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Diversity of interaction solutions to the (2+1)-dimensional Ito equation

Wen-Xiu Ma [a,b,c,d,*], Xuelin Yong [c,e], Hai-Qiang Zhang [c,f]

[a] College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China
[b] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China
[c] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
[d] Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa
[e] School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
[f] College of Science, P. O. Box 253, University of Shanghai for Science and Technology, Shanghai 200093, China

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ABSTRACT

We aim to show the diversity of interaction solutions to the (2+1)-dimensional Ito equation, based on its Hirota bilinear form. The proof is given through Maple symbolic computations. An interesting characteristic in the resulting interaction solutions is the involvement of an arbitrary function. Special cases lead to lump solutions, lump-soliton solutions and lump-kink solutions. Two illustrative examples of the resulting solutions are displayed by three-dimensional plots and contour plots.

1. Introduction

The Hirota formulation provides a direct approach to soliton equations [1]. The associated Hirota bilinear equations can be solved by the Wronskian technique and the resulting solutions are called Wronskian solutions [2,3]. Solitons, positons and complexitons are among typical Wronskian solutions [4,5]. Interaction solutions between two classes of such solutions are another class of Wronskian solutions, describing more diverse nonlinear physical phenomena [3]. Moreover, upon taking long wave limits, lump solutions, rationally localized solutions in all directions in space, can be computed from solitons [6–9]. Recently lumps solutions are generated through symbolic computations [10–14] and this raises more questions about interaction solutions. It is of current interest to study interaction solutions [3,15], particularly interaction solutions between lumps and either solitons or kinks [16,17]. More generally, if we start with lumps generated from quadratic functions, what interaction solutions can be formulated by combining other kinds of functions?

In this paper, we will focus on the (2+1)-dimensional Ito equation to show the diversity of such interaction solutions. The (2+1)-dimensional Ito equation reads

$$P_{Ito}(u,v) := u_{tt} + u_{xxxt} + 6u_x u_t + 3uu_{xt} + 3u_{xx}v_t + \alpha u_{yt} + \beta u_{xt} = 0, \tag{1.1}$$

where α and β are two given constants and $v_x = u$ [18–20]. It is known (see, e.g., [16]) that the Ito equation above possesses a Hirota bilinear form:

$$B_{Ito}(f) := (D_t^2 + D_x^3 D_t + \alpha D_y D_t + \beta D_x D_t) f \cdot f$$

* Corresponding author at: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA.
E-mail address: mawx@cas.usf.edu (W.-X. Ma).

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$$= 2\big[f_{tt}f - f_t^2 + f_{xxxt}f - 3f_{xxt}f_x + 3f_{xt}f_{xx} - f_{xxx}f_t + \alpha(f_{yt}f - f_yf_t) + \beta(f_{xt}f - f_xf_t)\big] = 0, \tag{1.2}$$

and the links are as follows:

$$u = 2(\ln f)_{xx} = \frac{2(f_{xx}f - f_x^2)}{f^2}, \ v = 2(\ln f)_x = \frac{2f_x}{f}. \tag{1.3}$$

Such characteristic transformations have been adopted in Bell polynomial theories of soliton equations (see, e.g., [21,22]), and a precise relation is

$$P_{Ito}(u, v) = \left[\frac{B_{Ito}(f)}{f^2}\right]_{xx}.$$

Thus, if f solves the bilinear Ito equation (1.2), then $u = 2(\ln f)_{xx}$ and $v = 2(\ln f)_x$ will solve the (2+1)-dimensional Ito equation (1.1).

