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SOLVING THE KORTEWEG-DE VRIES EQUATION BY ITS BILINEAR FORM: WRONSKIAN SOLUTIONS

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ABSTRACT. A broad set of sufficient conditions consisting of systems of linear partial differential equations is presented which guarantees that the Wronskian determinant solves the Korteweg-de Vries equation in the bilinear form. A systematical analysis is made for solving the resultant linear systems of second-order and third-order partial differential equations, along with solution formulas for their representative systems. The key technique is to apply variation of parameters in solving the involved non-homogeneous partial differential equations. The obtained solution formulas provide us with a comprehensive approach to construct the existing solutions and many new solutions including rational solutions, solitons, positons, negatons, breathers, complexitons and interaction solutions of the Korteweg-de Vries equation.

1. INTRODUCTION

Among integrable equations is the celebrated Korteweg-de Vries (KdV) equation, which serves as a model equation governing weakly nonlinear long waves whose phase speed attains a simple maximum for waves of infinite length [DJ]. It motivates us to explore beauty hidden in nonlinear differential (and difference) equations. The remarkable and exceptional discovery of the inverse scattering transform [ASe] is one of important developments in the field of applied mathematics, which comes from the study of the KdV equation. There are various algebraic and geometric characteristics that the KdV equation possesses, for example, infinitely many symmetries and infinitely many conserved densities [MGK], the Lax representation [L], bi-Hamiltonian structure [Mag], loop group [SW], and the Darboux-Bäcklund transformation [MS]. More significantly, many physically important solutions to the KdV equation can be presented explicitly through a simple, specific form, called the Hirota bilinear form. Such exact solutions contain solitons [H, S], rational solutions [ASa], positons [APP, Mat, K], negatons [RSK], breathers [J] and complexitons [M].

Let us consider the KdV equation in its standard form

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.1}$$

Hirota [H] introduced the transformation (here and in the rest of the paper, we use the notation $\ln f$ for simplicity, which does not imply that $f > 0$)

$$u = -2\partial_x^2 \ln f = -2(\ln f)_{xx} = -\frac{2(ff_{xx} - f_x^2)}{f^2},$$

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and transformed the KdV equation (1.1) into the bilinear form

$$(D_xD_t + D_x^4)f \cdot f = f_{xt}f - f_tf_x + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0, \tag{1.2}$$

which is called the bilinear KdV equation. Here D_x and D_t are Hirota's bilinear operators [BC], defined by

$$f(x+h,t+k)g(x-h,t-k) = \sum_{i,j=0}^{\infty} \frac{1}{i!j!}(D_x^iD_t^jf \cdot g)h^ik^j,$$

which are also closely related to the vertex operators from elementary particle theory. Strictly speaking, we have

$$u_t - 6uu_x + u_{xxx} = -\Big[\frac{2(D_xD_t + D_x^4)f \cdot f}{f^2}\Big]_x.$$

Therefore, if f solves the bilinear KdV equation (1.2), then $u = -2\partial_x^2 \ln f$ solves the original KdV equation (1.1). The bilinear form (1.2) looks a little bit more complicated than the KdV equation (1.1) itself, but its bilinear property, the nearest neighbor of the linear property, brings us great convenience in constructing explicit exact solutions. Such a bilinear form exists not only for the KdV equation, but also for many other physically important soliton equations, such as the Ablowitz-Kaup-Newell-Segur equations [AKNS], the Kadomtsev-Petviashvili equation [KP], and the Benney-Roskes equation [BR] (or the Davey-Stewartson equation [DS]).

The Wronskian technique is a powerful tool to construct exact solutions to bilinear differential (and difference) equations. To use this technique, we adopt the compact Freeman and Nimmo's notation [FN]:

$$\begin{aligned}(\widehat{N-1}) &= (\widehat{N-1};\Phi) = W(\phi_1,\phi_2,\cdots,\phi_N)\\ &= \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \ N \geq 1,\end{aligned} \tag{1.3}$$

where

$$\phi_i^{(0)} = \phi_i, \ \phi_i^{(j)} = \frac{\partial^j}{\partial x^j}\phi_i, \ j \geq 1, \ 1 \leq i \leq N. \tag{1.4}$$

Satsuma, Freeman and Nimmo [S, FN] proved that multi-soliton solutions to the KdV equation can be expressed through the above Wronskian determinant, and afterwards, Matveev [Mat] generalized the Wronskian determinant which allows us to present another important class of exact solutions, called positons, to the KdV equation. In using the Wronskian method to solve the KdV equation, one usually starts from

$$-\phi_{i,xx} = \lambda_i\phi_i, \ \phi_{i,t} = -4\phi_{i,xxx}, \ 1 \leq i \leq N, \tag{1.5}$$

where λ_i are arbitrary real constants, to guarantee that the Wronskian determinant (1.3) solves the bilinear KdV equation (1.2) and thus generates a solution

$$u = -2\partial_x^2 \ln W(\phi_1,\phi_2,\cdots,\phi_N) \tag{1.6}$$

to the KdV equation (1.1). The solution determined by a Wronskian determinant is called a Wronskian solution. For example, $f = W(\phi_1,\phi_2,\cdots,\phi_n)$ and $u = -2\partial_x^2 \ln W(\phi_1,\phi_2,\cdots,\phi_N)$ are Wronskian solutions to the bilinear KdV equation

(1.2) and the KdV equation (1.1), respectively, provided that $\phi_1, \phi_2, \cdots, \phi_n$ satisfy (1.5). Sirianunpiboon et al. [SHR] extended the conditions (1.5) to the following:

$$-\phi_{i,xx} = \sum_{j=1}^{i} \lambda_{ij}\phi_j, \ \phi_{i,t} = -4\phi_{i,xxx}, \ 1 \le i \le N, \tag{1.7}$$

where λ_{ij} are arbitrary real constants, in order that the Wronskian determinant can generate rational function solutions and their interaction solutions with multi-solitons. The first group of the above conditions (1.7) is a triangular system of ordinary differential equations. Thus, solving the system of differential equations (1.7) is exactly the same for scalar differential equations.

In this paper, we would like to construct explicit exact solutions to the KdV equation (1.1) by its bilinear form and to illustrate the entire process of construction with a broader set of sufficient conditions on the Wronskian solutions:

$$-\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \ \phi_{i,t} = -4\phi_{i,xxx} + \xi\phi_i, \ 1 \le i \le N, \tag{1.8}$$

where λ_{ij} and ξ are arbitrary real constants. Here the first group of conditions is a coupled system of ordinary differential equations of second-order, and thus needs a more general consideration for solving the involved differential equations. Note that the coefficient matrix $\Lambda = (\lambda_{ij})$ in (1.7) has only real eigenvalues, but the coefficient matrix $\Lambda = (\lambda_{ij})$ in (1.8) in general has complex eigenvalues. This is a significant difference between two sets of conditions, defined by (1.7) and (1.8). The substantial extension in (1.8) allows us to get a much broader class of explicit exact solutions to the KdV equation (1.1). In particular, new exact solutions called complexitons can be generated from the set of conditions (1.8), but not (1.7).

Now the whole problem of constructing the Wronskian solutions to the KdV equation (1.1) reduces to solve the coupled system of partial differential equations (1.8), which turns out to be an interesting mathematical problem itself. We shall analyze solution structures on the system (1.8) in detail, to obtain solution formulas for all cases of the system (1.8). The resultant fundamental solution formulas provide a direct and comprehensive approach to construct diverse exact solutions to the KdV equation (1.1), such as rational solutions, solitons, positons, negatons, breathers, complexitons, and more generally their interaction solutions.

2. Sufficient conditions on Wronskian solutions

We begin with stating a broad set of sufficient conditions which make the Wronskian determinant a solution to the bilinear KdV equation (1.2).

Theorem 2.1. *Assume that a group of functions $\phi_i = \phi_i(x,t)$, $1 \le i \le N$, satisfies the two sets of conditions*

$$-\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}(t)\phi_j, \ 1 \le i \le N, \tag{2.1}$$

$$\phi_{i,t} = -4\phi_{i,xxx} + \xi(t)\phi_i, \ 1 \le i \le N, \tag{2.2}$$

simultaneously, where $\lambda_{ij}(t)$ are arbitrary differentiable real functions of t and $\xi(t)$ is an arbitrary continuous real function of t. Then $f = (\widehat{N-1})$ defined by (1.3) *solves the bilinear KdV equation* (1.2).

The conditions (2.1) and (2.2) are a generalization to (1.7) presented by Sirianunpiboon et al. [SHR]. Actually, if $\xi = 0$ and the coefficient matrix $\Lambda = (\lambda_{ij})$ of (2.1) is lower-triangular and independent of time t, then the result of the above theorem boils down to the result in [SHR]. The proof of the theorem needs some basic equalities, and the following notation is helpful in our deduction and analysis:

$$\begin{aligned}(\widehat{N-j-1}, i_1, \cdots, i_j) &= |\Phi^{(0)}, \Phi^{(1)}, \cdots, \Phi^{(N-j-1)}, \Phi^{(i_1)}, \cdots, \Phi^{(i_j)}| \\ &= \det(\Phi^{(0)}, \Phi^{(1)}, \cdots, \Phi^{(N-j-1)}, \Phi^{(i_1)}, \cdots, \Phi^{(i_j)}),\ 1 \le j \le N-1, \end{aligned} \tag{2.3}$$

where $i_1, \cdots, i_j$ are non-negative integers, and the vectors of functions $\Phi^{(j)}$ are defined by

$$\Phi^{(j)} = (\phi_1^{(j)}, \phi_2^{(j)}, \cdots, \phi_N^{(j)})^T,\ 0 \le j \le N-1. \tag{2.4}$$

We also use the assumption for convenience that if $i < 0$, the column vector $\Phi^{(i)}$ will disappear in the determinant $\det(\cdots, \Phi^{(i)}, \cdots)$.

Lemma 2.2. *Under the conditions* (2.1)*, the following equalities hold:*

$$\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-1}) = (\widehat{N-3}, N-1, N) - (\widehat{N-2}, N+1), \tag{2.5}$$

$$\frac{\partial^2(\widehat{N-1})}{\partial x^2} = -\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-1}) + 2(\widehat{N-3}, N-1, N), \tag{2.6}$$

$$\begin{aligned}\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-3}, N-1, N) &= (\widehat{N-5}, N-3, N-2, N-1, N) \\ &+(\widehat{N-3}, N, N+1) - (\widehat{N-3}, N-1, N+2),\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\Big(\sum_{i=1}^{N} \lambda_{ii}(t)\Big)^2 (\widehat{N-1}) &= (\widehat{N-5}, N-3, N-2, N-1, N) \\ &-(\widehat{N-4}, N-2, N-1, N+1) - (\widehat{N-3}, N-1, N+2) \\ &+2(\widehat{N-3}, N, N+1) + (\widehat{N-2}, N+3).\end{aligned} \tag{2.8}$$

Proof. Note that we have

$$\sum_{k=1}^{N} |A|_{kl} = \sum_{k=1}^{N} |A|^{kl} = \sum_{i,j=1}^{N} A_{ij} \frac{\partial^l a_{ij}}{\partial x^l},$$

where $A = (a_{ij})_{N\times N}$, and $|A|_{kl}$, $|A|^{kl}$ and A_{ij} denote the determinant resulting from $|A|$ with its kth row differentiated l times with respect to x, the determinant resulting from $|A|$ with its kth column differentiated l times with respect to x, and the co-factor of a_{ij}, respectively. Choose $|A|$ as $(\widehat{N-1})$ and $(\widehat{N-3}, N-1, N)$, and use the above equality with $l = 2$ and the conditions (2.1). Then we obtain the required equalities (2.5) and (2.7) immediately. A combination of the equality (2.5) and the equality

$$\frac{\partial^2(\widehat{N-1})}{\partial x^2} = (\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1)$$

leads to the equality (2.6).

