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Soliton solutions to the B-type Kadomtsev–Petviashvili equation under general dispersion relations

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ABSTRACT

We analyze soliton solutions and verify the Hirota N-soliton condition for the B-type Kadomtsev–Petviashvili equation, within the Hirota bilinear formulation. A weight number is used in an algorithm to check the Hirota condition while transforming the Hirota function in N wave vectors to a homogeneous polynomial. Soliton solutions are presented under general dispersion relations.

1. Introduction

In mathematical physics, N-soliton solutions are very helpful in exploring nonlinear wave phenomena [1,2]. Within the Hirota bilinear formulation, we can present N-soliton solutions in a closed form [3,4]. Breather, complexiton, lump and rogue wave solutions are all special cases of N-soliton solutions.

Let us start by delineating the Hirota bilinear form. We focus on the $(2+1)$-dimensional case and denote the spatial variables by x and y, and the temporal variable by t. Hirota bilinear derivatives are defined by [5]:

$$D_x^m f\cdot g=\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(\partial_x^i f)(\partial_x^{m-i}g),\ \ m\geq 1, \tag{1.1}$$

and more generally, bilinear partial derivatives are similarly defined by

$$(D_x^m D_t^n f\cdot g)(x,t)=(\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t},\ \ m,n\geq 0,\ \ m+n\geq 1. \tag{1.2}$$

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Taking $f = g$, we obtain Hirota bilinear expressions:

$$D_x^m f \cdot f = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\partial_x^i f)(\partial_x^{m-i} f), \ m \geq 1, \tag{1.3}$$

and similarly, bilinear partial derivative expressions:

$$D_x^m D_t^n f \cdot f = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{m+n-i-j} \binom{m}{i} \binom{n}{j} (\partial_x^i \partial_t^j f)(\partial_x^{m-i} \partial_t^{n-j} f), \ m, n \geq 0, \ m+n \geq 1. \tag{1.4}$$

Obviously, any Hirota bilinear expression of odd order is zero, for example,

$$D_x f \cdot f = 0, \ D_x^2 D_t f \cdot f = 0.$$

Only Hirota bilinear expressions of even orders make sense, and particularly, we have

$$D_x^2 f \cdot f = 2(f_{xx} f - f_x^2), \ D_x^3 D_t f \cdot f = 2(f_{3x,t} f - 3 f_{xxt} f_x + 3 f_{xt} f_{xx} - f_t f_{3x}).$$

Now, in terms of Hirota bilinear expressions, we can define Hirota bilinear equations. Take an even polynomial $P(x, y, t)$ in x, y and t with $P(0, 0, 0) = 0$. The corresponding Hirota bilinear equation is defined as follows:

$$P(D_x, D_y, D_t) f \cdot f = 0, \tag{1.5}$$

all terms of which are Hirota bilinear expressions. If a given nonlinear partial differential equation can be transformed into a Hirota bilinear equation, we say that the equation possesses a Hirota bilinear form.

Among important integrable equations in (2+1)-dimensions is the Kadomtsev–Petviashvili equation [6]:

$$N(u) := (u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{1.6}$$

and its bilinear form reads

$$\begin{aligned} B(f) &:= (D_x^4 + D_x D_t - D_y^2) f \cdot f \\ &= 2(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 + f_{xt} f - f_x f_t - f_{yy} f + f_y^2) = 0. \end{aligned} \tag{1.7}$$

Under the logarithmic derivative transformation $u = 2(\ln f)_{xx}$, the two equations are linked together: $N(u) = (B(f)/f^2)_{xx}$. The Kadomtsev–Petviashvili equation is associated with the A-type infinite dimensional Lie algebra $\mathrm{gl}(\infty)$ and has N-soliton solutions [7].