A basic analysis with Maple symbolic computations can show that the (2+1)-dimensional bilinear Ito equation (1.2) has a class of quadratic function solutions given by

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \tag{1.4}$$

where the parameters need to satisfy

$$\beta a_1 + \alpha a_2 + a_3 = 0, \ \beta a_5 + \alpha a_6 + a_7 = 0, \ a_1a_3 + a_5a_7 = 0. \tag{1.5}$$

By the transformations in (1.3), this can generate a large class of lump solutions to the (2+1)-dimensional Ito equation (1.1). The condition

$$a_1a_6 - a_2a_5 \neq 0, \tag{1.6}$$

is necessary and sufficient for a function f by (1.4) to generate a lump solution through (1.3). Under this condition (1.6), we can solve the polynomial system

$$f_x(x(t), y(t)) = 0, \ f_y(x(t), y(t)) = 0, \tag{1.7}$$

to get all critical points of the function f:

$$x(t) = \frac{(a_2a_7 - a_3a_6)t + (a_2a_8 - a_4a_6)}{a_1a_6 - a_2a_5}, \ y(t) = -\frac{(a_1a_7 - a_3a_5)t + (a_1a_8 - a_4a_5)}{a_1a_6 - a_2a_5}, \tag{1.8}$$

at a fixed time t. Obviously, the sum of two squares, $f - a_9$, vanishes at this set of critical points, and so $f > 0$ if and only if $a_9 > 0$. This way, u and v defined by (1.3) are analytical if and only if $a_9 > 0$. At any given time t, $(x(t), y(t))$ determined by (1.8) is also a critical point in the (x, y)-plane for the function $u = 2(\ln f)_{xx}$. Thus, based on the second partial derivative test, the lump solution u has a peak at the critical point $(x(t), y(t))$, because we have

$$u_{xx} = -\frac{24(a_1^2 + a_5^2)^2}{a_9^2} < 0, \ u_{xx}u_{yy} - u_{xy}^2 = \frac{192(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^2}{a_9^4} > 0$$

at the critical point $(x(t), y(t))$. The traveling speeds of the peak in the x- and y-directions, which can be computed from (1.8), are constant.

The resulting lump solutions above cover the two classes of lump solutions previously presented by symbolic computations in [16]:

$$u_1 = \frac{4\,a_5^2}{f_1} - \frac{8\,a_5^2(a_5x - \frac{\beta\,a_5}{\alpha}y + a_8)^2}{f_1^2}, \ v_1 = \frac{4\,a_5(a_5x - \frac{\beta\,a_5}{\alpha}y + a_8)}{f_1},$$

where

$$f_1 = (-\frac{a_3}{\alpha}y + a_3t + a_4)^2 + (a_5x - \frac{\beta\,a_5}{\alpha}y + a_8)^2 + a_9;$$

and

$$\begin{cases} u_2 = \dfrac{4\,a_1^2(a_3^2 + a_7^2)}{a_7^2f_2} - \dfrac{8\,a_1^2\big[a_1(a_3^2 + a_7^2)(\alpha\,x - \beta\,y) - \alpha\,a_7(a_3a_8 - a_4a_7)\big]^2}{\alpha^2\,a_7^4f_2^2}, \\ v_2 = \dfrac{4\,a_1\big[a_1(a_3^2 + a_7^2)(\alpha\,x - \beta\,y) - \alpha\,a_7(a_3a_8 - a_4a_7)\big]}{\alpha\,a_7f_2}, \end{cases}$$

where

$$f_2 = \big(a_1x - \frac{\beta\,a_1 + a_3}{\alpha}y + a_3t + a_4\big)^2 + \big(-\frac{a_1a_3}{a_7}x + \frac{\beta\,a_1a_3 - a_7^2}{\alpha\,a_7}y + a_7t + a_8\big)^2 + a_9.$$

In this paper, we would like to analyze interaction solutions between lump solutions and other kinds of solutions to the (2+1)-dimensional Ito equation, and present two general classes of interaction solutions through Maple symbolic computations, which could exhibit diverse nonlinear phenomena. We will begin with the Hirota bilinear form of the (2+1)-dimensional Ito equation, and test if combinations of quadratic functions with other kinds of functions can solve the bilinear Ito equation. In the first class of the resulting interaction solutions, there is an arbitrary function involved, and in the second class of the resulting interaction solutions, the other function only needs to satisfy a linear third-order ordinary differential equation. All this shows the diversity of interaction solutions to the (2+1)-dimensional Ito equation. A few of concluding remarks will be finally given in the last section.

