarchies, see [27].

To derive a Magri recursion formulation of the hierarchy specified by (28) and (29), we define a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ that links successive flows via the recursion formula:

$$P^{[m+1]} = \Phi P^{[m]}.$$

Direct computation confirms that Φ is hereditary (see [28] for more discussion) and its entries are explicitly given as follows:

$$\begin{cases} \Phi_{11} = \frac{1}{\alpha}(-2\sigma_1 v_1 \partial^{-1} v_4 - 2\sigma_1 v_2 \partial^{-1} v_3), \\ \Phi_{12} = \frac{1}{\alpha}(\frac{1}{\sigma_1}\partial_x - 2\sigma_1 v_1 \partial^{-1} v_3 - 2\sigma_2 v_2 \partial^{-1} v_4), \\ \Phi_{13} = \frac{1}{\alpha}(-2\sigma_1 v_1 \partial^{-1} v_2 - 2\sigma_1 v_2 \partial^{-1} v_1), \\ \Phi_{14} = \frac{1}{\alpha}(-2\sigma_1 v_1 \partial^{-1} v_1 - 2\sigma_2 v_2 \partial^{-1} v_2); \end{cases} \tag{52}$$

$$\begin{cases} \Phi_{21} = \frac{1}{\alpha}(\frac{1}{\sigma_2}\partial_x - \frac{2}{\sigma_2}\sigma_1^2 v_1 \partial^{-1} v_3 - 2\sigma_1 v_2 \partial^{-1} v_4), \\ \Phi_{22} = \frac{1}{\alpha}(-2\sigma_1 v_1 \partial^{-1} v_4 - 2\sigma_1 v_2 \partial^{-1} v_3), \\ \Phi_{23} = \frac{1}{\alpha}(-\frac{2}{\sigma_2}\sigma_1^2 v_1 \partial^{-1} v_1 - 2\sigma_1 v_2 \partial^{-1} v_2), \\ \Phi_{24} = \frac{1}{\alpha}(-2\sigma_1 v_1 \partial^{-1} v_2 - 2\sigma_1 v_2 \partial^{-1} v_1); \end{cases} \tag{53}$$

$$\begin{cases} \Phi_{31} = \frac{1}{\alpha}(2\sigma_1 v_3 \partial^{-1} v_4 + 2\sigma_1 v_4 \partial^{-1} v_3), \\ \Phi_{32} = \frac{1}{\alpha}(2\sigma_1 v_3 \partial^{-1} v_3 + 2\sigma_2 v_4 \partial^{-1} v_4), \\ \Phi_{33} = \frac{1}{\alpha}(2\sigma_1 v_3 \partial^{-1} v_2 + 2\sigma_1 v_4 \partial^{-1} v_1), \\ \Phi_{34} = \frac{1}{\alpha}(-\frac{1}{\sigma_1}\partial_x + 2\sigma_1 v_3 \partial^{-1} v_1 + 2\sigma_2 v_4 \partial^{-1} v_2); \end{cases} \tag{54}$$

$$\begin{cases} \Phi_{41} = \frac{1}{\alpha}(\frac{2}{\sigma_2}\sigma_1^2 v_3 \partial^{-1} v_3 + 2\sigma_1 v_4 \partial^{-1} v_4), \\ \Phi_{42} = \frac{1}{\alpha}(2\sigma_1 v_3 \partial^{-1} v_4 + 2\sigma_1 v_4 \partial^{-1} v_3), \\ \Phi_{43} = \frac{1}{\alpha}(-\frac{1}{\sigma_2}\partial_x + \frac{2}{\sigma_2}\sigma_1^2 v_3 \partial^{-1} v_1 + 2\sigma_1 v_4 \partial^{-1} v_2), \\ \Phi_{44} = \frac{1}{\alpha}(2\sigma_1 v_3 \partial^{-1} v_2 + 2\sigma_1 v_4 \partial^{-1} v_1). \end{cases} \tag{55}$$

The hereditariness of Φ implies it satisfies the identity

$$\Phi'(v)[\Phi Q^{[1]}]Q^{[2]} - \Phi'(v)[\Phi Q^{[2]}]Q^{[1]} - \Phi\Phi'(v)[Q^{[1]}]Q^{[2]} + \Phi\Phi'(v)[Q^{[2]}]Q^{[1]} = 0 \tag{56}$$

for arbitrary vector fields $Q^{[1]}$ and $Q^{[2]}$. This property is essential for generating an infinite hierarchy of commuting flows.

