Available online at www.sciencedirect.com
SCIENCE DIRECT®

Chaos, Solitons and Fractals 22 (2004) 395–406

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Rational solutions of the Toda lattice equation in Casoratian form

Wen Xiu Ma *, Yuncheng You

Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA

Accepted 24 February 2004

Abstract

A recursive procedure is presented for constructing rational solutions to the Toda lattice equation through the Casoratian formulation. It allows us to compute a broad class of rational solutions directly, without computing long wave limits in soliton solutions. All rational solutions arising from the Taylor expansions of the generating functions of soliton solutions are special ones of the general class, but only a Taylor expansion containing even or odd powers leads to non-constant rational solutions. A few rational solutions of lower order are worked out.

1. Introduction

Many integrable equations possess rational solutions, besides soliton solutions. Among the celebrated examples are the Korteweg–de Vries (KdV) equation [1,2], the Burgers equation [3] and the Kadomtsev–Petviashvili (KP) and modified KP equations [4]. Ablowitz and Satsuma recognized that rational solutions could be obtained by taking the long wave limits in soliton solutions, and considered the KdV equation in detail in such a manner [5]. The technique of computing long wave limits can also be applied to discrete integrable equations. For the Toda lattice equation [6], a few rational solutions have been presented by this technique but hard computation of long wave limits is often involved [7,8].

There are some other approaches to finding rational solutions of integrable equations, for example, the Wronskian or Casorati determinant technique [9], Bäcklund transformation [10,11], and the symmetry reduction [3,12]. While applying the Wronskian or Casorati determinant technique, the Taylor expansion about the modified spectral parameter plays an important role in constructing a kind of eigenfunctions required in determinant rational solutions [13,14]. Moreover, the determinant technique can lead to a novel class of exact solutions, called complexitons, to the KdV equation and the Toda lattice equation [15,16]. Very recently, Wu and Zhang have successfully applied the idea of using the Taylor expansion to the Toda lattice equation and constructed a large class of rational and mixed rational-soliton solutions [17]. It is natural to ask why the technique of using the Taylor expansion works well for constructing rational solutions to the Toda lattice equation. Are there any other rational solutions to the Toda lattice equation?

In this paper, we would like to answer these questions by establishing a formulation of a broad class of rational solutions to the Toda lattice equation through the Casorati determinant. The paper is structured as follows. In Section 2, a recursive procedure will be presented for constructing eigenfunctions and rational solutions, together with a formulation of rational solutions. This yields a broad class of rational solutions expressed by the Casorati determinant. In Section 3, all rational solutions arising from the Taylor expansions of the generating functions of soliton solutions will be analyzed within our formulation, and only a Taylor expansion containing even or odd powers leads to non-constant

* Corresponding author. Tel.: +1-813-974-3140; fax: +1-813-974-2700.
E-mail addresses: mawx@math.usf.edu (W.X. Ma), you@math.usf.edu (Y. You).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2004.02.011

rational solutions, thereby yielding two specific series of rational solutions. In Section 4, a few concluding remarks will be given.

2. Rational solutions

2.1. Casoratian formulation

Let us consider the Toda lattice equation in the following form:

$$a_{n,t} = a_n(b_{n-1} - b_n), \qquad b_{n,t} = a_n - a_{n+1}, \tag{2.1}$$

where (and throughout the paper) the subscript t denotes the derivative with respect to t. It can be reduced to the periodic case ($a_{n+N} = a_n$ and $b_{n+N} = b_n$ for some positive integer N) and the finite case (only finitely many a_n and b_n are non-zero). The square form of the Toda lattice equation [18]:

$$A_{n,t} = A_n(B_n - B_{n+1}), \qquad B_{n,t} = 2(A_{n-1}^2 - A_n^2), \tag{2.2}$$

presents solutions to the Toda lattice equation (2.1) through

$$(a_n(t), b_n(t)) = ((A_{n-1}(\tfrac{1}{2}t))^2, B_n(\tfrac{1}{2}t)).$$

The Toda lattice equation (2.1) has the Lax representation:

$$\dot{L} = [A, L], \tag{2.3}$$

where the Lax pair is defined by

$$\begin{cases} L_{nm} = a_n\delta_{n+1,m} + b_{n-1}\delta_{nm} + \delta_{n-1,m}, \\ A_{nm} = \delta_{n+1,m} + b_{n-1}\delta_{nm}. \end{cases} \tag{2.4}$$

Equivalently, (2.1) is determined by the isospectral ($\lambda_t = 0$) compatibility condition of the following spectral problem:

$$\begin{cases} (\phi(n))_t = b_{n-1}\phi(n) + \phi(n-1), \\ a_n\phi(n+1) + b_{n-1}\phi(n) + \phi(n-1) = \lambda\phi(n), \end{cases} \tag{2.5}$$

where λ is a spectral parameter.