We now differentiate (2.5) twice with respect to x and utilize (2.6) and (2.7). Then a further computation yields the equality (2.8). This completes the proof of the lemma. □

Proof of Theorem 2.1. By using the conditions (2.2), we obtain that

$$\begin{aligned} f_t &= N\xi f - 4[(\widehat{N-4}, N-2, N-1, N) \\ &\quad -(\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2)], \\ f_{xt} &= N\xi f_x - 4[(\widehat{N-5}, N-3, N-2, N-1, N) \\ &\quad -(\widehat{N-3}, N, N+1) + (\widehat{N-2}, N+3)]. \end{aligned}$$

Therefore, we can further deduce that

$$\begin{aligned} \Delta &:= D_x(D_t + D_x^3)f \cdot f = f_{xt}f - f_t f_x + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 \\ &= [-3(\widehat{N-5}, N-3, N-2, N-1, N) + 6(\widehat{N-3}, N, N+1) - 3(\widehat{N-2}, N+3) \\ &\quad + 3(\widehat{N-4}, N-2, N-1, N+1) + 3(\widehat{N-3}, N-1, N+2)](\widehat{N-1}) \\ &\quad - 12(\widehat{N-3}, N-1, N+1)(\widehat{N-2}, N) \\ &\quad + 3[(\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1)]^2. \end{aligned}$$

Now using the equalities (2.5) and (2.8) in Lemma 2.2 and the Laplace expansion of determinants about the last N rows, we have

$$\begin{aligned} \Delta &= -12(\widehat{N-3}, N-1, N+1)(\widehat{N-2}, N) \\ &\quad +12(\widehat{N-3}, N, N+1)(\widehat{N-1}) + 12(\widehat{N-3}, N-1, N)(\widehat{N-2}, N+1) \\ &= 6 \begin{vmatrix} \widehat{N-3} & 0 & N-2 & N-1 & N & N+1 \\ 0 & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix} = 0. \end{aligned}$$

This shows that $f = (\widehat{N-1})$ solves (1.2). Note that the entries in the last abbreviated $2N \times 2N$ determinant above are derivatives of Φ, e.g., $N-1$ denotes $\Phi^{(N-1)}$. The proof is finished. □

Theorem 2.1 tells us that if a group of functions $\phi_i(x,t),\ 1 \le i \le N$, satisfies the conditions (2.1) and (2.2), then we can get a solution $f = W(\phi_1, \phi_2, \cdots, \phi_N)$ to the bilinear KdV equation (1.2). The conditions (2.2) and (2.1) are two linear systems of second-order and third-order partial differential equations. We are going to solve these linear systems explicitly. It is rather difficult to deal with the case where the coefficient matrix Λ of (2.1) has complex eigenvalues. However, when $-\Lambda$ is a positive definite constant matrix, the situation is easy and the solution formula of (2.1) and (2.2) can be expressed as

$$\Phi = \mathrm{e}^{\int_0^t \xi(t')\,dt'} \mathrm{e}^{\sqrt{-\Lambda}\,x + 4\Lambda\sqrt{-\Lambda}\,t} \Phi_0, \tag{2.9}$$

where $\sqrt{-\Lambda}$ is the square root of $-\Lambda$ and Φ_0 is an arbitrary initial vector. The corresponding Wronskian solutions to the KdV equation (1.1),

$$u = -2\partial_x^2 \ln f = -2\partial_x^2 \ln(\widehat{N-1}; \Phi), \tag{2.10}$$

contain soliton and negaton solutions.

Before we proceed to solve (2.1) and (2.2), let us observe the Wronskian determinants and solutions more carefully.

Oservation I. From the compatibility conditions $\phi_{i,txx} = \phi_{i,xxt},\ 1 \le i \le N$, of the conditions (2.1) and (2.2), we have the equality

$$\sum_{j=1}^{N} \lambda_{ij,t}\phi_j = 0,\ 1 \le i \le N, \tag{2.11}$$

and thus it is easy to see that the Wronskian determinant $W(\phi_1, \phi_2, \cdots, \phi_N)$ becomes zero if there is at least one entry λ_{ij} satisfying $\lambda_{ij,t} \neq 0$.

Oservation II. If we make use of the transformation

$$\tilde{\phi}_i = \mathrm{e}^{-\int_0^t \xi(s)\,ds}\phi_i,\ 1 \le i \le N, \tag{2.12}$$

the differential equations in (2.2) can be put into

$$\tilde{\phi}_{i,t} = -4\tilde{\phi}_{i,xxx},\ 1 \le i \le N,$$

but the differential equations (2.1) do not change. Obviously, the resultant Wronskian solutions to the KdV equation are the same:

$$u = -2\partial_x^2 \ln W(\phi_1, \phi_2, \cdots, \phi_N) = \tilde{u} = -2\partial_x^2 \ln W(\tilde{\phi}_1, \tilde{\phi}_2, \cdots, \tilde{\phi}_N). \tag{2.13}$$

But the Wronskian determinants $W(\phi_1, \phi_2, \cdots, \phi_N)$ and $W(\tilde{\phi}_1, \tilde{\phi}_2, \cdots, \tilde{\phi}_N)$ are different, and thus they gives rise to different solutions to the bilinear KdV equation (1.2).

Oservation III. If the coefficient matrix $\Lambda = (\lambda_{ij})$ is similar to another matrix $M = (\mu_{ij})$ under an invertible constant matrix P, let us say $\Lambda = P^{-1}MP$, then $\tilde{\Phi} = P\Phi$ solves

$$\tilde{\Phi}_{xx} = M\tilde{\Phi},\ \tilde{\Phi}_t = -4\tilde{\Phi}_{xxx} + \xi\tilde{\Phi}, \tag{2.14}$$

and the resultant Wronskian solutions to the KdV equation are also the same:

$$\begin{aligned} u(\Lambda) &= -2\partial_x^2 \ln |\Phi^{(0)}, \Phi^{(1)}, \cdots, \Phi^{(N-1)}| \\ &= -2\partial_x^2 \ln |P\Phi^{(0)}, P\Phi^{(1)}, \cdots, P\Phi^{(N-1)}| = u(M). \end{aligned} \tag{2.15}$$

It follows from Observations II and III that different Wronskian determinants may lead to the same solution to the KdV equation. Therefore, based on Observation I, in order to construct Wronskian solutions to the KdV equation by its bilinear form, we only need to consider the reduced case of (2.2) and (2.1) under $\xi = 0$ and $d\Lambda/dt = 0$, i.e., the following conditions:

$$-\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j,\ \phi_{i,t} = -4\phi_{i,xxx},\ 1 \le i \le N, \tag{2.16}$$

where λ_{ij} are arbitrary real constants. On the other hand, the Jordan form of a real matrix has the following two types of blocks:

$$\begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i \times k_i}, \tag{2.17}$$

$$\begin{bmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{bmatrix}_{l_i \times l_i}, \quad A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.18}$$

where λ_i, α_i and $\beta_i > 0$ are all real constants. The first type of blocks has the real eigenvalue λ_i with algebraic multiplicity k_i, and the second type of blocks has the complex eigenvalues $\lambda_i^{\pm} = \alpha_i \pm \beta_i\sqrt{-1}$ with algebraic multiplicity l_i. Note that an eigenvalue of the coefficient matrix $\Lambda = (\lambda_{ij})$ is also an eigenvalue of the Schrödinger operator $-\frac{\partial^2}{\partial x^2} + u$ with zero potential. We will present solution formulas for the system of differential equations defined by (2.16), according to the situations of eigenvalues of the coefficient matrix. Now, based on Observation III, all we need to do is to solve a group of subsystems of the sufficient conditions on the Wronskian solutions, whose coefficient matrices are of the forms (2.17) and (2.18).

Let us summarize the above analysis as the following theorem.

Theorem 2.3. *For the KdV equation* (1.1)*, all Wronskian solutions generated from the conditions* (2.1) *and* (2.2) *are among the Wronskian solutions associated with the special cases of the conditions* (2.16) *whose coefficient matrices* $\Lambda = (\lambda_{ij})$ *are of the Jordan form consisting of two types of Jordan blocks in* (2.17) *and* (2.18)*.*

In the next section, we shall consider how to solve the two types of subsystems associated with the Jordan blocks in (2.17) and (2.18), and present solution formulas for their representative systems generating Wronskian solutions to the KdV equation (1.1).

3. Solution formulas for the representative systems

In this section, we would like to consider the construction of solutions to the associated system of differential equations defined by (2.16). Based on the form of two types of Jordan blocks in (2.17) and (2.18), a basic idea to solve the system (2.16) is to use a recursion process. That is to solve (2.16) from ϕ_1 to ϕ_N, one by one when the coefficient matrix $\Lambda = (\lambda_{ij})$ has real eigenvalues or pair by pair when the coefficient matrix $\Lambda = (\lambda_{ij})$ has complex eigenvalues. Therefore, the entire problem is divided into two subproblems—to solve the following two representative systems of non-homogeneous differential equations:

$$-\phi_{xx} = \lambda\phi + f, \ \phi_t = -4\phi_{xxx}, \tag{3.1}$$

and

$$\begin{cases} -\phi_{1,xx} = \alpha\phi_1 - \beta\phi_2 + f_1, \\ -\phi_{2,xx} = \beta\phi_1 + \alpha\phi_2 + f_2, \\ \phi_{i,t} = -4\phi_{i,xxx}, \ i = 1, 2, \end{cases} \tag{3.2}$$

where λ, α and $\beta > 0$ are real constants, and $f = f(x,t)$, $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are three given functions satisfying the compatibility condition $g_t = -4g_{xxx}$. The whole system (2.16) whose coefficient matrix is of Jordan form decouples into a group of systems determined by (3.1) and (3.2).

3.1. **The case of real eigenvalues.** First, let us consider the first representative system (3.1). In terms of the eigenvalue λ, we will establish solution formulas for three situations of the representative system (3.1).

Zero eigenvalue: The representative system (3.1) in this case is

$$\phi_{xx} = f, \ \phi_t = -4\phi_{xxx}, \tag{3.3}$$

where $f = f(x,t)$ is a given function satisfying the compatibility condition $f_t = -4f_{xxx}$. Immediately from the first equation in (3.3), we can get

$$\phi = \int_0^x \int_0^{x'} f(x'', t) dx'' dx' + c(t)x + d(t). \tag{3.4}$$

Note that $f_t = -4f_{xxx}$, and thus the second equation of (3.3) equivalently requires

$$c_t = -4f_{xx}|_{x=0}, \ d_t = -4f_x|_{x=0}.$$

Then, we obtain

$$c(t) = -4\int_0^t f_{xx}(0,t')\,dt' + c_0, \ d(t) = -4\int_0^t f_x(0,t')\,dt' + d_0, \tag{3.5}$$

where c_0 and d_0 are arbitrary real constants. Summing up, we have the following theorem.

Theorem 3.1. *Let $f = f(x,t)$ be a given function satisfying the compatibility condition $f_t = -4f_{xxx}$. Then the system of differential equations* (3.3) *has the general solution given by* (3.4) *and* (3.5).