Another important example is the B-type Kadomtsev–Petviashvili (BKP) equation associated with the B-type infinite dimensional Lie algebra $\mathrm{o}(\infty)$ [8,9]:

$$N(u) := (15u_x^3 + 15u_x u_{3x} + u_{5x})_x + 5[u_{3x,y} + 3(u_x u_y)_x] + u_{xt} - 5u_{yy} = 0, \tag{1.8}$$

and its Hirota bilinear form is

$$\begin{aligned} B(f) &:= (D_x^6 + 5D_x^3 D_y + D_x D_t - 5D_y^2) f \cdot f = 2[f_{6x} f - 6 f_{5x} f_x + 15 f_{4x} f_{xx} - 10 f_{3x}^2 \\ &+ 5(f_{3x,y} f - 3 f_{xxy} f_x + 3 f_{xy} f_{xx} - f_y f_{3x}) + f_{xt} f - f_x f_t - 5(f_{yy} f - f_y^2)] = 0. \end{aligned} \tag{1.9}$$

This is equivalent to the above BKP equation, under the logarithmic derivative transformation $u = 2(\ln f)_x$, and the link is $N(u) = (B(f)/f^2)_x$. Soliton solutions are formulated for the BKP equation via the τ-function [9] and the Pfaffian [10].

We would like to analyze a general class of N-soliton solutions and verify the corresponding Hirota condition for the BKP equation, within the Hirota bilinear formulation. By applying an algorithm using a weight to check the Hirota N-soliton condition, a verification of the Hirota N-soliton condition will be given for the BKP equation, and thus, N-soliton solitons will be presented explicitly, under more general dispersion relations than the ones chosen in the τ-function and Pfaffian theories.

2. Formulation of soliton solutions

We express N wave vectors as follows:

$$\mathbf{k}_i = (k_i, l_i, -\omega_i), \ 1 \leq i \leq N. \tag{2.1}$$

Let $P(x, y, t)$ be an even polynomial in x, y and t, satisfying $P(0, 0, 0) = 0$. An N-soliton solution to a Hirota bilinear equation

$$P(D_x, D_y, D_t) f \cdot f = 0 \tag{2.2}$$

is given by [11]:

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} a_{ij} \mu_i \mu_j\Big), \tag{2.3}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, $\mu = 0, 1$ means that each μ_i takes 0 or 1, and

$$\eta_i = k_i x + l_i y - \omega_i t + \eta_{i,0}, \ 1 \le i \le N, \tag{2.4}$$

$$e^{a_{ij}} = A_{ij} := -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)}, \ 1 \le i < j \le N, \tag{2.5}$$

$\eta_{i,0}$'s being arbitrary constant phase shifts.

Let us check what conditions an N-soliton solution should satisfy. Introduce

$$H(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_n}) = \sum_{\sigma=\pm 1} P(\sum_{r=1}^{n} \sigma_r \mathbf{k}_{i_r}) \prod_{1\le r<s\le n} P(\sigma_r \mathbf{k}_{i_r} - \sigma_s \mathbf{k}_{i_s}) \sigma_r \sigma_s, \ 1 \le n \le N, \tag{2.6}$$

where $1 \le i_1 < \cdots < i_n \le N$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, and $\sigma = \pm 1$ means that each σ_i takes 1 or -1. Particularly, we have $H(\mathbf{k}_1) = P(\mathbf{k}_1) + P(-\mathbf{k}_1) = 2P(\mathbf{k}_1)$. We call these functions the Hirota functions.

Applying the basic properties

$$P(D_x, D_y, D_t) e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j) e^{\eta_i + \eta_j}, \tag{2.7}$$

and

$$P(D_x, D_y, D_t) e^{\eta_n} f \cdot e^{\eta_n} g = e^{2\eta_n} P(D_x, D_y, D_t) f \cdot g, \tag{2.8}$$

where η_i, η_j and η_n are arbitrary linear functions, we can formulate the following expression [12,13].

Theorem 2.1. *Let f be defined by (2.3), and $\hat{\xi}$ mean that no ξ is involved. Then we have*

$$\begin{aligned} & P(D_x, D_y, D_t) f \cdot f \\ = & (-1)^{\frac{1}{2}N(N-1)} \frac{H(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)}{\prod_{1\le i<j\le N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \eta_2 + \cdots + \eta_N} \\ & + \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1\le i_1<\cdots<i_n\le N} \frac{H(\mathbf{k}_1, \ldots, \hat{\mathbf{k}}_{i_1}, \ldots, \hat{\mathbf{k}}_{i_n}, \ldots, \mathbf{k}_N)}{\prod_{\substack{1\le i<j\le N \\ i,j\notin\{i_1,\ldots,i_n\}}} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\eta_1 + \cdots + \hat{\eta}_{i_1} + \cdots + \hat{\eta}_{i_n} + \cdots + \eta_N} \\ & + \sum_{n=1}^{N-1} \sum_{1\le i_1<\cdots<i_n\le N} e^{2(\eta_{i_1} + \cdots + \eta_{i_n} + \sum_{1\le r<s\le n} a_{i_r i_s})} P(D_{x_1}, \ldots, D_{x_M}) \tilde{f}_{i_1\cdots i_n} \cdot \tilde{f}_{i_1\cdots i_n} \end{aligned}$$