Note that the dependent variable transformation

$$a_n = 1 + \frac{d^2}{dt^2}\log\tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad b_n = \frac{d}{dt}\log\frac{\tau_n}{\tau_{n+1}} = \frac{\tau_{n,t}\tau_{n+1} - \tau_n\tau_{n+1,t}}{\tau_n\tau_{n+1}}, \tag{2.6}$$

leads to $a_{n,t} - a_n(b_{n-1} - b_n) = 0$ but

$$b_{n,t} - a_n + a_{n+1} = \frac{\tau_{n,tt}\tau_n - (\tau_{n,t})^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2}{\tau_n^2} - \frac{\tau_{n+1,tt}\tau_{n+1} - (\tau_{n+1,t})^2 - \tau_{n+2}\tau_n + \tau_{n+1}^2}{\tau_{n+1}^2}.$$

Therefore, the Toda lattice equation (2.1) holds if so does the bilinear equation

$$\left[\frac{1}{2}D_t^2 - 2\sinh^2\left(\frac{D_n}{2}\right)\right]\tau_n\cdot\tau_n = \tau_{n,tt}\tau_n - (\tau_{n,t})^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2 = 0, \tag{2.7}$$

where D_t and D_n are Hirota's operators. This is called the bilinear Toda lattice equation.

The transformation

$$e^{-(y_n - y_{n-1})} = (\ln\tau_n)_{tt},$$

links the above bilinear form to the dimensionless form of the Toda lattice equation [19]

$$y_{n,tt} = e^{-(y_n - y_{n-1})} - e^{-(y_{n+1} - y_n)}. \tag{2.8}$$

This is transformed into the square form (2.2) if we put

$$A_n = \tfrac{1}{2}e^{-(y_{n+1} - y_n)/2}, \qquad B_n = \tfrac{1}{2}y_{n,t}.$$

It is known that multi-soliton solutions of the Toda lattice equation (2.1) can be presented through the τ-function determined by the Casorati determinant [13,20]:

$$\mathrm{Cas}(\phi_1,\phi_2,\ldots,\phi_N) := \begin{vmatrix} \phi_1(n,t) & \phi_1(n+1,t) & \cdots & \phi_1(n+N-1,t) \\ \phi_2(n,t) & \phi_2(n+1,t) & \cdots & \phi_2(n+N-1,t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n,t) & \phi_N(n+1,t) & \cdots & \phi_N(n+N-1,t) \end{vmatrix}, \quad N \geqslant 1, \tag{2.9}$$

provided that the functions $\phi_i = \phi_i(n,t)$, $1 \leqslant i \leqslant N$, solve

$$\phi_i(n+1,t) + \phi_i(n-1,t) = \lambda_i \phi_i(n,t), \quad (\phi_i(n,t))_t = \phi_i(n-1,t), \ \ 1 \leqslant i \leqslant N, \tag{2.10}$$

where $\lambda_i = 2\cosh(k_i)$ and the k_i's are arbitrary distinct real constants. Note that the functions determined by (2.10) are eigenfunctions of the spectral problem (2.5) under a special solution $a_n = 1$ and $b_n = 0$. Therefore, the Casorati determinant solution is actually resulted from the Darboux transformation of the Toda lattice equation [20], associated with the spectral problem (2.5), from the special solution $a_n = 1$ and $b_n = 0$.

In what follows, we would first like to show that the Casorati determinant presents a very broad class of exact solutions to the Toda lattice equation (2.1), among which solitons, positons, negatons and complexitons are special examples.