Negative eigenvalue: In this case, the representative system (3.1) is

$$\phi_{xx} = \alpha\phi + f, \ \phi_t = -4\phi_{xxx}, \ \alpha > 0, \tag{3.6}$$

where α is a constant and f is a given function satisfying the compatibility condition $f_t = -4f_{xxx}$. From two basic solutions $\mathrm{e}^{\sqrt{\alpha}x}$ and $\mathrm{e}^{-\sqrt{\alpha}x}$ of $\phi_{xx} = \alpha\phi$, we get a general solution of the first differential equation of (3.6):

$$\begin{aligned} \phi \ = \ & \Big[\frac{1}{2\sqrt{\alpha}}\int_0^x f(x',t)\mathrm{e}^{-\sqrt{\alpha}\,x'}dx' + c(t)\Big]\mathrm{e}^{\sqrt{\alpha}\,x} \\ & -\Big[\frac{1}{2\sqrt{\alpha}}\int_0^x f(x',t)\mathrm{e}^{\sqrt{\alpha}\,x'}dx' + d(t)\Big]\mathrm{e}^{-\sqrt{\alpha}\,x}, \end{aligned} \tag{3.7}$$

by variation of parameters. Then noting that $f_t = -4f_{xxx}$, by a direct but lengthy computation, we find that the second differential equation of (3.6) equivalently

requires

$$c_t = -4\alpha\sqrt{\alpha}\,c - 2(\frac{1}{\sqrt{\alpha}}\,f_{xx} + f_x + \sqrt{\alpha}\,f)|_{x=0},$$
$$d_t = 4\alpha\sqrt{\alpha}\,d + 2(-\frac{1}{\sqrt{\alpha}}\,f_{xx} + f_x - \sqrt{\alpha}\,f)|_{x=0}.$$

These are two linear systems for c and d, respectively. Hence, we can immediately obtain the solution formulas for c and d:

$$c(t) = \mathrm{e}^{-4\alpha\sqrt{\alpha}\,t}\Big[c_0 - 2\int_0^t \mathrm{e}^{4\alpha\sqrt{\alpha}\,t'}(\frac{1}{\sqrt{\alpha}}\,f_{xx} + f_x + \sqrt{\alpha}\,f)(0,t'))\,dt'\Big], \tag{3.8}$$

$$d(t) = \mathrm{e}^{4\alpha\sqrt{\alpha}\,t}\Big[d_0 + 2\int_0^t \mathrm{e}^{-4\alpha\sqrt{\alpha}\,t'}(-\frac{1}{\sqrt{\alpha}}\,f_{xx} + f_x - \sqrt{\alpha}\,f)(0,t'))\,dt'\Big], \tag{3.9}$$

where c_0 and d_0 are arbitrary real constants. We conclude this result as follows.

Theorem 3.2. *Let $f = f(x,t)$ be a given function satisfying the compatibility condition $f_t = -4f_{xxx}$. Then the system of differential equations* (3.6) *has the general solution given by* (3.7), (3.8) *and* (3.9).

Positive eigenvalue: The representative system (3.1) in this case is

$$\phi_{xx} = -\alpha\phi + f,\ \alpha > 0, \tag{3.10}$$

$$\phi_t = -4\phi_{xxx}, \tag{3.11}$$

where α is a constant and $f = f(x,t)$ is a given function satisfying the compatibility condition $f_t = -4f_{xxx}$. To solve the differential equation (3.10), we can start from two basic solutions $\sin\sqrt{\alpha}x$ and $\cos\sqrt{\alpha}x$ of the corresponding homogeneous equation of (3.10), and then use the method of variation of parameters. A general solution is listed in the following lemma for reference.

Lemma 3.3. *For any given function $g = g(x)$ and any positive constant α, the second-order ordinary differential equation $\psi_{xx} = -\alpha\psi + g$ has the general solution*

$$\psi = \int_0^x \frac{\sin\sqrt{\alpha}(x-x')}{\sqrt{\alpha}} g(x')\,dx' + c\cos\sqrt{\alpha}x + d\sin\sqrt{\alpha}x,$$

where c and d are arbitrary real constants.

Now to solve the partial differential equation (3.11), we start with the solution of (3.10) given by Lemma 3.3 and use the method of variation of parameters again. Therefore, the solution of (3.11) is assumed to be

$$\phi = \int_0^x \frac{\sin\sqrt{\alpha}(x-x')}{\sqrt{\alpha}} f(x',t)\,dx' + c(t)\cos\sqrt{\alpha}x + d(t)\sin\sqrt{\alpha}x. \tag{3.12}$$

Directly we can compute that

$$\begin{aligned}
\phi_{xxx} &= f_x - \int_0^x f(x',t)(\alpha\cos\sqrt{\alpha}(x-x'))\,dx' \\
&\quad + c(t)\alpha\sqrt{\alpha}\sin\sqrt{\alpha}x - d(t)\alpha\sqrt{\alpha}\cos\sqrt{\alpha}x, \\
\phi_t &= 4f_{xx}|_{x=0}\frac{\sin\sqrt{\alpha}x}{\sqrt{\alpha}} + 4f_x|_{x=0}\cos\sqrt{\alpha}x - 4f|_{x=0}(\sqrt{\alpha}\sin\sqrt{\alpha}x) \\
&\quad - 4f_x + \int_0^x 4f(x',t)(\alpha\cos\sqrt{\alpha}(x-x'))\,dx' \\
&\quad + c_t\cos\sqrt{\alpha}x + d_t\sin\sqrt{\alpha}x,
\end{aligned}$$

where $f_t = -4f_{xxx}$ has been used for the computation of ϕ_t. Therefore, the differential equation (3.11) equivalently requires

$$\frac{4}{\sqrt{\alpha}} f_{xx}|_{x=0} - 4\sqrt{\alpha} f|_{x=0} + d_t = -4\alpha\sqrt{\alpha}\, c, \tag{3.13}$$

$$4f_x|_{x=0} + c_t = 4\alpha\sqrt{\alpha}\, d. \tag{3.14}$$

Then it follows that

$$c_{tt} + 16\alpha^3 c + h = 0,$$

where the function $h = h(t)$ is defined by

$$h := 16\alpha f_{xx}|_{x=0} - 16\alpha^2 f|_{x=0} + 4f_{xt}|_{x=0}. \tag{3.15}$$

Again by Lemma 3.3, we obtain

$$c(t) = -\int_0^t \frac{\sin 4\alpha\sqrt{\alpha}(t-t')}{4\alpha\sqrt{\alpha}} h(t')\, dt' + c_0 \cos 4\alpha\sqrt{\alpha} t + d_0 \sin 4\alpha\sqrt{\alpha} t,$$

where c_0 and d_0 are arbitrary real constants. Noting that h is given by (3.15), we have

$$\begin{aligned} c(t) \;=\; & f_x(0,0)\frac{\sin 4\alpha\sqrt{\alpha} t}{\alpha\sqrt{\alpha}} - 4\int_0^t f_x(0,t')\cos 4\alpha\sqrt{\alpha}(t-t')\, dt' \\ & -4\int_0^t (\frac{1}{\sqrt{\alpha}} f_{xx} - \sqrt{\alpha} f)(0,t')\sin 4\alpha\sqrt{\alpha}(t-t')\, dt' \\ & +c_0 \cos 4\alpha\sqrt{\alpha} t + d_0 \sin 4\alpha\sqrt{\alpha} t, \end{aligned} \tag{3.16}$$

and then by (3.14), we have

$$\begin{aligned} d(t) \;=\; & f_x(0,0)\frac{\cos 4\alpha\sqrt{\alpha} t}{\alpha\sqrt{\alpha}} + 4\int_0^t f_x(0,t')\sin 4\alpha\sqrt{\alpha}(t-t')\, dt' \\ & -4\int_0^t (\frac{1}{\sqrt{\alpha}} f_{xx} - \sqrt{\alpha} f)(0,t')\cos 4\alpha\sqrt{\alpha}(t-t')\, dt' \\ & -c_0 \sin 4\alpha\sqrt{\alpha} t + d_0 \cos 4\alpha\sqrt{\alpha} t, \end{aligned} \tag{3.17}$$

where c_0 and d_0 are independent of x and t.

Finally, summing up, we have the following theorem on the general solution of the system of differential equations (3.10) and (3.11).

Theorem 3.4. *Let $f = f(x,t)$ be a given function satisfying the compatibility condition $f_t = -4f_{xxx}$, and α a positive constant. Then the system of differential equations* (3.11) *and* (3.10) *has the general solution given by* (3.12)*) with $c(t)$ and $d(t)$ being shown by* (3.16) *and* (3.17)*, respectively.*

3.2. **The case of complex eigenvalues.** Let us now consider the second representative system (3.2). To present its solution formula, we first solve the following coupled system of second-order, non-homogeneous ordinary differential equations

$$\begin{cases} -\phi_{1,xx} = \alpha\phi_1 - \beta\phi_2 + f_1, \\ -\phi_{2,xx} = \beta\phi_1 + \alpha\phi_2 + f_2, \end{cases} \tag{3.18}$$

where α and $\beta > 0$ are real constants, and f_1 and f_2 are two given functions of x. Since $\beta > 0$, its coefficient matrix

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

has two complex eigenvalues $\lambda^{\pm} = \alpha \pm \beta i$.

Theorem 3.5. *The coupled non-homogenous system of second-order ordinary differential equations* (3.18) *has the general solution*

$$\phi_1 = \phi_1^h + \phi_1^s,\ \phi_2 = \phi_2^h + \phi_2^s, \tag{3.19}$$

$$\phi_1^h = (D_1 \cos \delta x + D_2 \sin \delta x) e^{\Delta x} + (D_3 \cos \delta x + D_4 \sin \delta x) e^{-\Delta x}, \tag{3.20}$$

$$\phi_2^h = (-D_2 \cos \delta x + D_1 \sin \delta x) e^{\Delta x} + (-D_4 \cos \delta x + D_3 \sin \delta x) e^{-\Delta x}, \tag{3.21}$$

$$\phi_1^s = -\frac{1}{\Delta} \int_0^x \big[f_1(y) \cos \delta(x-y) - f_2(y) \sin \delta(x-y) \big] \sinh \Delta(x-y)\, dy, \tag{3.22}$$

$$\phi_2^s = -\frac{1}{\Delta} \int_0^x \big[f_1(y) \sin \delta(x-y) + f_2(y) \cos \delta(x-y) \big] \sinh \Delta(x-y)\, dy, \tag{3.23}$$

where two constants Δ and δ are given by

$$\Delta = \sqrt{\frac{\sqrt{\alpha^2+\beta^2} - \alpha}{2}},\ \delta = \sqrt{\frac{\sqrt{\alpha^2+\beta^2} + \alpha}{2}}, \tag{3.24}$$

and D_i, $1 \le i \le 4$, are arbitrary real constants.

Proof. First let us consider the homogenous case of (3.18), i.e., the case of (3.18) with $f_1 = f_2 = 0$. Therefore, we have

$$\phi_2 = \frac{1}{\beta} \phi_{1,xx} + \frac{\alpha}{\beta} \phi_1. \tag{3.25}$$

Then replacing ϕ_2 with this expression in the second equation of the homogenous case leads to the following fourth-order differential equation

$$\phi_{1,xxxx} + 2\alpha \phi_{1,xx} + (\alpha^2 + \beta^2) \phi_1 = 0. \tag{3.26}$$

Its characteristic polynomial

$$p(\lambda) = \lambda^4 + 2\alpha\lambda^2 + (\alpha^2 + \beta^2) = 0$$

has four complex roots

$$\lambda_1^{\pm} = \Delta \pm \delta\sqrt{-1},\ \lambda_2^{\pm} = -\Delta \pm \delta\sqrt{-1},$$

where Δ and δ are defined by (3.24). Therefore, we have the general solution ϕ_1^h and ϕ_2^h, as given in the theorem, for the homogeneous case of (3.18).