with

$$\tilde{f}_{i_1\cdots i_n} = \sum_{\tilde{\mu}_{i_1\cdots i_n}=0,1} exp(\sum_{\substack{1\le i\le N \\ i\notin\{i_1,\ldots,i_n\}}} \mu_i \tilde{\eta}_i + \sum_{\substack{1\le i<j\le N \\ i,j\notin\{i_1,\ldots,i_n\}}} a_{ij}\mu_i\mu_j), \ \tilde{\eta}_i = \eta_i + \sum_{r=1}^{n} a_{i i_r},$$

where $\tilde{\mu}_{i_1\cdots i_n} = (\mu_1, \ldots, \hat{\mu}_{i_1} \cdots, \hat{\mu}_{i_n}, \ldots, \mu_N)$ and $\tilde{\mu}_{i_1\cdots i_n} = 0, 1$ means that each μ_i in $\tilde{\mu}_{i_1\cdots i_n}$ takes 0 or 1.

Following this basic theorem, we can find by a recursive procedure that a Hirota bilinear equation (2.2) possesses an N-soliton solution (2.3) if and only if the condition

$$H(\mathbf{k}_{i_1}, \ldots, \mathbf{k}_{i_n}) = 0, \ 1 \le i_1 < \cdots, < i_n \le N, \ 1 \le n \le N, \tag{2.9}$$

is satisfied. This is called the Hirota condition for an N-soliton solution, or simply, the N-soliton condition [14,15]. The case of $n = 1$ leads to the dispersion relations

$$P(\mathbf{k}_i) = P(k_i, l_i, -\omega_i) = 0, \ 1 \le i \le N. \tag{2.10}$$

Few studies are available in the literature on the Hirota N-soliton condition, due to its high complexity [14].

Examples: The one-soliton condition is just the dispersion relation: $P(\mathbf{k}_1) = 0$, which means that $f = 1 + e^{\eta_1}$ is a solution. Besides the dispersion relations, the two-soliton condition requires

$$2(P(\mathbf{k}_1 + \mathbf{k}_2)P(\mathbf{k}_1 - \mathbf{k}_2) - P(\mathbf{k}_1 - \mathbf{k}_2)P(\mathbf{k}_1 + \mathbf{k}_2)) = 0, \tag{2.11}$$

which is automatically satisfied. Therefore, we always have a two-soliton solution:

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \tag{2.12}$$

to a Hirota bilinear equation. Taking $N = 3$, we see that in addition to the dispersion relations, the three-soliton condition requires [16]:

$$\sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0,$$

which is equivalent to

$$\sum_{(\sigma_1,\sigma_2,\sigma_3)\in S} P(\sigma_1\mathbf{k}_1+\sigma_2\mathbf{k}_2+\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_2\mathbf{k}_2) \times P(\sigma_2\mathbf{k}_2-\sigma_3\mathbf{k}_3)P(\sigma_1\mathbf{k}_1-\sigma_3\mathbf{k}_3)=0, \tag{2.13}$$

where $S = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$. The three-soliton solution reads

$$f = 1+\mathrm{e}^{\eta_1}+\mathrm{e}^{\eta_2}+\mathrm{e}^{\eta_3}+A_{12}\mathrm{e}^{\eta_1+\eta_2}+A_{13}\mathrm{e}^{\eta_1+\eta_3} +A_{23}\mathrm{e}^{\eta_2+\eta_3}+A_{123}\mathrm{e}^{\eta_1+\eta_2+\eta_3},\ A_{123}=A_{12}A_{13}A_{23}. \tag{2.14}$$

It is generally agreed that the three-soliton condition implies the N-soliton condition, but no proof of its accuracy has yet been given.