2. Abundant interaction solutions

Basic approaches to soliton solutions and dromion-type solutions include the Hirota perturbation technique and symmetry reductions and constraints (see, e.g., [23–27]). Recently, the multiple exp-function algorithm has also been used to compute multiple wave solutions to both integrable and non-integrable equations (see, e.g., [20,28,29]). We aim to present interaction solutions between lump solutions and other kinds of solutions to the (2+1)-dimensional Ito equation (1.1), by determining when combined functions of quadratic functions and other kinds of functions will solve the (2+1)-dimensional Ito equation (1.2).

Through Maple symbolic computations, we look for combined solutions to the bilinear Ito equation (1.2). We start from an ansatz

$$f = {\xi_1}^2 + {\xi_2}^2 + w(\xi_3) + a_{13}, \tag{2.1}$$

where w is a function and three linear wave variables are

$$\begin{cases} \xi_1 = a_1x + a_2y + a_3t + a_4, \\ \xi_2 = a_5x + a_6y + a_7t + a_8, \\ \xi_3 = a_9x + a_{10}y + a_{11}t + a_{12}. \end{cases} \tag{2.2}$$

The parameters a_i, $1 \le i \le 13$, are all real constants to be determined. With the help of Maple symbolic computations, we can determine the possibilities for the function w and the parameters a_i's, when we impose the conditions (1.5). This is stated in the following two theorems.

Theorem 2.1. *If we take the following choice for the parameters:*

$$\begin{cases} \beta a_1 + \alpha a_2 + a_3 = 0, \\ \beta a_5 + \alpha a_6 + a_7 = 0, \\ a_1a_3 + a_5a_7 = 0, \\ a_9 = 0, \ \alpha a_{10} + a_{11} = 0, \end{cases} \tag{2.3}$$

then the function f determined by (2.1) with an arbitrary function w solves the bilinear Ito equation (1.2).

Theorem 2.2. *If we take the following choice for the parameters:*

$$\begin{cases} \beta a_1 + \alpha a_2 + a_3 = 0, \\ \beta a_5 + \alpha a_6 + a_7 = 0, \\ a_1a_3 + a_5a_7 = 0, \\ \alpha a_{10} + \beta a_9 + {a_9}^3 = 0, \ a_{11} = 0, \end{cases} \tag{2.4}$$

then the function f determined by (2.1) with a function w satisfying $w'''(z) = w'(z)$ solves the bilinear Ito equation (1.2).

The proof is direct by applying Maple in resolution theorem proving. These sets of solutions for the parameters generate two classes of combined solutions f_1 and f_2 to the bilinear Ito equation (1.2), defined by (2.1) and (2.2) with (2.3) or (2.4), and then the resulting combined solutions present two classes of interaction solutions u_1 and u_2 to the (2+1)-dimensional Ito equation (1.1), under the transformations in (1.3). The analyticity of the interactions solutions will be guaranteed, if we require $w(\xi_3) + a_{13} > 0$, which can be easily achieved. These interaction solutions reduce to the lump solutions [16] when the function w disappears, and the soliton solutions [19] when w is taken to be the hyperbolic cosine and the quadratic function disappears. Special choices for w generate some interesting interaction solutions, for example, lump-like solutions with periodic perturbations.

An interesting characteristic is the involvement of an arbitrary function w in the first class of the resulting interaction solutions, but the function w has to be

$$w(z) = c_1e^z + c_2e^{-z} + c_3, \tag{2.5}$$

where c_1, c_2 and c_3 are arbitrary constants, in the second class of the resulting interaction solutions.