To construct a special solution, ϕ_1^s and ϕ_2^s, for the non-homogenous case of (3.18), we adopt the method of variation of parameters for D_i, $1 \le i \le 4$, and thus assume that

$$\begin{cases} \phi_1^s = (D_1 \cos \delta x + D_2 \sin \delta x) \mathrm{e}^{\Delta x} + (D_3 \cos \delta x + D_4 \sin \delta x) \mathrm{e}^{-\Delta x}, \\ \phi_2^s = (-D_2 \cos \delta x + D_1 \sin \delta x) \mathrm{e}^{\Delta x} + (-D_4 \cos \delta x + D_3 \sin \delta x) \mathrm{e}^{-\Delta x}, \end{cases} \tag{3.27}$$

where D_i, $1 \leq i \leq 4$, are viewed as functions of x. Then we can make the following computation:

$$\begin{aligned}
(3.28)\quad \phi^s_{1,x} &= \big[(\Delta D_1 + \delta D_2)\cos\delta x + (\Delta D_2 - \delta D_1)\sin\delta x\big]\mathrm{e}^{\Delta x} \\
&\quad +(\delta D_4 - \Delta D_3)\cos\delta x - (\delta D_3 + \Delta D_4)\sin\delta x\big]\mathrm{e}^{-\Delta x}, \\
(3.29)\quad \phi^s_{1,xx} &= \big[\Delta(\Delta D_1 + \delta D_2) + \delta(\Delta D_2 - \delta D_1)\big](\cos\delta x)\mathrm{e}^{\Delta x} \\
&\quad +\big[\Delta(\Delta D_2 - \delta D_1) - \delta(\Delta D_1 + \delta D_2)\big](\sin\delta x)\mathrm{e}^{\Delta x} \\
&\quad +\big[-\delta(\delta D_3 + \Delta D_4) - \Delta(\delta D_4 - \Delta D_3)\big](\cos\delta x)\mathrm{e}^{-\Delta x} \\
&\quad +\big[-\delta(\delta D_4 - \Delta D_3) + \Delta(\delta D_3 + \Delta D_4)\big](\sin\delta x)\mathrm{e}^{-\Delta x} \\
&= (\alpha D_1 + \beta D_2)(\cos\delta x)\mathrm{e}^{\Delta x} + (-\beta D_1 + \alpha D_2)(\sin\delta x)\mathrm{e}^{\Delta x} \\
&\quad +(\alpha D_3 - \beta D_4)(\cos\delta x)\mathrm{e}^{-\Delta x} + (\beta D_3 + \alpha D_4)(\sin\delta x)\mathrm{e}^{-\Delta x} \\
&:= (\tilde{D}_1\cos\delta x + \tilde{D}_2\sin\delta x)\mathrm{e}^{\Delta x} + (\tilde{D}_3\cos\delta x + \tilde{D}_4\sin\delta x)\mathrm{e}^{-\Delta x}, \\
(3.30)\quad \phi^s_{2,x} &= \big[(-\Delta D_2 + \delta D_1)\cos\delta x + (\Delta D_1 + \delta D_2)\sin\delta x\big]\mathrm{e}^{\Delta x} \\
&\quad +(\delta D_3 + \Delta D_4)\cos\delta x + (\delta D_4 - \Delta D_3)\sin\delta x\big]\mathrm{e}^{-\Delta x}, \\
(3.31)\quad \phi^s_{2,xx} &= (-\alpha D_2 + \beta D_1)(\cos\delta x)\mathrm{e}^{\Delta x} + (\beta D_2 + \alpha D_1)(\sin\delta x)\mathrm{e}^{\Delta x} \\
&\quad -(\alpha D_4 + \beta D_3)(\cos\delta x)\mathrm{e}^{-\Delta x} - (\beta D_4 - \alpha D_3)(\sin\delta x)\mathrm{e}^{-\Delta x}.
\end{aligned}$$

In the above computation, the method of variation of parameters required that

$$\begin{aligned}
&(D_{1,x}\cos\delta x + D_{2,x}\sin\delta x)\mathrm{e}^{\Delta x} + (D_{3,x}\cos\delta x + D_{4,x}\sin\delta x)\mathrm{e}^{-\Delta x} = 0, \\
&\big[(\Delta D_{1,x} + \delta D_{2,x})\cos\delta x + (\Delta D_{2,x} - \delta D_{1,x})\sin\delta x\big]\mathrm{e}^{\Delta x} \\
&+(\delta D_{4,x} - \Delta D_{3,x})\cos\delta x - (\delta D_{3,x} + \Delta D_{4,x})\sin\delta x\big]\mathrm{e}^{-\Delta x} = -f_1, \\
&(-D_{2,x}\cos\delta x + D_{1,x}\sin\delta x)\mathrm{e}^{\Delta x} - (D_{4,x}\cos\delta x - D_{3,x}\sin\delta x)\mathrm{e}^{-\Delta x} = 0, \\
&\big[(-\Delta D_{2,x} + \delta D_{1,x})\cos\delta x + (\Delta D_{1,x} + \delta D_{2,x})\sin\delta x\big]\mathrm{e}^{\Delta x} \\
&+(\delta D_{3,x} + \Delta D_{4,x})\cos\delta x + (\delta D_{4,x} - \Delta D_{3,x})\sin\delta x\big]\mathrm{e}^{-\Delta x} = -f_2.
\end{aligned}$$

Solving this linear system for $D_{i,x}$ leads to the expressions:

$$\begin{aligned}
D_{1,x} &= -\frac{1}{2\Delta}\big(f_1\cos\delta x + f_2\sin\delta x\big)\mathrm{e}^{-\Delta x}, \\
D_{2,x} &= -\frac{1}{2\Delta}\big(f_1\sin\delta x - f_2\cos\delta x\big)\mathrm{e}^{-\Delta x}, \\
D_{3,x} &= \frac{1}{2\Delta}\big(f_1\cos\delta x + f_2\sin\delta x\big)\mathrm{e}^{-\Delta x} = -D_{1,x}, \\
D_{4,x} &= \frac{1}{2\Delta}\big(f_1\sin\delta x - f_2\cos\delta x\big)\mathrm{e}^{-\Delta x} = -D_{2,x}.
\end{aligned}$$

Then integrating them to obtain the expressions for D_i and inserting the resultant expressions of D_i into (3.27) yield the special solution, ϕ^s_1 and ϕ^s_2, given in the theorem. The proof is finished. □

Let us then present the solution formula for the second representative system (3.2), which can be used to generate a new type of Wronskian solutions to the KdV equation.

Theorem 3.6. *Assume that $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are two given functions satisfying the compatibility conditions $f_{1,t} = -4f_{1,xxx}$ and $f_{2,t} = -4f_{2,xxx}$. Then*

the general solution to the system of differential equations (3.2) *is given by* (3.19)–(3.23) *with the coefficients* $D_i = D_i(t)$, $1 \le i \le 4$, *being determined by*

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = e^{\tilde{\alpha}t} \begin{bmatrix} \cos\tilde{\beta}t & -\sin\tilde{\beta}t \\ \sin\tilde{\beta}s & \cos\tilde{\beta}t \end{bmatrix} \begin{bmatrix} D_{10} \\ D_{20} \end{bmatrix} + \int_0^t e^{\tilde{\alpha}(t-s)} \begin{bmatrix} \cos\tilde{\beta}(t-s) & -\sin\tilde{\beta}(t-s) \\ \sin\tilde{\beta}(t-s) & \cos\tilde{\beta}(t-s) \end{bmatrix} \begin{bmatrix} p_1(s) \\ p_2(s) \end{bmatrix} ds, \tag{3.32}$$

$$\begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = e^{-\tilde{\alpha}t} \begin{bmatrix} \cos\tilde{\beta}t & -\sin\tilde{\beta}t \\ \sin\tilde{\beta}s & \cos\tilde{\beta}t \end{bmatrix} \begin{bmatrix} D_{30} \\ D_{40} \end{bmatrix} + \int_0^t e^{-\tilde{\alpha}(t-s)} \begin{bmatrix} \cos\tilde{\beta}(t-s) & -\sin\tilde{\beta}(t-s) \\ \sin\tilde{\beta}(t-s) & \cos\tilde{\beta}(t-s) \end{bmatrix} \begin{bmatrix} p_3(s) \\ p_4(s) \end{bmatrix} ds, \tag{3.33}$$

where D_{i0}, $1 \le i \le 4$, *are arbitrary real constants, and two constants* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *are given by*

$$\tilde{\alpha} = -4\Delta(\sqrt{\alpha^2+\beta^2}+2\alpha),\ \tilde{\beta} = 4\delta(\sqrt{\alpha^2+\beta^2}-2\alpha), \tag{3.34}$$

and four functions, p_i, $1 \le i \le 4$, *are defined by*

$$\begin{aligned}
p_1(t) &= \frac{2}{\Delta}f_{1,xx}|_{x=0} + 2f_{1,x}|_{x=0} + \frac{2\alpha}{\Delta}f_1|_{x=0} - \frac{2\delta}{\Delta}f_{2,x}|_{x=0} - 4\delta f_2|_{x=0},\\
p_2(t) &= -\frac{2\delta}{\Delta}f_{1,x}|_{x=0} - 4\delta f_1|_{x=0} - \frac{2}{\Delta}f_{2,xx}|_{x=0} - 2f_{2,x}|_{x=0} - \frac{2\alpha}{\Delta}f_2|_{x=0},\\
p_3(t) &= -\frac{2}{\Delta}f_{1,xx}|_{x=0} + 2f_{1,x}|_{x=0} - \frac{2\alpha}{\Delta}f_1|_{x=0} + \frac{2\delta}{\Delta}f_{2,x}|_{x=0} - 4\delta f_2|_{x=0},\\
p_4(t) &= \frac{2\delta}{\Delta}f_{1,x}|_{x=0} - 4\delta f_1|_{x=0} + \frac{2}{\Delta}f_{2,xx}|_{x=0} - 2f_{2,x}|_{x=0} + \frac{2\alpha}{\Delta}f_2|_{x=0},
\end{aligned}$$

with Δ *and* δ *being defined by* (3.24).

Proof. Note that now $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are functions of two variables x and t. But in the solution formula (3.22) and (3.23), t is the dummy variable. Let us first compute that

$$\begin{aligned}
\phi^h_{1,t} &= (D_{1,t}\cos\delta x + D_{2,t}\sin\delta x)\mathrm{e}^{\Delta x} + (D_{3,t}\cos\delta x + D_{4,t}\sin\delta x)\mathrm{e}^{-\Delta x}\\
\phi^s_{1,t} &= -\int_0^x \left[g_1(x-y)f_{1,t}(y,t) + g_2(x-y)f_{2,t}(y,t)\right] dy\\
&= 4\int_0^x \left[g_1(x-y)f_{1,yyy}(y,t) + g_2(x-y)f_{2,yyy}(y,t)\right] dy\\
&= -4g_1(x)(f_{1,xx}|_{x=0}) - 4g_{1,x}(x)(f_{1,x}|_{x=0}) - 4g_{1,xx}(x)(f_1|_{x=0})\\
&\quad -4g_2(x)(f_{2,xx}|_{x=0}) - 4g_{2,x}(x)(f_{2,x}|_{x=0}) - 4g_{2,xx}(x)(f_2|_{x=0}) - 4\phi^s_{1,xxx},
\end{aligned}$$

where

$$g_1 = \frac{1}{\Delta}(\cos\delta x)(\sinh\Delta x),\ g_2 = -\frac{1}{\Delta}(\sin\delta x)\,(\sinh\Delta x).$$

Second, based on (3.28) and (3.29), we have

$$\begin{aligned}
\phi^h_{1,xxx} = &\left[(\Delta\tilde{D}_1 + \delta\tilde{D}_2)\cos\delta x + (\Delta\tilde{D}_2 - \delta\tilde{D}_1)\sin\delta x\right]\mathrm{e}^{\Delta x}\\
&+\left[(\delta\tilde{D}_4 - \Delta\tilde{D}_3)\cos\delta x - (\delta\tilde{D}_3 + \Delta\tilde{D}_4)\sin\delta x\right]\mathrm{e}^{-\Delta x},
\end{aligned}$$

and a direct computation gives rise to

$$g_{1,x}(x) = \frac{1}{2}(\cos\delta x)(\mathrm{e}^{\Delta x} + \mathrm{e}^{-\Delta x}) - \frac{\delta}{2\Delta}(\sin\delta x)(\mathrm{e}^{\Delta x} - \mathrm{e}^{-\Delta x}),$$
$$g_{1,xx}(x) = (-\delta\sin\delta x + \frac{\alpha}{2\Delta}\cos\delta x)\mathrm{e}^{\Delta x} - (\delta\sin\delta x + \frac{\alpha}{2\Delta}\cos\delta x)\mathrm{e}^{-\Delta x},$$
$$g_{2,x}(x) = -\frac{1}{2}(\sin\delta x)(\mathrm{e}^{\Delta x} + \mathrm{e}^{-\Delta x}) - \frac{\delta}{2\Delta}(\cos\delta x)(\mathrm{e}^{\Delta x} - \mathrm{e}^{-\Delta x}),$$
$$g_{2,xx}(x) = -(\delta\cos\delta x + \frac{\alpha}{2\Delta}\sin\delta x)\mathrm{e}^{\Delta x} - (\delta\cos\delta x - \frac{\alpha}{2\Delta}\sin\delta x)\mathrm{e}^{-\Delta x}.$$