Furthermore, the operators J_1 and $J_2 = \Phi J_1$ can be shown to form a Hamiltonian pair, indicating the compatibility of these Poisson structures and ensuring that any linear combination of them remains a valid Hamiltonian operator. Therefore, the hierarchy admits a bi-Hamiltonian structure [26]:

$$v_{t_m} = P^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta v} = J_2 \frac{\delta \mathcal{H}^{[m-1]}}{\delta v}, \ \ m \geq 1. \tag{57}$$

In addition, the associated Hamiltonian functionals commute:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = 0, \tag{58}$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = 0, \tag{59}$$

for all $n_1, n_2 \geq 0$, with respect to both Poisson brackets induced by the two Hamiltonial operators:

$$\{\mathcal{I}^{[1]}, \mathcal{I}^{[2]}\}_{J_j} = \int (\frac{\delta \mathcal{I}^{[1]}}{\delta v})^T J_j \frac{\delta \mathcal{I}^{[2]}}{\delta v}\, dx,\ j = 1, 2. \tag{60}$$

This mutual commutativity of conserved quantities is a key feature of Liouville integrability [6].

Thus, the entire hierarchy characterized by (28) and (29) is Liouville integrable, with a bi-Hamiltonian formulation complemented by a hereditary recursion operator that produces two infinite sequences of commuting symmetries and conserved quantities. As illustrative examples, we present two representative nonlinear, combined Liouville integrable Hamiltonian systems, explicitly formulated as the coupled NLS-type model in (30) and (31) and the coupled mKdV-type model in (32) and (33).

Based on the established symmetries, both the NLS-type and mKdV-type models admit a class of exact solutions parameterized by arbitrary integers $n_1, \cdots, n_k$:

$$v = \exp(\varepsilon_1 P^{[n_1]}) \cdots \exp(\varepsilon_k P^{[n_k]}) v_0,$$

where exp denotes the exponential map, $\varepsilon_1, \cdots, \varepsilon_n$ are small constants, and $P^{[n]}$ denotes the symmetry generator of order n. In the NLS-type case, the initial vector v_0 can be chosen as

$$v_0 = (\sqrt{\frac{\sigma_2}{\sigma_1}}\, v_{2,0}, v_{2,0}, -\sqrt{\frac{\sigma_2}{\sigma_1}}\, v_{4,0}, v_{4,0})^T,$$

where $v_{2,0}$ and $v_{4,0}$ are arbitrary constants. In the mKdV-type case, v_0 is an arbitrary constant vector:

$$v_0 = (v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0})^T.$$

4. Conclusions

A Liouville integrable hierarchy involving four distinct potentials has been constructed from a newly introduced 4×4 matrix spectral problem, along with its corresponding bi-Hamiltonian formulation. The foundation of this technique relies on a well-structured Laurent series solution that formally solves the associated stationary zero-curvature equation. The resulting integrable models involve four arbitrary constants, allowing for a diverse range of special four-component integrable models.

A deeper study into the algebraic and geometric properties characterizing the soliton solutions of these systems would provide valuable insights. Powerful analytical tools in soliton theory for this purpose include the Riemann–Hilbert problem framework (see, e.g., [29,30]), the Darboux transformation (see, e.g., [31–33]), the Zakharov–Shabat dressing method [34], and the determinant method [35,36]. All of these techniques are intimately connected to the associated Lax pairs.

Apart from solitons, nonlinear wave structures such as breathers, kinks, rogue waves and lumps, and their interactions (see, e.g., [37–46]), are also of considerable interest. These waves play important roles in various applications and often arise from general soliton solutions via suitable wave number reduction techniques.

In addition, applying nonlocal group reductions and similarity transformations to the matrix spectral problem offers a systematic method for deriving novel integrable models featuring reflection points. By employing established techniques from soliton theory, one can likewise determine the soliton-type solution structures and associated dynamical properties of these models.

Integrable models continue to attract significant attention due to their profound connections with diverse areas of mathematical physics, including advanced theories of nonlinear dynamics, algebraic geometry, and supersymmetry. Advancing research on multi-component integrable models promises to greatly deepen our understanding of complex nonlinear phenomena across both mathematics and physics.

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