To illustrate the resulting interaction solutions, we take the following two special choices for the parameters:

$$\begin{cases} \alpha = -1, \ \beta = 2, \ a_1 = 2, \ a_2 = 1, \ a_3 = -3, \ a_4 = -1, \ a_5 = -2, \\ a_6 = -7, \ a_7 = -3, \ a_8 = 1, \ a_9 = 0, \ a_{10} = 1, \ a_{11} = 1, \ a_{12} = -2, \ a_{13} = 2, \end{cases} \tag{2.6}$$

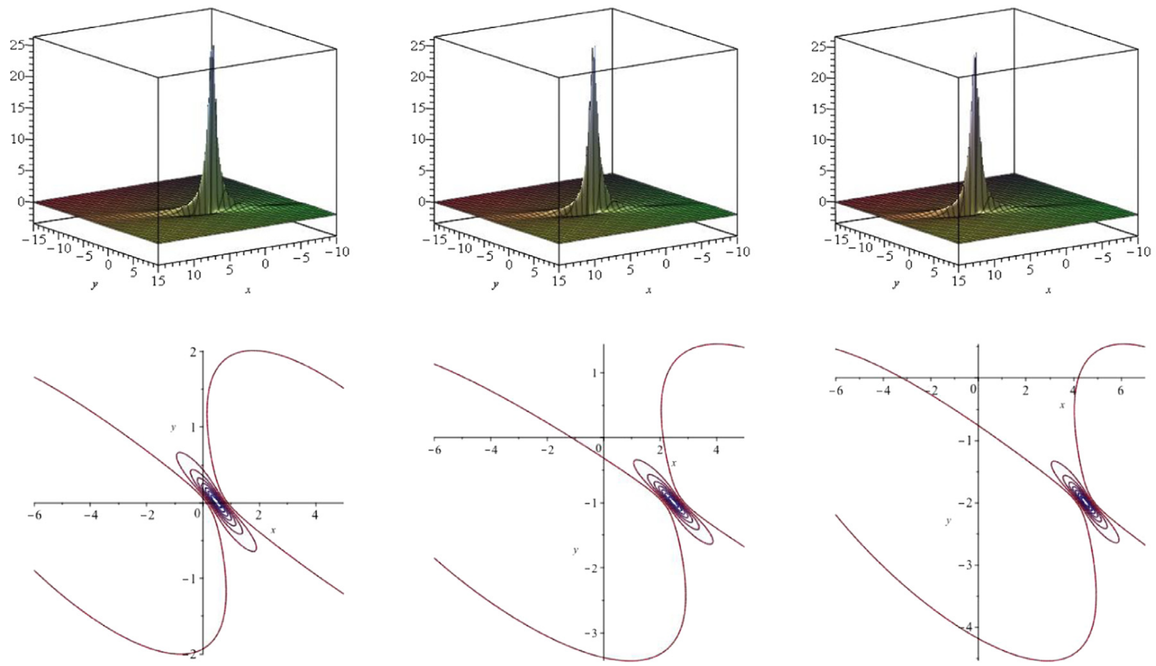

Fig. 1. Profiles of u_1 in (2.8) with $t = 0, 1, 2$: 3d plots (top) and contour plots (bottom).

and

$$\begin{cases} \alpha = -2,\ \beta = 1,\ a_1 = -1,\ a_2 = 2,\ a_3 = 5,\ a_4 = -1,\ a_5 = 5, \\ a_6 = 3,\ a_7 = 1,\ a_8 = 3,\ a_9 = 1,\ a_{10} = 1,\ a_{11} = 0,\ a_{12} = -2,\ a_{13} = 1. \end{cases} \tag{2.7}$$