Now a comparison of coefficients of four functionally independent terms

$$(\cos\delta x)\mathrm{e}^{\Delta x},\ (\sin\delta x)\mathrm{e}^{\Delta x},\ (\cos\delta x)\mathrm{e}^{-\Delta x},\ (\sin\delta x)\mathrm{e}^{-\Delta x}$$

in the equation $\phi_{1,t} = -4\phi_{1,xxx}$, i.e., $\phi^h_{1,t} + \phi^s_{1,t} = -4\phi^h_{1,xxx} - 4\phi^s_{1,xxx}$, yields that

$$\begin{aligned} D_{1,t} &= -4(\Delta\tilde{D}_1 + \delta\tilde{D}_2) + p_1(t) \\ &= 4(3\Delta\delta^2 - \Delta^3)D_1 - 4(3\Delta^2\delta - \delta^3)D_2 + p_1(t) \\ &= \tilde{\alpha}D_1 - \tilde{\beta}D_2 + p_1(t), \end{aligned} \tag{3.35}$$

$$\begin{aligned} D_{2,t} &= -4(\Delta\tilde{D}_2 - \delta\tilde{D}_1) + p_2(t) \\ &= 4(3\Delta^2\delta - \delta^3)D_1 + 4(3\Delta\delta^2 - \Delta^3)D_2 + p_2(t) \\ &= \tilde{\beta}D_1 + \tilde{\alpha}D_2 + p_2(t), \end{aligned} \tag{3.36}$$

$$\begin{aligned} D_{3,t} &= -4(\delta\tilde{D}_4 - \Delta\tilde{D}_3) + p_3(t) \\ &= -4(3\Delta\delta^2 - \Delta^3)D_3 - 4(3\Delta^2\delta - \delta^3)D_4 + p_3(t) \\ &= -\tilde{\alpha}D_3 - \tilde{\beta}D_4 + p_3(t), \end{aligned} \tag{3.37}$$

$$\begin{aligned} D_{4,t} &= -4(-\delta\tilde{D}_3 - \Delta\tilde{D}_4) + p_4(t) \\ &= 4(3\Delta^2\delta - \delta^3)D_3 - 4(3\Delta\delta^2 - \Delta^3)D_4 + p_4(t) \\ &= \tilde{\beta}D_3 - \tilde{\alpha}D_4 + p_4(t). \end{aligned} \tag{3.38}$$

These are two linear systems of first-order ordinary differential equations. One is for D_1 and D_2 and the other is for D_3 and D_4. Solving these two coupled linear systems for D_i, $1 \le i \le 4$, we obtain the general solution given by (3.32) and (3.33) in the theorem. For example, we rewrite (3.35) and (3.36) as

$$\begin{bmatrix} D_{1,t} \\ D_{2,t} \end{bmatrix} = \tilde{A}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},\ \tilde{A} = \begin{bmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \tilde{\beta} & \tilde{\alpha} \end{bmatrix}.$$

Then it follows that

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \mathrm{e}^{\tilde{A}t}\begin{bmatrix} D_{10} \\ D_{20} \end{bmatrix} + \mathrm{e}^{\tilde{A}t}\int_0^t \mathrm{e}^{-\tilde{A}s}\begin{bmatrix} p_1(s) \\ p_2(s) \end{bmatrix} ds.$$

Since $\tilde{A}$ has two complex eigenvalues

$$\lambda^{\pm} = \tilde{\alpha} \pm \tilde{\beta}\sqrt{-1}\,,$$

the evolution operator $\mathrm{e}^{\tilde{A}t}$ reads as [HS, AP]

$$\mathrm{e}^{\tilde{A}t} = \mathrm{e}^{\tilde{\alpha}t}\begin{bmatrix} \cos\tilde{\beta}t & -\sin\tilde{\beta}t \\ \sin\tilde{\beta}t & \cos\tilde{\beta}t \end{bmatrix}.$$

The derivation of the formulas for D_3 and D_4 is completely similar. Note that the other equation $\phi_{2,t} = -4\phi_{2,xxx}$ is automatically satisfied, once $D_i,\ 1 \le i \le 4$, are determined as above. The proof is finished. □

4. Wronskian solutions

In principle, we can construct general solutions associated with two types of Jordan blocks of the coefficient matrix through the solution formulas established in the previous section. In what follows, we present a few special but interesting Wronskian solutions, rather than general ones which are too complicated and lengthy to present.

4.1. **Rational solutions.** Let us start with

$$\psi(\eta) = \mathrm{e}^{\eta x - 4\eta^3 t} - \mathrm{e}^{-\eta x + 4\eta^3 t} = 2\sinh(\eta x - 4\eta^3 t), \tag{4.1}$$

where η is a real constant. It is easy to see that this function $\psi(\eta)$ solves

$$\psi_{xx} = \eta^2 \psi,\ \psi_t = -4\psi_{xxx}. \tag{4.2}$$

Upon expanding

$$\psi(\eta) = \sum_{i=0}^{\infty} \phi_i \eta^{2i+1}, \tag{4.3}$$

from (4.2) we obtain

$$\phi_{0,xx} = 0,\ \phi_{i+1,xx} = \phi_i,\ \phi_{i,t} = -4\phi_{i,xxx},\ i \ge 0. \tag{4.4}$$

The expressions of ϕ_i can be calculated through the series expansion of the hyperbolic function $\sinh x$:

$$\phi_i = \sum_{j=0}^{[(2i+1)/3]} \frac{(-4)^j}{(2i-3j+1)!j!} x^{2i-3j+1} t^j,\ i \ge 0. \tag{4.5}$$

Then for each $k \ge 1$, the associated Wronskian solution to the KdV equation (1.1) is given by

$$u = -2\partial_x^2 \ln W(\phi_0, \phi_1, \cdots, \phi_{k-1}). \tag{4.6}$$

Obviously, this solution is rational and it corresponds to the following Jordan block:

$$\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{k\times k}. \tag{4.7}$$

We call it a rational solution of order $k-1$. This type of solutions was also discussed in [ASa, SHR, AM] and obtained by Freeman and Nimmo [FN] by taking the limit $\eta_i \to 0$ of the multi-soliton solution. Two rational solutions of lower-order are

$$u = -2\partial_x^2 \ln W(\phi_0) = -2\partial_x^2 \ln W(x) = \frac{2}{x^2},$$

$$u = -2\partial_x^2 \ln W(\phi_0, \phi_1) = -2\partial_x^2 \ln W(x, \frac{1}{6}x^3 - 4t) = \frac{6x^4 - 144xt}{(x^3 + 12t)^2}.$$

4.2. **Solitons, positons and negatons.** For a nonzero real eigenvalue λ_i, we start from the eigenfunction $\phi_i(\lambda_i)$ determined by

$$-(\phi_i(\lambda_i))_{xx} = \lambda_i \phi_i(\lambda_i),\ (\phi_i(\lambda_i))_t = -4(\phi_i(\lambda_i))_{xxx}. \tag{4.8}$$

General solutions to this system in two cases of $\lambda_i > 0$ and $\lambda_i < 0$ are

$$\phi_i(\lambda_i) = C_{1i}\sin(\eta_i x + 4\eta_i^3 t) + C_{2i}\cos(\eta_i x + 4\eta_i^3 t),\ \eta_i = \sqrt{\lambda_i}\,, \tag{4.9}$$

$$\phi_i(\lambda_i) = C_{1i}\sinh(\eta_i x - 4\eta_i^3 t) + C_{2i}\cosh(\eta_i x - 4\eta_i^3 t),\ \eta_i = \sqrt{-\lambda_i}\,, \tag{4.10}$$

respectively, where C_{1i} and C_{2i} are arbitrary real constants. By an inspection, we find that

$$-\begin{bmatrix} \phi_i(\lambda_i) \\ \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i) \end{bmatrix}_{xx} = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i\times k_i} \begin{bmatrix} \phi_i(\lambda_i) \\ \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i) \end{bmatrix},$$

and

$$(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i))_t = -4(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i))_{xxx},\ 0 \le j \le k_i - 1,$$

where ∂_{λ_i} denotes the derivative with respect to λ_i and k_i is an arbitrary nonnegative integer. Therefore, through this set of eigenfunctions, we obtain a Wronskian solution to the KdV equation (1.1):

$$u = -2\partial_x^2 \ln W(\phi_i(\lambda_i), \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i), \cdots, \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i)), \tag{4.11}$$

which corresponds to the first type of Jordan blocks with a nonzero real eigenvalue.

When $\lambda_i > 0$, we get positon solutions [Mat], and when $\lambda_i < 0$, we get negaton solutions [RSK]. A more general positon or negaton can be obtained by combining n sets of eigenfunctions associated with different $\lambda_i > 0$ or different $\lambda_i < 0$:

$$\begin{aligned} u \ = \ & -2\partial_x^2 \ln W(\phi_1(\lambda_1), \cdots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1); \\ & \cdots; \phi_n(\lambda_n), \cdots, \frac{1}{(k_n-1)!}\partial_{\lambda_n}^{k_n-1}\phi_n(\lambda_n)). \end{aligned} \tag{4.12}$$

This solution is called an n-position or n-negaton of order $(k_1 - 1, k_2 - 1, \cdots, k_n - 1)$, to reflect the number of different eigenvalues and the orders of derivatives of eigenfunctions. If $k_i = 1$, $1 \le i \le n$, we simply say that it is an n-position or n-negaton.