If we require a sufficient Hirota N-soliton condition [17]:

$$P(\mathbf{k}_i-\mathbf{k}_j)=0,\ 1\le i<j\le N, \tag{2.15}$$

we obtain the resonant N-soliton solution:

$$f = 1+c_1\mathrm{e}^{\eta_1}+c_2\mathrm{e}^{\eta_2}+\cdots+c_N\mathrm{e}^{\eta_N}, \tag{2.16}$$

where c_i's are arbitrary constants. Those wave vectors $\mathbf{k}_i$'s associated with resonant solutions constitute an affine space in $\mathbb{R}^3$ [18].

Directly from the definition, we can see the following properties of the Hirota functions.

Theorem 2.2. *The Hirota functions defined by (2.6) are symmetric and even functions in the involved wave vectors.*

Taking $\mathbf{k}_2 = \pm\mathbf{k}_1$, we have

$$P(\sigma_i\mathbf{k}_i-\mathbf{k}_2)P(\sigma_i\mathbf{k}_i\pm\mathbf{k}_1)=P(\mathbf{k}_i-\mathbf{k}_1)P(\mathbf{k}_i+\mathbf{k}_1) \tag{2.17}$$

in both cases of $\sigma_i = \pm 1$, owing to the even property of the polynomial P. Applying this property, we can explore the following consequence [12,13].

Theorem 2.3. *If $\mathbf{k}_2 = \pm\mathbf{k}_1$, then we have*

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=2H(\mathbf{k}_3,\ldots,\mathbf{k}_N)P(2\mathbf{k}_1)\prod_{i=3}^{N}P(\mathbf{k}_i-\mathbf{k}_1)P(\mathbf{k}_i+\mathbf{k}_1). \tag{2.18}$$

This result will be used to factor out as more common factors out of the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ as possible, while verifying the Hirota condition.

3. Verifying the hirota N-soliton condition

Note that the dispersion relations (2.10) determine all frequencies $\omega_i = \omega(k_i, l_i),\ 1 \le i \le N$. Therefore, $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ are, actually, functions of k_i, l_i and k_j, l_j,

On one hand, we assume that under the substitution

$$l_i = l_i k_i^w,\ 1\le i\le N, \tag{3.1}$$

for some integer weight w, $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ and $P(\sigma_1\mathbf{k}_1 + \cdots + \sigma_N\mathbf{k}_N)$ can be simplified into rational functions as follows:

$$P(\sigma_i\mathbf{k}_i-\sigma_j\mathbf{k}_j)=\frac{\sigma_i\sigma_j k_i k_j Q_1(k_i,l_i,k_j,l_j,\sigma_i,\sigma_j)}{Q_2(k_i,l_i,k_j,l_j)}, \tag{3.2}$$

where Q_1 and Q_2 are polynomial functions, and

$$P(\sigma_1\mathbf{k}_1+\cdots+\sigma_N\mathbf{k}_N)=\frac{Q_3(k_1,l_1,\ldots,k_N,l_N,\sigma_1,\ldots,\sigma_N)}{Q_4(k_1,l_1,\ldots,k_N,l_N)}, \tag{3.3}$$

where Q_3 and Q_4 are polynomial functions. The factor of k_ik_j in $P(\sigma_i\mathbf{k}_i - \sigma_j\mathbf{k}_j)$ is a characteristic condition for existence of N-soliton solutions.

On the other hand, Theorem 2.3 tells that under the induction assumption, the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ will be zero, if two of the wave vectors are equal. Applying the symmetric property in Theorem 2.2, we know that under the transforms in (3.1), $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ will still be even with respect to k_i, l_i $1\le i\le N$, when w is even, but $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ will be even only with respect to $k_i,\ 1\le i\le N$, when w is odd. However, in each case, we can have the simplified form

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=(k_i^2-k_j^2)^2g_{ij}+(l_i-l_j)^2h_{ij},\ \text{for } 1\le i<j\le N,$$

where g_{ij} and h_{ij} are rational functions of $k_n, l_n,\ 1\le n\le N$.