When we require $w(\xi_3) + a_{13} > 0$ in these two cases, the analyticity is guaranteed for the two corresponding specific interaction solutions. Upon taking $w(z) = \sin(z)$ and $w(z) = e^z + \frac{1}{2}e^{-z}$ respectively, both of which satisfy $w(\xi_3) + a_{13} > 0$ indeed, it is direct to work out these two interaction solutions:

$$u_1 = \frac{32}{g_1} - \frac{2(16\,x + 32\,y - 8)^2}{{g_1}^2},\ v_1 = \frac{2(16\,x + 32\,y - 8)}{g_1}, \tag{2.8}$$

with

$$g_1 = (-3\,t + 2\,x + y - 1)^2 + (-3\,t - 2\,x - 7\,y + 1)^2 + \sin(t + y - 2) + 2,$$

and

$$\begin{cases} u_2 = \dfrac{2\left(52 + e^{\xi_3} + \frac{1}{2}\,e^{-\xi_3}\right)}{g_2} - \dfrac{2\,(52\,x + 26y + 32 + e^{\xi_3} - \frac{1}{2}\,e^{-\xi_3})^2}{{g_2}^2}, \\ v_2 = \dfrac{2\,(52\,x + 26\,y + 32 + e^{\xi_3} - \frac{1}{2}\,e^{-\xi_3})}{g_2}, \end{cases} \tag{2.9}$$

with

$$g_2 = (5\,t - x + 2\,y - 1)^2 + (t + 5\,x + 3\,y + 3)^2 + e^{\xi_3} + \frac{1}{2}\,e^{-\xi_3} + 1,\ \xi_3 = x + y - 2.$$

Three 3-dimensional plots and contour plots of the solutions u_1 and v_1 at $t = 0, 1, 2$ are shown in Figs. 1 and 2, respectively. Similarly, three 3-dimensional plots and contour plots of the solutions u_2 and v_2 at $t = 0, 1, 2$ are shown in Figs. 3 and 4, respectively.

3. Concluding remarks

Based on the Hirota bilinear form of the (2+1)-dimensional Ito equation, we presented two classes of interaction solutions between lumps and other kinds of solutions to the (2+1)-dimensional Ito equation through Maple symbolic computations. The first class of the resulting interaction solutions contains an arbitrary function, and the second class of the resulting

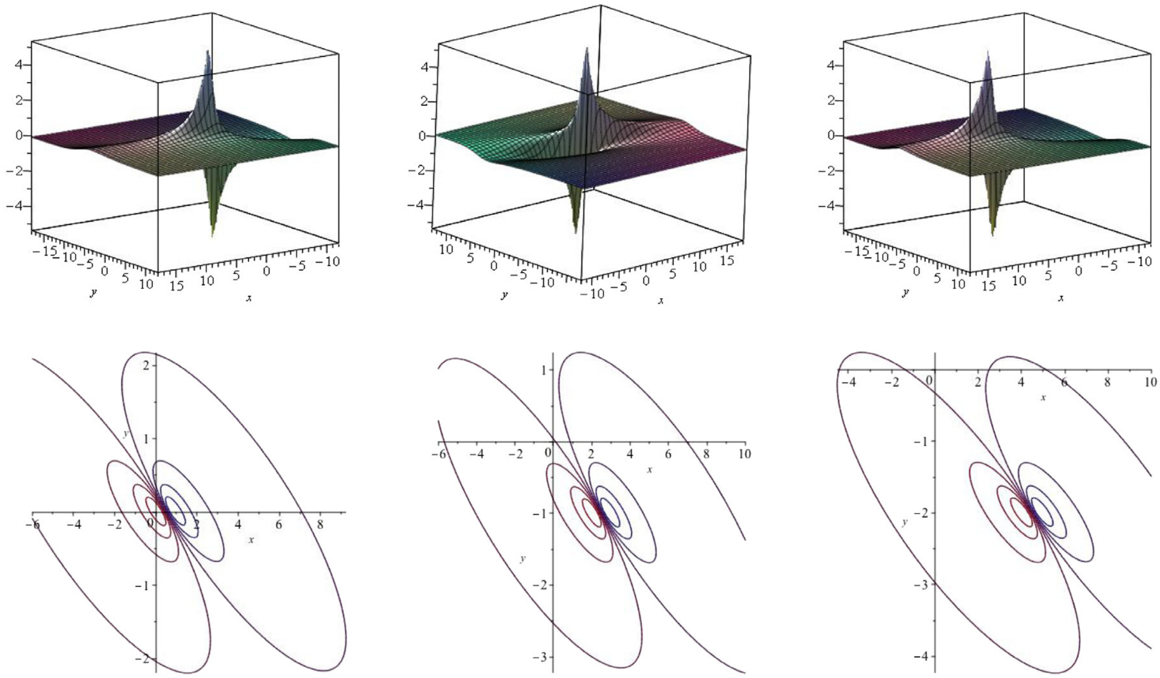

Fig. 2. Profiles of v_1 in (2.8) with $t = 0, 1, 2$: 3d plots (top) and contour plots (bottom).