An n-soliton solution is a special n-negaton:

$$u = -2\partial_x^2 \ln(\phi_1, \phi_2, \cdots, \phi_n) \tag{4.13}$$

with ϕ_i being given by

$$\phi_i = \cosh(\eta_i x - 4\eta_i^3 t + \gamma_i),\ i \text{ odd}, \tag{4.14a}$$

$$\phi_i = \sinh(\eta_i x - 4\eta_i^3 t + \gamma_i),\quad i \text{ even}, \tag{4.14b}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_n$ and γ_i, $1 \le i \le n$, are arbitrary real constants. There are other representations of multi-soliton solutions to the KdV equation (see, for

example, [H, BS]). Two kinds of special positons of order k are

$$u = -2\partial_x^2 \ln W(\phi, \partial_\lambda \phi, \cdots, \partial_\lambda^{k-1}\phi),\ \phi = \cos(\eta x + 4\eta^3 t + \gamma(\eta)), \tag{4.15}$$

$$u = -2\partial_x^2 \ln W(\phi, \partial_\lambda \phi, \cdots, \partial_\lambda^{k-1}\phi),\ \phi = \sin(\eta x + 4\eta^3 t + \gamma(\eta)), \tag{4.16}$$

where $\lambda > 0$, $\eta = \sqrt{\lambda}$ and γ is an arbitrary function of η. But these two kinds of positons are equivalent to each other, due to the existence of the arbitrary function γ. Similarly, two kinds of special negatons of order k are

$$u = -2\partial_x^2 \ln W(\phi, \partial_\lambda \phi, \cdots, \partial_\lambda^{k-1}\phi),\ \phi = \cosh(\eta x - 4\eta^3 t + \gamma(\eta)), \tag{4.17}$$

$$u = -2\partial_x^2 \ln W(\phi, \partial_\lambda \phi, \cdots, \partial_\lambda^{k-1}\phi),\ \phi = \sinh(\eta x - 4\eta^3 t + \gamma(\eta)), \tag{4.18}$$

where $\lambda < 0$, $\eta = \sqrt{-\lambda}$ and γ is an arbitrary function of η. These solutions exhibited above were also discussed in a slightly different way in [APP, Mat, RSK]. The following are solitons, positons and negatons of lower-order:

$$\begin{aligned}
u &= -2\partial_x^2 \ln(\cosh(\eta x - 4\eta^3 t + \gamma)) = \frac{-2\eta^2}{\cosh^2(\eta x - 4\eta^3 t + \gamma)},\\
u &= -2\partial_x^2 \ln(\cos(\eta x + 4\eta^3 t + \gamma)) = \frac{2\eta^2}{\cos^2(\eta x + 4\eta^3 t + \gamma)},\\
u &= -2\partial_x^2 \ln(\sinh(\eta x - 4\eta^3 t + \gamma)) = \frac{2\eta^2}{\sinh^2(\eta x - 4\eta^3 t + \gamma)};\\
u &= -2\partial_x^2 \ln W(\cosh(\eta_1 x - 4\eta_1^3 t + \gamma_1), \sinh(\eta_2 x - 4\eta_2^3 t + \gamma_2))\\
&= \frac{4(\eta_1^2 - \eta_2^2)[(\eta_2^2 - \eta_1^2) + \eta_1^2 \cosh 2\theta_2 + \eta_2^2 \cosh 2\theta_1]}{[(\eta_2 - \eta_1)\cosh(\theta_1 + \theta_2) + (\eta_2 + \eta_1)\cosh(\theta_1 - \theta_2)]^2},\\
u &= -2\partial_x^2 \ln W(\cos(\eta x + 4\eta^3 t + \gamma), \partial_\lambda \cos(\eta x + 4\eta^3 t + \gamma))\\
&= \frac{16\eta^2[2\cos^2(\eta x + 4\eta^3 t + \gamma) + (\eta x + 12\eta^3 t)\sin 2(\eta x + 4\eta^3 t + \gamma)]}{[2(\eta x + 12\eta^3 t) + \sin 2(\eta x + 4\eta^3 t + \gamma)]^2},\\
u &= -2\partial_x^2 \ln W(\cosh(\eta x - 4\eta^3 t + \gamma), \partial_\lambda \cosh(\eta x - 4\eta^3 t + \gamma))\\
&= \frac{16\eta^2[2\cosh^2(\eta x - 4\eta^3 t + \gamma) - (\eta x - 12\eta^3 t)\sinh 2(\eta x - 4\eta^3 t + \gamma)]}{[2(\eta x - 12\eta^3 t) + \sinh 2(\eta x - 4\eta^3 t + \gamma)]^2};
\end{aligned}$$

where η, γ, η_i and γ_i are arbitrary real constants, and

$$\theta_i = \eta_i x - 4\eta_i^3 t + \gamma_i,\ i = 1, 2.$$

Some parts of the graphs of these solutions with $\gamma = \gamma_i = 0$ are displayed in Figures 1 and 2.

4.3. **Complexitons.** For the second type of Jordan blocks of the coefficient matrix, we start from a pair of eigenfunctions $\Phi_i(\alpha_i, \beta_i) = (\phi_{i1}(\alpha_i, \beta_i), \phi_{i2}(\alpha_i, \beta_i))^T$ determined by

$$-\Phi_{i,xx} = A\Phi_i,\ \Phi_i = \begin{bmatrix} \phi_{i1}(\alpha_i, \beta_i) \\ \phi_{i2}(\alpha_i, \beta_i) \end{bmatrix},\ A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}, \tag{4.19}$$

and

$$(\phi_{ij}(\alpha_i, \beta_i))_t = -4(\phi_{ij}(\alpha_i, \beta_i))_{xxx},\ j = 1, 2. \tag{4.20}$$

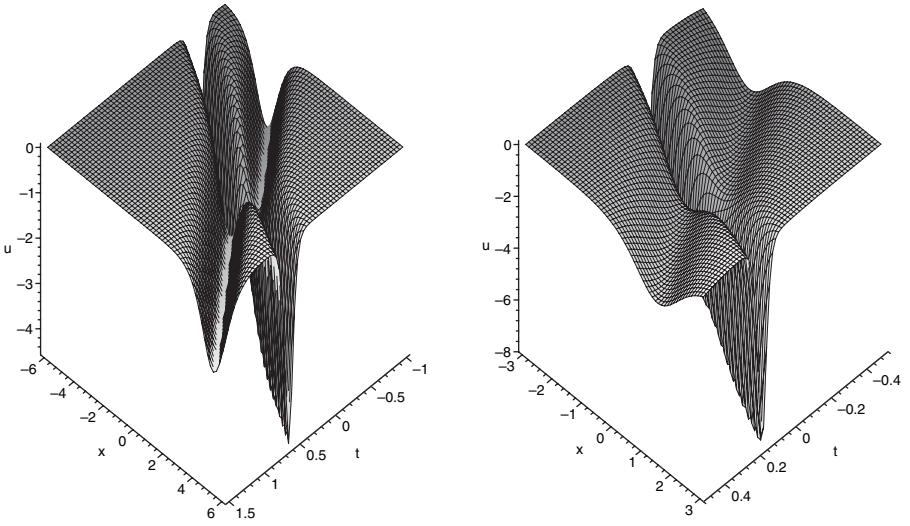

FIGURE 1. 2-soliton - $\eta_1 = 1$, $\eta_2 = 1.5$ (left) 2-soliton - $\eta_1 = 1$, $\eta_2 = 2$ (right)

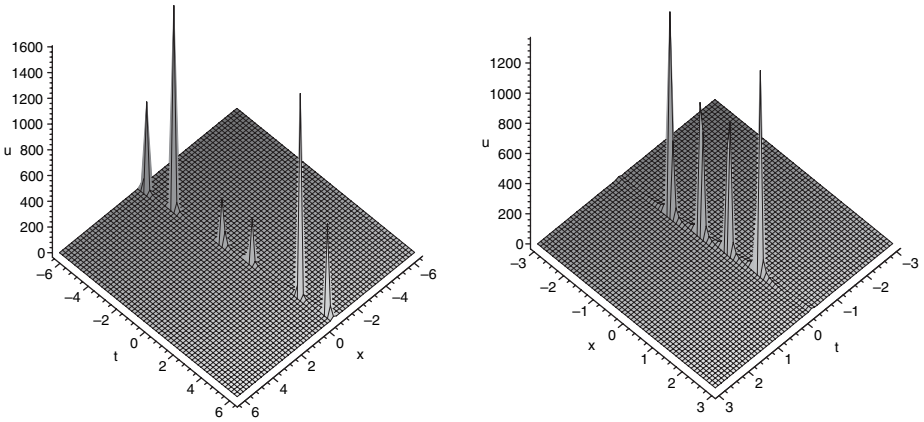

FIGURE 2. 1-positon of order 1 - $\lambda = 1$ (left) 1-negaton of order 1 - $\lambda = -1$ (right)

The coefficient matrix A_i has two complex eigenfunctions: $\alpha_i \pm \beta_i\sqrt{-1}$. By Theorem 3.6, a general solution to the homogenous system of (4.19) and (4.20) is

$$\begin{aligned} \phi_{i1} &= \frac{1}{2}[\cos(\delta_i(x-\bar{\beta}_i t)+\kappa_{1i})]\,\mathrm{e}^{\Delta_i(x+\bar{\alpha}_i t)+\gamma_{1i}} \\ &\quad +\frac{1}{2}[\cos(\delta_i(x-\bar{\beta}_i t)+\kappa_{2i})]\,\mathrm{e}^{-\Delta_i(x+\bar{\alpha}_i t)-\gamma_{2i}}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \phi_{i2} &= \frac{1}{2}[\sin(\delta_i(x-\bar{\beta}_i t)+\kappa_{1i})]\,\mathrm{e}^{\Delta_i(x+\bar{\alpha}_i t)+\gamma_{1i}} \\ &\quad -\frac{1}{2}[\sin(\delta_i(x-\bar{\beta}_i t)+\kappa_{2i})]\,\mathrm{e}^{-\Delta_i(x+\bar{\alpha}_i t)-\gamma_{2i}}, \end{aligned} \tag{4.22}$$

where κ_{ji} and γ_{ji} are arbitrary real constants and

$$\Delta_i = \sqrt{\frac{\sqrt{\alpha_i^2+\beta_i^2}-\alpha_i}{2}},\ \delta_i = \sqrt{\frac{\sqrt{\alpha_i^2+\beta_i^2}+\alpha_i}{2}}, \tag{4.23}$$

$$\bar{\alpha}_i = 4\sqrt{\alpha_i^2+\beta_i^2}+8\alpha_i,\ \bar{\beta}_i = 4\sqrt{\alpha_i^2+\beta_i^2}-8\alpha_i. \tag{4.24}$$

Similarly, by an inspection, we can see that

$$-\begin{bmatrix} \Phi_i \\ \frac{1}{1!}\partial_{\alpha_i}\Phi_i \\ \vdots \\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{l_i-1}\Phi_i \end{bmatrix}_{xx} = \begin{bmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{bmatrix}_{l_i\times l_i} \begin{bmatrix} \Phi_i \\ \frac{1}{1!}\partial_{\alpha_i}\Phi_i \\ \vdots \\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{l_i-1}\Phi_i \end{bmatrix},$$

and

$$(\frac{1}{j!}\partial_{\alpha_i}^j\Phi_i)_t = -4(\frac{1}{j!}\partial_{\alpha_i}^j\Phi_i)_{xxx},\ 0\le j\le l_i-1,$$

where ∂_{α_i} denotes the derivative with respect to α_i. Taking the derivative with respect to β_i, we can have

$$-\begin{bmatrix} \Phi_i \\ \frac{1}{1!}\partial_{\beta_i}\Phi_i \\ \vdots \\ \frac{1}{(l_i-1)!}\partial_{\beta_i}^{l_i-1}\Phi_i \end{bmatrix}_{xx} = \begin{bmatrix} A_i & & & 0 \\ \Sigma_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & \Sigma_2 & A_i \end{bmatrix}_{l_i\times l_i} \begin{bmatrix} \Phi_i \\ \frac{1}{1!}\partial_{\beta_i}\Phi_i \\ \vdots \\ \frac{1}{(l_i-1)!}\partial_{\beta_i}^{l_i-1}\Phi_i \end{bmatrix}$$

and

$$(\frac{1}{j!}\partial_{\beta_i}^j\Phi_i)_t = -4(\frac{1}{j!}\partial_{\beta_i}^j\Phi_i)_{xxx},\ 0\le j\le l_i-1,$$

where

$$\Sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, we obtain two Wronskian solutions to the KdV equation (1.1):

$$u = -2\partial_x^2 \ln W(\Phi_i^T, \frac{1}{1!}\partial_{\alpha_i}\Phi_i^T, \cdots, \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{l_i-1}\Phi_i^T), \tag{4.25}$$

and

$$u = -2\partial_x^2 \ln W(\Phi_i^T, \frac{1}{1!}\partial_{\beta_i}\Phi_i^T, \cdots, \frac{1}{(l_i-1)!}\partial_{\beta_i}^{l_i-1}\Phi_i^T), \tag{4.26}$$

which correspond to the same Jordan block of the second type but are different. Such a general Wronskian solution is

$$u = -2\partial_x^2 \ln W(\Phi_1^T, \cdots, \frac{1}{(l_1-1)!}\partial_{\zeta_1}^{l_1-1}\Phi_1^T; \cdots; \Phi_n^T, \cdots, \frac{1}{(l_n-1)!}\partial_{\zeta_n}^{l_n-1}\Phi_n^T), \tag{4.27}$$

where ∂_{ζ_i} can be either of ∂_{α_i} and ∂_{β_i}. This solution is called an n-complexiton solution of order $(l_1-1, l_2-1, \cdots, l_n-1)$, to reflect the orders of derivatives of eigenfunctions with respect to eigenvalues. If $l_i = 1,\ 1\le i\le n$, we simply say that it is an n-complexiton.