Further using (3.2) and (3.3), we see that the Hirota function $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ can be written as

$$H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=\frac{\prod_{1\le i<j\le N}k_i^2k_j^2[\prod_{1\le i<j\le N}(k_i^2-k_j^2)^2g+\prod_{1\le i<j\le N}(l_i-l_j)^2h]}{Q_4(k_1,l_1,\ldots,k_N,l_N)\prod_{1\le i<j\le N}Q_2(k_i,l_i,k_j,l_j)} \tag{3.4}$$

under (3.1), where g and h are homogeneous polynomials of $k_n, l_n,\ 1\le n\le N$. If $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\neq 0$, then we can have a nonzero function g at least. Define the homogeneous polynomial

$$\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)=H(\mathbf{k}_1,\ldots,\mathbf{k}_N)Q_4(k_1,l_1,\ldots,k_N,l_N)\prod_{1\le i<j\le N}Q_2(k_i,l_i,k_j,l_j). \tag{3.5}$$

Then, since

$$\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)=\prod_{1\le i<j\le N}k_i^2k_j^2[\prod_{1\le i<j\le N}(k_i^2-k_j^2)^2g+\prod_{1\le i<j\le N}(l_i-l_j)^2h],$$

we can find that if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\neq 0$ and so $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)\neq 0$, then the degree of $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is at least $2N(N-1)+2N(N-1)=4N(N-1)$. In other words, if the degree of $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is less than $4N(N-1)$, then $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=0$, which is what we need to prove. Therefore, based on

$$\begin{aligned}&\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)\\&=\sum_{\sigma=\pm1}Q_3(k_1,l_1,\ldots,k_N,l_N,\sigma_1,\ldots,\sigma_N)\prod_{1\le i<j\le N}\sigma_i\sigma_jk_ik_jQ_1(k_i,l_i,k_j,l_j,\sigma_i,\sigma_j),\end{aligned} \tag{3.6}$$

a final proof task is to compute Q_1 and Q_3 to check if the degree of $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is less than $4N(N-1)$.

Let us recall that the BKP equation is associated with

$$P(x,y,t)=x^6+5x^3y+xt-5y^2, \tag{3.7}$$

which gives the bilinear BKP equation:

$$\begin{aligned}B(f):=(D_x^6+5D_x^3D_y+D_xD_t-5D_y^2)f\cdot f&=2[f_{6x}f-6f_{5x}f_x+15f_{4x}f_{xx}-10f_{3x}^2\\&+5(f_{3x,y}f-3f_{xxy}f_x+3f_{xy}f_{xx}-f_yf_{3x})+f_{xt}f-f_xf_t-5(f_{yy}f-f_y^2)]=0.\end{aligned} \tag{3.8}$$

This is equivalent to the BKP equation:

$$N(u):=(15u_x^3+15u_xu_{3x}+u_{5x})_x+5[u_{3x,y}+3(u_xu_y)_x]+u_{xt}-5u_{yy}=0, \tag{3.9}$$

since we have $N(u)=(B(f)/f^2)_x$ under $u=2(\ln f)_x$.

It is easy to evaluate that

$$\omega_i=\frac{k_i^6+5k_i^3l_i-5l_i^2}{k_i},\ 1\le i\le N, \tag{3.10}$$

and

$$\begin{cases}Q_1=-5[k_i^4-3\sigma_i\sigma_jk_i^3k_j+4k_i^2k_j^2-3\sigma_i\sigma_jk_ik_j^3+k_j^4\\ \quad -3\sigma_i\sigma_j(l_i+l_j)k_ik_j+(2l_i+l_j)k_i^2+(l_i+2l_j)k_j^2+(l_i-l_j)^2],\\ \deg Q_3=6,\ Q_2=1,\ Q_4=1,\end{cases} \tag{3.11}$$

under the substitution (3.1) with $w=1$. Now if $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)\neq 0$, then based on (3.6), the degree of the polynomial $\tilde{H}(\mathbf{k}_1,\ldots,\mathbf{k}_N)$ is $3N(N-1)+6$, which could not be greater than $4N(N-1)$ when $N\ge 4$. A direct check by symbolic computation can show $H(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=0$. Therefore, we obtain $H(\mathbf{k}_1,\ldots,\mathbf{k}_N)=0,\ N\ge 1$, which completes the proof.