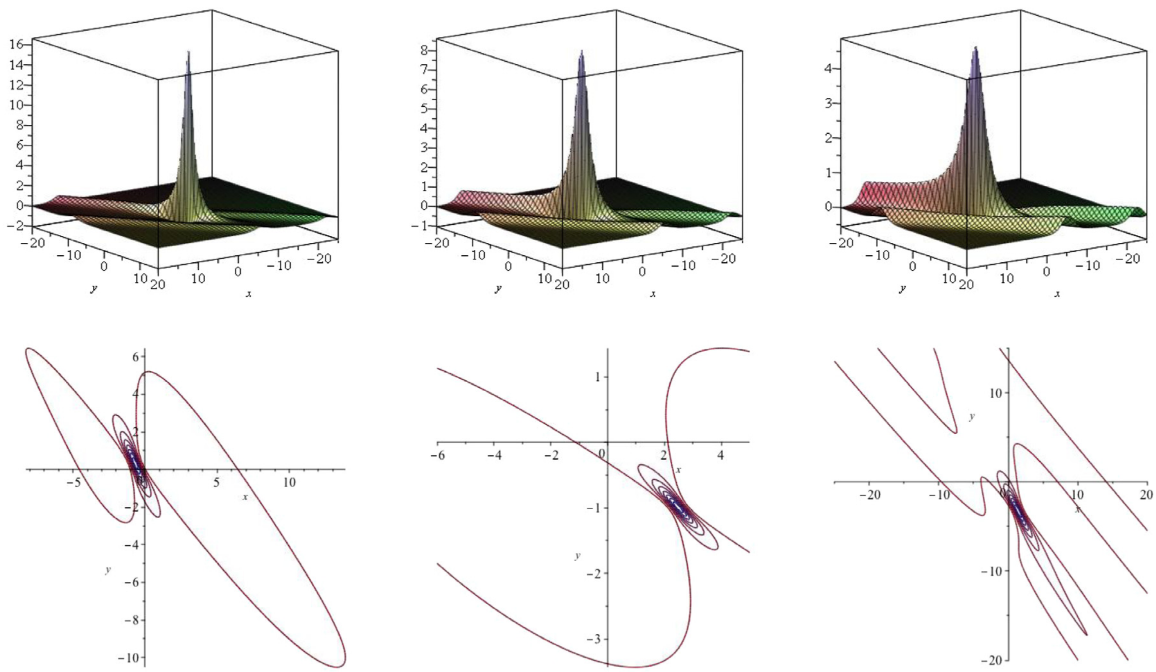

Fig. 3. Profiles of u_2 in (2.9) with $t = 0, 1, 2$: 3d plots (top) and contour plots (bottom).

interaction solutions requires a function satisfying a linear third-order ordinary differential equation. All this provides abundant interaction solutions, supplementing existing lump and soliton solutions.

We remark that under the conditions in (1.5), we can also show that the function f determined by (2.1) solves the bilinear Ito equation (1.2) for any function w if and only if the fourth condition in (2.3) is satisfied, and the function f determined by