In particular, one-complexiton is [M]

$$
\begin{aligned}
u &= -2\partial_x^2 \ln W(\phi_{11}, \phi_{12}) \\
&= \frac{-4\beta_1^2\big[1+\cos(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)\cosh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)\big]}{\big[\Delta_1\sin(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)+\delta_1\sinh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)\big]^2} \\
&\quad + \frac{4\alpha_1\beta_1\sin(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)\sinh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)}{\big[\Delta_1\sin(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)+\delta_1\sinh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)\big]^2},
\end{aligned} \tag{4.28}
$$

where $\alpha_1,\ \beta_1 > 0$, $\kappa_1 = \kappa_{11} = \kappa_{12}$, and $\gamma_1 = \gamma_{11} = \gamma_{12}$ are arbitrary real constants, and Δ_1, δ_1, $\bar\alpha_1$, and $\bar\beta_1$ are given by

$$
\Delta_1 = \sqrt{\frac{\sqrt{\alpha_1^2+\beta_1^2}-\alpha_1}{2}},\ \delta_1 = \sqrt{\frac{\sqrt{\alpha_1^2+\beta_1^2}+\alpha_1}{2}},
$$

$$
\bar\alpha_1 = 4\sqrt{\alpha_1^2+\beta_1^2}+8\alpha_1,\ \bar\beta_1 = 4\sqrt{\alpha_1^2+\beta_1^2}-8\alpha_1.
$$

The subcase of $\alpha_1 = 0$ leads to the following solution:

$$
u = \frac{8\beta_1 + 8\beta_1\cos(\sqrt{2\beta_1}\,x - 4\beta_1\sqrt{2\beta_1}\,t + 2\kappa_1)\cosh(\sqrt{2\beta_1}\,x + 4\beta_1\sqrt{2\beta_1}\,t + 2\gamma_1)}{\big[\sin(\sqrt{2\beta_1}\,x - 4\beta_1\sqrt{2\beta_1}\,t + 2\kappa_1) + \sinh(\sqrt{2\beta_1}\,x + 4\beta_1\sqrt{2\beta_1}\,t + 2\gamma_1)\big]^2}.
$$

This solution is associated with purely imaginary eigenvalues of the Schrödinger spectral problem with zero potential. In addition, if we fix

$$
\kappa_1 = \frac{\pi}{4},\ \gamma_1 = \frac{1}{2}\ln(\frac{b}{a}),\ \Delta_1 = -a,\ \delta_1 = b,
$$

where a and b are arbitrary real constants, then our one-complexiton solution (4.28) boils down to the breather-like or spike-like solution presented in [J]:

$$
u = \frac{8\{(a^2-b^2)(b/a)\cos\nu\sinh(\eta+p) + 2b^2[1+\sin\nu\cosh(\eta+p)]\}}{[\cos\nu - (b/a)\sinh(\eta+p)]^2},
$$

where $\nu = -2bx + 8(3a^2b - b^3)t$, $\eta = -2ax + 8(a^3 - 3ab^2)t$ and $p = \ln(b/a)$. Figure 3 depicts some singularities of the single complexiton solution with $\kappa = \gamma = 0$.

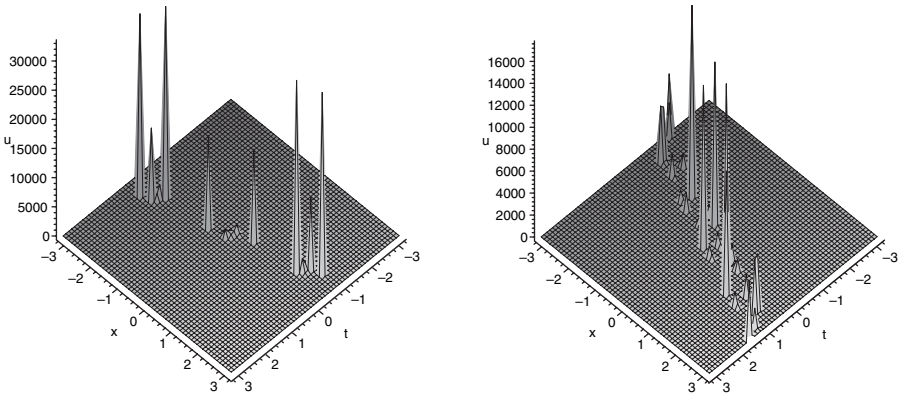

FIGURE 3. 1-complexiton - $\alpha_1 = 0$, $\beta_1 = 1$ (left) 1-complexiton - $\alpha_1 = 1$, $\beta_1 = 1$ (right)

4.4. **Interaction solutions.** We are now presenting examples of Wronskian interaction solutions among different kinds of Wronskian solutions to the KdV equation.

Let us assume that there are two sets of eigenfunctions

$$\phi_1(\lambda), \phi_2(\lambda), \cdots, \phi_k(\lambda);\ \psi_1(\mu), \psi_2(\mu), \cdots, \psi_l(\mu) \tag{4.29}$$

associated with two different eigenvalues λ and μ, respectively. A Wronskian solution

$$u = -2\partial_x^2 \ln W(\phi_1(\lambda), \cdots, \phi_k(\lambda); \psi_1(\mu), \cdots, \psi_l(\mu)) \tag{4.30}$$

is said to be a Wronskian interaction solution between two solutions determined by the two sets of eigenfunctions in (4.29). Of course, we can have more general Wronskian interaction solutions among three or more kinds of solutions such as rational solutions, positons, solitons, negatons, breathers and complexitons. Roughly speaking, it increases the complexities of rational solutions, positons, negatons and complexitons, respectively, to add zero, positive, negative or complex eigenvalues to the spectrum of the coefficient matrix.

In what follows, we would like to show a few special Wronskian interaction solutions. Let us first choose different sets of eigenfunctions:

$$\begin{aligned}
&\phi_{\text{rational}} = cx + d,\ \ c, d = \text{consts.};\\
&\phi_{\text{soliton}} = \cosh(\eta x - 4\eta^3 t + \gamma),\ \ \eta, \gamma = \text{consts.};\\
&\phi_{\text{positon}} = \cos(\eta x + 4\eta^3 t + \gamma),\ \ \eta, \gamma = \text{consts.};\\
&\phi_{\text{complexiton},1} = \cos(\frac{\sqrt{2}}{2}(x - 4t) + \kappa)\cosh(\frac{\sqrt{2}}{2}(x + 4t) + \gamma),\ \ \kappa, \gamma = \text{consts.},\\
&\phi_{\text{complexiton},2} = \sin(\frac{\sqrt{2}}{2}(x - 4t) + \kappa)\sinh(\frac{\sqrt{2}}{2}(x + 4t) + \gamma),\ \ \kappa, \gamma = \text{consts.}
\end{aligned}$$

Three Wronskian interaction determinants between any two of a rational solution, a single soliton and a single positon are

$$\begin{aligned}
W(\phi_{\text{rational}}, \phi_{\text{soliton}}) &= \eta(cx + d)\sinh(\eta x - 4\eta^3 t + \gamma) - c\cosh(\eta x - 4\eta^3 t + \gamma),\\
W(\phi_{\text{rational}}, \phi_{\text{positon}}) &= -\eta(cx + d)\sin(\eta x + 4\eta^3 t + \gamma) - c\cos(\eta x + 4\eta^3 t + \gamma),\\
W(\phi_{\text{soliton}}, \phi_{\text{positon}}) &= -\eta\cosh(\eta x - 4\eta^3 t + \gamma)\sin(\eta x + 4\eta^3 t + \gamma)\\
&\quad -\eta\cos(\eta x + 4\eta^3 t + \gamma)\sinh(\eta x - 4\eta^3 t + \gamma).
\end{aligned}$$

Further, the corresponding Wronskian interaction solutions are

$$\begin{aligned}
u_{rs} &= -2\partial_x^2 \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}) = \frac{2\eta^2[\eta^2(cx + d)^2 + c^2\cosh^2\xi_-]}{[\eta(cx + d)\sinh\xi_- - c\cosh\xi_-]^2},\\
u_{rp} &= -2\partial_x^2 \ln W(\phi_{\text{rational}}, \phi_{\text{positon}}) = \frac{2\eta^2[\eta^2(cx + d)^2 - c^2\cos^2\xi_+]}{[\eta(cx + d)\sin\xi_+ + c\cos\xi_+]^2},\\
u_{sp} &= -2\partial_x^2 \ln W(\phi_{\text{soliton}}, \phi_{\text{positon}}) = \frac{4\eta^2(\cosh^2\xi_- + \cos^2\xi_+)}{(\cosh\xi_-\sin\xi_+ + \cos\xi_+\sinh\xi_-)^2},
\end{aligned}$$

where

$$\xi_\pm = \eta x \pm 4\eta^3 t + \gamma.$$

Some singularities of these solutions are shown in Figures 4 and 5.

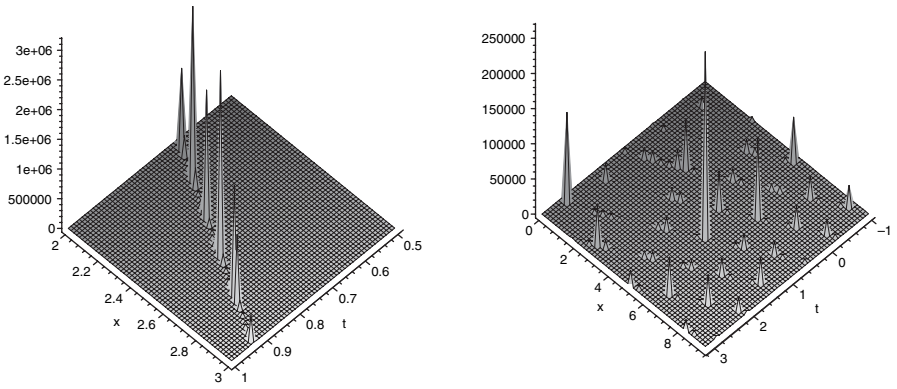

FIGURE 4. u_{rs} - $\eta = \gamma = c = d = 1$ (left) u_{rp} - $\eta = \gamma = c = d = 1$ (right)

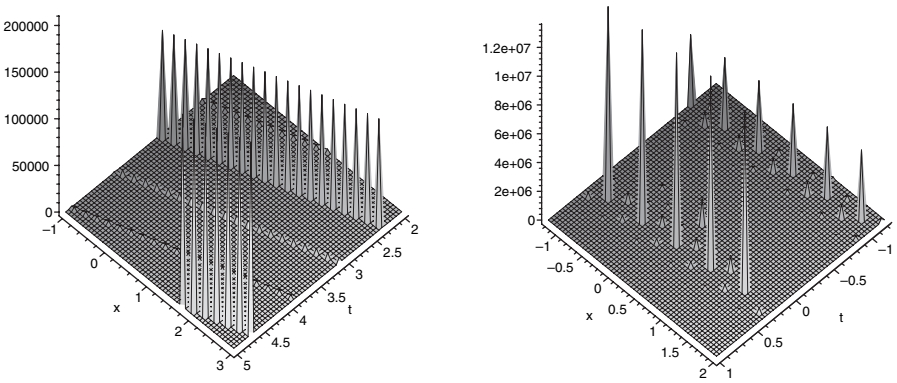

FIGURE 5. u_{sp} - $\eta = \gamma = c = d = 1$ (left) u_{sp} - $\eta = 2$, $\gamma = c = d = 1$ (right)