Note that soliton solutions of the BKP equation has also been formulated by τ-functions and Pfaffians in [9,10], respectively. However, the dispersion relations chosen in [9,10] are all special cases of (2.10) (or more specifically, (3.10) in the above BKP case), since we have

$$k_i=p_i-q_i,\ l_i=q_i^3-p_i^3,\ \omega_i=9(q_i^5-p_i^5),\ 1\le i\le N, \tag{3.12}$$

produced by the τ-function and Pfaffian techniques. This is not a one-to-one correspondence between (k_i,l_i) and (p_i,q_i) in $\mathbb{R}^2$, and so the dispersion relations in (3.10) are more general. Additionally, lump solutions [19,20], lump-soliton solutions [21], rogue wave solutions [22], and special N-soliton solutions [23] have been presented for the BKP equation. Those are specific reductions of the N-soliton solutions in Section 2 as well.

4. Concluding remarks

We have verified the Hirota N-soliton condition for the BKP equation, and therefore, presented soliton solutions explicitly, under general dispersion relations. From the general N-soliton solutions, we can work out many other special kinds of closed form solutions, such as lump, breather, rogue wave and interactions solutions.

It would be interesting to search for other bilinear equations in (2+1)-dimensions, which possess N-soliton solutions. One more interesting question is to explore bilinear equations in (3+1)-dimensions or higher dimensions, to which there exist N-soliton solutions. Symbolic computations and theoretical proofs could be used together to determine new such equations in higher dimensional cases.

Another direction for further research is to check generalized bilinear equations, which is certainly more challenging. Let us briefly state the generalized bilinear formulation.

Let p be a natural number. The $D_{p,x}$-operators are defined as follows [24]:

$$D_{p,x}^m D_{p,t}^n f\cdot g=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\,\alpha_p^{i+j}(\partial_x^{m-i}\partial_t^{n-j}f)(\partial_x^i\partial_t^j g),\ \ m,n\ge 0,\ \ m+n\ge 1, \tag{4.1}$$

where the powers of α_p are determined by

$$\alpha_p^i=(-1)^{r(i)},\ \ i=r(i) \bmod p,\ \ i\ge 0, \tag{4.2}$$

with $0\le r(i)<p$. The patterns of those powers for $i=1,2,3,\ldots$ read

$$p=3:\ \ -,+,+,-,+,+,\ldots;$$

$$p=5:\ \ -,+,-,+,+,-,+,-,+,+,\ldots;$$

$$p=7:\ \ -,+,-,+,-,+,+,-,+,-,+,-,+,+,\ldots.$$

For example, we can have $D_{3,x}$ and $D_{5,x}$ associated with the two smallest odd prime numbers: $p=3,5$. The cases of $p=2k$, $k\in\mathbb{N}$, just present the Hirota bilinear derivatives. The corresponding generalized bilinear expressions exhibit new characteristics. For instance, we have

$$D_{3,x}^3 f\cdot f=2f_{xxx}f,\ \ D_{3,x}^4 f\cdot f=6f_{xx}^2, \tag{4.3}$$

which amend the Hirota case, since the Hirota bilinear formulation only works with bilinear differential equations consisting of even-order differential terms. Other generalized bilinear derivatives such as $D_{6,x}$ and $D_{9,x}$ should exhibit more interesting phenomena.

We are particularly interested in knowing any example of generalized bilinear equations, which has N-soliton solutions. Focus on the (2+1)-dimensional case again. It is known that a generalized bilinear equation

$$P(D_{p,x},D_{p,y},D_{p,t})f\cdot f=0 \tag{4.4}$$

can possess a resonant N-soliton solution [25]:

$$f=1+c_1e^{\eta_1}+c_2e^{\eta_2}+\cdots+c_Ne^{\eta_N} \tag{4.5}$$

where c_i's are arbitrary constants and $\eta_i=k_ix+l_iy-\omega_it+\eta_{i,0},\ \ 1\le i\le N$, if and only if

$$P(\mathbf{k}_i+\alpha_p\mathbf{k}_j)+P(\mathbf{k}_j+\alpha_p\mathbf{k}_i)=0,\ \ 1\le i\le j\le N, \tag{4.6}$$

where $\mathbf{k}_i=(k_i,l_i,-\omega_i),\ \ 1\le i\le N$. What is a generalized N-soliton condition, i.e., an N-soliton condition for a generalized bilinear equation? How can one find generalized bilinear equations, for instance,