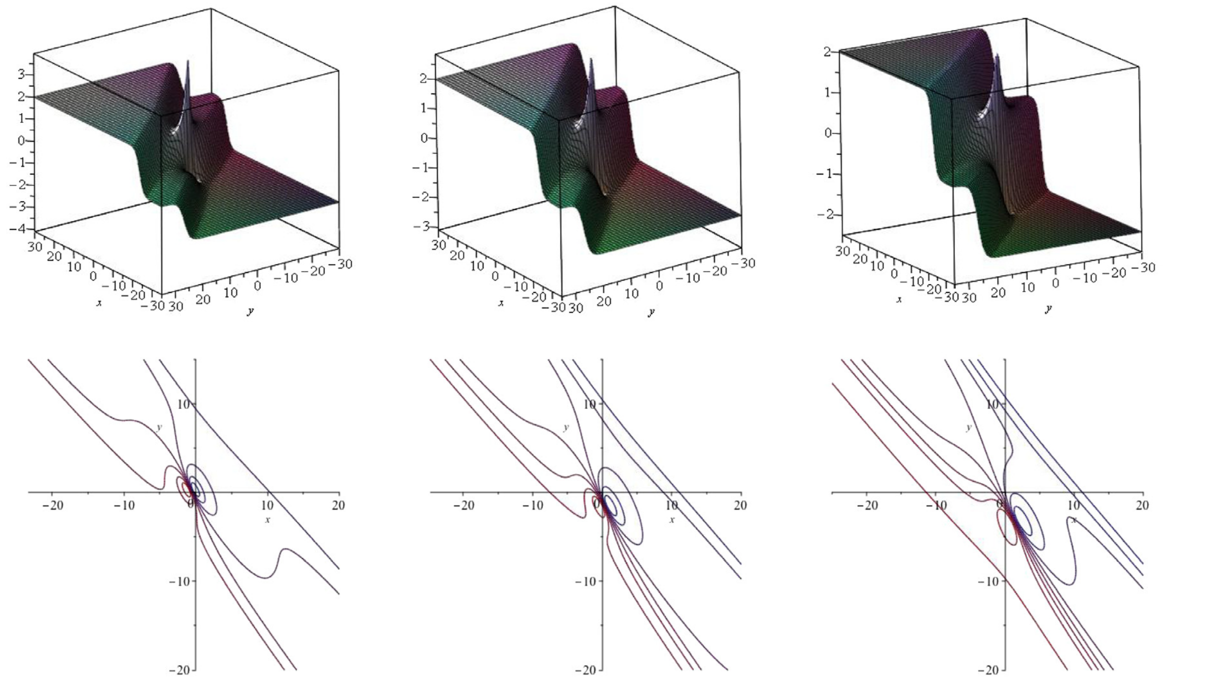

Fig. 4. Profiles of v_2 in (2.9) with $t = 0, 1, 2$: 3d plots (top) and contour plots (bottom).

(2.1) solves the bilinear Ito equation (1.2) for any function w with $w'''(z) = w'(z)$ if and only if the fourth condition in (2.4) is satisfied. Moreover, even if the fourth condition in (2.4) holds, the functions

$$f = {\xi_1}^2 + {\xi_2}^2 + \sin\xi_3 + a_{13},\ f = {\xi_1}^2 + {\xi_2}^2 + \cos\xi_3 + a_{13},$$

do not solve the bilinear Ito equation (1.2).

It should be interesting to consider the generalized bilinear Ito-like equations

$$(D_{3,t}^2 + D_{3,x}^3 D_{3,t} + \alpha D_{3,y} D_{3,t} + \beta D_{3,x} D_{3,t}) f \cdot f = 0,$$

$$(D_{5,t}^2 + D_{5,x}^3 D_{5,t} + \alpha D_{5,y} D_{5,t} + \beta D_{5,x} D_{5,t}) f \cdot f = 0,$$

where $D_{3,z}$ and $D_{5,z}$ are two kinds of generalized bilinear derivatives [30]. All previous computations would be different in those two cases, but lump solutions generated from quadratic functions remain the same. It is also interesting to determine combined solutions for other generalized bilinear and tri-linear differential equations or if they can be generated from symmetry constraints [24–27]. This kind of interaction solutions is quite different from resonant solutions formulated by the linear superposition principle [31,32] and could bring insights to help solve nonlinear problems we face today.

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