The following are two Wronskian interaction determinants and solutions involving three eigenfunctions. The first Wronskian determinant is

$$\begin{aligned} & W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) \\ = \; & -\eta^2[(cx+d)\eta(\cos\xi_+ \sinh\xi_- - \sin\xi_+ \cosh\xi_-) - 2c\cos\xi_+ \cosh\xi_-], \end{aligned}$$

so that its corresponding Wronskian solution is

$$\begin{aligned} u_{rsp} &= -2\partial_x^2 \ln W(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}}) \\ &= \frac{4(cx+d)\eta^3(\sin\xi_+ \cosh\xi_- + \cos\xi_+ \sinh\xi_-)}{(cx+d)\eta(\cos\xi_+ \sinh\xi_- - \sin\xi_+ \cosh\xi_-) - 2c\cos\xi_+ \cosh\xi_-} \\ &\quad + \frac{2\eta^2[c\eta\cos\xi_+ \sinh\xi_- + 2(cx+d)\eta\sin\xi_+ \sinh\xi_- - c\eta\sin\xi_+ \cosh\xi_-]^2}{[(cx+d)\eta(\cos\xi_+ \sinh\xi_- - \sin\xi_+ \cosh\xi_-) - 2c\cos\xi_+ \cosh\xi_-]^2}, \end{aligned}$$

where

$$\xi_\pm = \eta x \pm 4\eta^3 t + \gamma.$$

The second Wronskian determinant and its corresponding Wronskian solution are

$$\begin{aligned} & W(x, \phi_{\text{complexiton},1}, \phi_{\text{complexiton},2}) \\ = \quad & -\frac{\sqrt{2}}{4} x \sin 2\xi + \frac{\sqrt{2}}{4} x \sinh 2\zeta - \cosh^2 \zeta + \sin^2 \xi, \end{aligned}$$

and

$$\begin{aligned} u_{rc} &= -2\partial_x^2 \ln W(x, \phi_{\text{complexiton},1}, \phi_{\text{complexiton},2}) \\ &= [\cosh^4 \zeta - \cos^4 \xi - 4x^2 \cos^2 \xi \cosh^2 \zeta + (2x^2 - 1) \cosh^2 \zeta \\ &\quad + (2x^2 + 1) \cos^2 \xi + 4\sqrt{2}\, x \cosh^2 \zeta \sin 2\xi + 4\sqrt{2}\, x \cos^2 \xi \sinh 2\zeta \\ &\quad - \sqrt{2}\, x \sinh 2\zeta - \sqrt{2}\, x \sin 2\xi - \frac{1}{2} \sin 2\xi \sinh 2\zeta] \Big/ (-\frac{\sqrt{2}}{4} x \sin 2\xi \\ &\quad + \frac{\sqrt{2}}{4} x \sinh 2\zeta - \cosh^2 \zeta + \sin^2 \xi)^2, \end{aligned}$$

where

$$\xi = \frac{1}{2}\sqrt{2}\, x - 2\sqrt{2}\, t + \kappa, \ \zeta = \frac{1}{2}\sqrt{2}\, x + 2\sqrt{2}\, t + \gamma.$$

Figure 6 depicts some singularities of these two solutions.

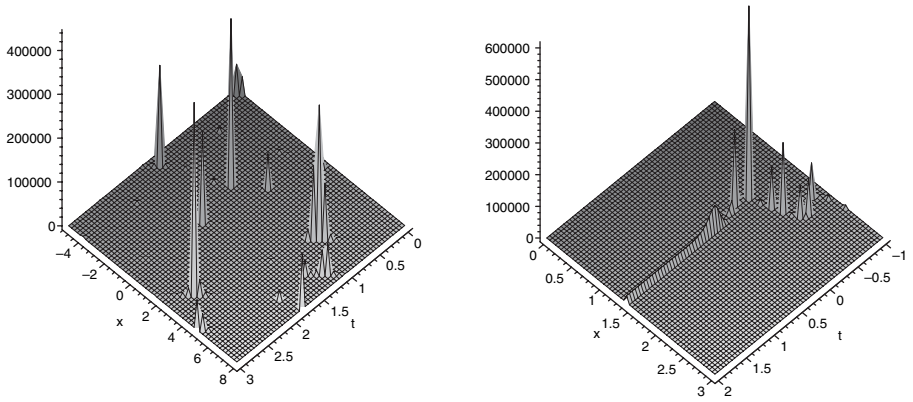

FIGURE 6. u_{rsp} - $\eta = \gamma = c = d = 1$ (left) $\quad$ u_{rc} - $\kappa = \gamma = 1$ (right)

5. Concluding remarks

A broad set of sufficient conditions consisting of coupled systems of linear partial differential equations has been presented, which guarantees that the Wronskian determinant solves the Korteweg-de Vries equation in the bilinear form. A systematical analysis has been made for solving the resultant coupled systems of linear partial differential equations, and solution formulas for their representative systems have been explicitly presented. The key technique is to apply the variation of parameters in solving the involved non-homogeneous partial differential equations of second-order and third-order. The whole analysis also offers an approach to solve

the coupled system of partial differential equations:

$$-\phi_{i,xx} = \sum_{i=1}^{N} \lambda_{ij}\phi_j,\ \phi_{i,t} = -4\phi_{i,xxx},\ 1 \le i \le N,$$

where λ_{ij} are arbitrary real constants.

Moreover, for each type of Jordan blocks of the coefficient matrix $\Lambda = (\lambda_{ij})$, special sets of eigenfunctions have been constructed and used to generate rational solutions, solitons, positons, negatons, breathers, complexitons and their interaction solutions to the Korteweg-de Vries equation. Of course, the obtained solution formulas of the representative systems allow us to construct more general Wronskian solutions than rational solutions, positons, negatons, complexitons and their interaction solutions presented in this paper. The resultant solution structures also show us the rich diversity that the KdV equation carries. We believe that any $(2+1)$-dimensional counterpart of the KdV equation will have much more general solution structures. This is because higher dimensional equations have bigger spaces of initial data to choose.

However, any new explicit exact solutions, even Wronskian solutions, to the KdV equation will still be very interesting. Two open questions follow:

(1) What about the following case?

$$-\phi_{i,xx} = \sum_{j=1}^{N} \mu_{ij}\phi_{j,x} + \sum_{j=1}^{N} \lambda_{ij}\phi_j,\ 1 \le i \le N,$$

$$\phi_{i,t} = \sum_{j=1}^{N} \zeta_{ij}\phi_{j,xxx} + \sum_{j=1}^{N} \xi_{ij}\phi_j,\ 1 \le i \le N,$$

where all coefficients are real constants. Are there any conditions, rather than the ones established in this paper, which will guarantee the Wronskian solutions for the KdV equation?

(2) Are there any similar conditions to guarantee the double Wronskian solutions for the KdV equation?

On the other hand, it deserves more investigation whether there exist Wronskian solutions for the generalized KdV equations (see, say, [B, Y, SY] for examples) and what differential conditions on Wronskian solutions one can have if there exist Wronskian solutions.

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References

[AKNS] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math. **53** (1974), 249–315. MR0450815 (56:9108)

[ASa] M. J. Ablowitz and J. Satsuma, *Solitons and rational solutions of nonlinear evolution equations*, J. Math. Phys. **19** (1978), 2180–2186. MR0507515 (80b:35121)

[ASe] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981. MR0642018 (84a:35251)

[AM] M. Adler and J. Moser, *On a class of polynomials connected with the Korteweg-de Vries equation*, Commun. Math. Phys. **61** (1978), 1–30. MR0501106 (58:18554)

[APP] V. A. Arkad'ev, A. K. Pogrebkov and M. K. Polivanov, *Singular solutions of the KdV equation and the method of the inverse problem*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **133** (1984), 17–37. MR0742146 (86c:35127)

[AP] D. K. Arrowsmith and C. M. Place, *Dynamical Systems*, Chapman & Hall, London, 1992. MR1195127 (93j:58040)

[BR] D. J. Benney and D. J. Roskes, *Wave instabilities*, Studies in Appl. Math. **48** (1969), 377–385.

[BS] A. C. Bryan and A. E. G. Stuart, *Representations of the multisoliton solutions of the Korteweg-de Vries equation*, Nonlinear Anal. **22** (1994), 561–566. MR1266543 (94m:35260)

[BC] R. K. Bullough and P. J. Caudrey (eds.), *Solitons*, Springer-Verlag, Berlin, 1980. MR0625877 (82m:35001)

[B] N. J. Burroughs, *A loop algebra co-adjoint orbit construction of the generalized KdV hierarchies*, Nonlinearity **6** (1993), 583–616. MR1231775 (94j:58078)

[DS] A. Davey and K. Stewarton, *On three-dimensional packets of surface waves*, Proc. R. Soc. A **338** (1974), 101–110. MR0349126 (50:1620)

[DJ] P. G. Drazin and R. S. Johnson, *Solitons: an Introduction*, Cambridge University Press, Cambridge, 1989. MR0985322 (90j:35166)

[FN] N. C. Freeman and J. J. C. Nimmo, *Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique*, Phys. Lett. A **95** (1983), 1–3. MR0700477 (85j:35168)

[H] R. Hirota, *Exact solution of the Korteweg de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett. **27** (1971), 1192–1194.

[HS] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, San Diego, California, 1974. MR0486784 (58:6484)

[J] M. Jaworski, *Breather-like solutions to the Korteweg-de Vries equation*, Phys. Lett. A **104** (1984), 245–247. MR0758224 (85h:35198)

[KP] B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersing media*, Sov. Phys. Dokl. **15** (1970), 539–541.

[K] M. Kovalyov, *Basic motions of the Korteweg-de Vries equation*, Nonlinear Anal. **31** (1998), 599–619. MR1487849 (99f:35177)

[L] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968), 467–490. MR0235310 (38:3620)

[M] W. X. Ma, *Complexiton solutions to the Korteweg-de Vries equation*, Phys. Lett. A **301** (2002), 35–44. MR1927047 (2003h:35237)

[Mag] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. **19** (1978), 1156–1162. MR0488516 (80a:35112)

[Mat] V. B. Matveev, *Generalized Wronskian formula for solutions of the KdV equations: first applications; Positon-positon and soliton-positon collisions: KdV case*, Phys. Lett. A **166** (1992), 205–208; 209–212. MR1170966 (93g:35125a); MR1170967 (93c:35141)

[MS] V. B Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer-Verlag, Berlin, 1991. MR1146435 (93d:35136)

[MGK] R. M. Miura, C. S. Gardner and M. D. Kruskal, *Korteweg-de Vries equation and generalizations II: Existence of conservation laws and constants of motion*, J. Math. Phys. **9** (1968), 1204–1209. MR0252826 (40:6042b)

[RSK] C. Rasinariu, U. Sukhatme and A. Khare, *Negaton and positon solutions of the KdV and mKdV hierarchy*, J. Phys. A: Math. Gen. **29** (1996), 1803–1823. MR1395807 (97c:35182)

[S] J. Satsuma, *A Wronskian representation of N-soliton solutions of nonlinear evolution equations*, J. Phys. Soc. Jpn. **46** (1979) 359–360.

[SW] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. No. **61** (1985), 5–65. MR0783348 (87b:58039)

[SY] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer-Verlag, New-York, 2002. MR1873467 (2003f:37001b)

[SHR] S. Sirianunpiboon, S. D. Howard and S. K. Roy, *A note on the Wronskian form of solutions of the KdV equation*, Phys. Lett. A **134** (1988), 31–33. MR0972621 (89k:35230)

[Y] Y. You, *Global dynamics of dissipative generalized Korteweg-de Vries equations*, Chin. Ann. of Math. **17B** (1996), 389–402. MR1441652 (97k:35230)

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