$$P(D_{3,x},D_{3,t})=0\ \text{ or }\ P(D_{3,x},D_{3,y},D_{3,t})=0, \tag{4.7}$$

where $p=3$, which possess N-soliton solutions? These are basic questions one needs to answer to establish a more general bilinear theory on soliton equations, even lump, breather and rogue wave solutions (see, e.g., [26–28]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Consultants Bureau, New York, 1984.
[3] R. Hirota, Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[4] A.-M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Springer-Verlag, Berlin, Heidelberg, 2009.
[5] R. Hirota, A new form of Bäcklund transformations and its relation to the inverse scattering problem, Progr. Theoret. Phys. 52 (1974) 1498–1512.
[6] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. 15 (1970) 539–541.
[7] J. Satsuma, N-Soliton solution of the two-dimensional Kortweg-de Vries equation, J. Phys. Soc. Japan 40 (1976) 286–290.
[8] E. Date, M. Kashiwara, T. Miwa, Vertex operators and τ functions - transformation groups for soliton equations II, Proc. Japan Acad. Ser. A 57 (1981) 387–392.
[9] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, KP Hierarchies of orthogonal and symplectic type - transformation groups for soliton equations VI, J. Phys. Soc. Japan 50 (1981) 3813–3818.
[10] R. Hirota, Soliton solutions to the BKP equations I - the Pfaffian technique, J. Phys. Soc. Japan 58 (1989) 2285–2296.
[11] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192–1194.
[12] W.X. Ma, N-Soliton solutions and the Hirota conditions in (1+1)-dimensions, Int. J. Nonlinear Sci. Numer. Simul. 22 (2021) in press, DOI: https://doi.org/10.1515/ijnsns-2020-0214.
[13] W.X. Ma, N-Soliton solutions and the Hirota conditions in (2+1)-dimensions, Opt. Quantum Electron. 52 (2020) 511.
[14] R. Hirota, Direct methods in soliton theory, in: R.K. Bullough, P. Caudrey (Eds.), Solitons, Springer-Verlag, Berlin, Heidelberg, 1980, pp. 157–176.
[15] A.C. Newell, Y.B. Zeng, The Hirota conditions, J. Math. Phys. 27 (1986) 2016–2021.
[16] J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition I - KdV-type bilinear equations, J. Math. Phys. 28 (1987) 1732–1742.
[17] W.X. Ma, E.G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011) 950–959.
[18] W.X. Ma, Y. Zhang, Y.N. Tang, J.Y. Tu, Hirota bilinear equations with linear subspaces of solutions, Appl. Math. Comput. 218 (2012) 7174–7183.
[19] C.R. Gilson, J.J.C. Nimmo, Lump solutions of the BKP equation, Phys. Lett. A 147 (1990) 472–476.
[20] J.Y. Yang, W.X. Ma, Lump solutions to the BKP equation by symbolic computation, Internat. J. Modern Phys. B 30 (2016) 1640028.
[21] J.Y. Yang, W.X. Ma, Z.Y. Qin, Abundant mixed lump-soliton solutions to the BKP equation, East Asian J. Appl. Math. 8 (2018) 224–232.
[22] W.Q. Peng, S.F. Tian, Y.Y. Zhang, Analysis on lump, lumpoff and rogue waves with predictability to the (2+1)-dimensional B-type Kadomtsev–Petviashvili equation, Phys. Lett. A 382 (2018) 2701–2708.
[23] Y.L. Kang, Y. Zhang, L.G. Jin, Soliton solution to BKP equation in Wronskian form, Appl. Math. Comput. 224 (2013) 250–258.
[24] W.X. Ma, Generalized bilinear differential equations, Stud. Nonlinear Sci. 2 (2011) 140–144.
[25] W.X. Ma, Bilinear equations and resonant solutions characterized by Bell polynomials, Rep. Math. Phys. 72 (2013) 41–56.
[26] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Differential Equations 264 (2018) 2633–2659.
[27] N. Akhmediev, A. Ankiewicz, M. Taki, Waves that appear from nowhere and disappear without a trace, Phys. Lett. A 373 (2009) 675–678.
[28] Y. Zhang, J.W. Yang, K.W. Chow, C.F. Wu, Solitons, breathers and rogue waves for the coupled Fokas-Lenells system via Darboux transformation, Nonlinear Anal. RWA 33 (2017) 237–252.