Theorem 2.1. *Let $\delta = \pm 1$. Suppose that a set of functions $\psi_i(n,t)$, $1 \leqslant i \leqslant N$, solve a system of differential-difference equations*

$$\psi_i(n+1,t) + \psi_i(n-1,t) = \sum_{j=1}^{N} \lambda_{ij}\psi_j(n,t), \quad 1 \leqslant i \leqslant N, \tag{2.11}$$

$$(\psi_i(n,t))_t = \psi_i(n+\delta,t) + \xi(t)\psi_i(n,t), \quad 1 \leqslant i \leqslant N, \tag{2.12}$$

where the λ_{ij}'s are arbitrary constants and $\xi(t)$ is an arbitrary function. Then the corresponding Casorati determinant

$$\tau_n = \mathrm{Cas}(\psi_1,\ldots,\psi_N) = \mathrm{Cas}(\psi_1(n,t),\ldots,\psi_N(n,t)) \tag{2.13}$$

solves the bilinear Toda lattice equation (2.7), *and thus the dependent variable transformation* (2.6) *presents a solution to the Toda lattice equation* (2.1).

Proof. Introduce a new set of functions

$$\phi_i(n,t) = \mathrm{e}^{-\int_0^t \xi(s)\,\mathrm{d}s}\psi_i(n,t), \quad 1 \leqslant i \leqslant N. \tag{2.14}$$

Obviously, these functions solve the following system of differential-difference equations

$$\phi_i(n+1,t) + \phi_i(n-1,t) = \sum_{j=1}^{N} \lambda_{ij}\phi_j(n,t), \quad 1 \leqslant i \leqslant N, \tag{2.15}$$

$$(\phi_i(n,t))_t = \phi_i(n+\delta,t), \quad 1 \leqslant i \leqslant N, \tag{2.16}$$

if and only if the functions $\psi_i(n,t)$, $1 \leqslant i \leqslant N$, solve the system of differential-difference equations, (2.11) and (2.12).

On the other hand, we have

$$\tau_n = \mathrm{Cas}(\psi_1,\ldots,\psi_N) = \mathrm{e}^{N\int_0^t \xi(s)\,\mathrm{d}s}\tilde{\tau}_n, \quad \tilde{\tau}_n := \mathrm{Cas}(\phi_1,\ldots,\phi_N),$$

which leads to

$$\tau_{n,tt}\tau_n - (\tau_{n,t})^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2 = \mathrm{e}^{2N\int_o^t \xi(s)\,\mathrm{d}s}(\tilde{\tau}_{n,tt}\tilde{\tau}_n - (\tilde{\tau}_{n,t})^2 - \tilde{\tau}_{n+1}\tilde{\tau}_{n-1} + \tilde{\tau}_n^2).$$

Thus, τ_n is a solution to the bilinear Toda lattice equation (2.7) if and only if so is $\tilde{\tau}_n$. But they both present the same solution to the Toda lattice equation (2.1).

Now recall that such a τ-function $\tilde{\tau}_n$ solves the Toda bilinear lattice equation (2.7) if (2.15) and (2.16) hold [16]. Therefore, it follows that the corresponding τ-function τ_n solves the Toda bilinear lattice equation (2.7) as well. This completes the proof. □

Now, the entire problem of constructing explicit solutions is reduced to the problem of solving the system, (2.11) and (2.12) [or (2.15) and (2.16)]. The first half conditions in (2.5) are only a special case of the conditions in (2.11). Therefore, we can expect to generate other solutions to the Toda lattice equation (2.1) by solving the system of differential-difference equations, (2.11) and (2.12) [or (2.15) and (2.16)], as in the KdV case [21].

In what follows, we will focus on the system of (2.15) and (2.16). This system can be compactly written as

$$\Phi_N(n+1,t)+\Phi_N(n-1,t)=\Lambda\Phi_N(n,t),\quad (\Phi_N(n,t))_t=\Phi_N(n+\delta,t), \tag{2.17}$$

where Φ_N and Λ are defined by

$$\Phi_N=\Phi_N(n,t):=(\phi_1(n,t),\dots,\phi_N(n,t))^{\mathrm{T}},\qquad \Lambda:=(\lambda_{ij})_{N\times N}. \tag{2.18}$$

Note that a constant similar transformation for the coefficient matrix Λ does not change the resulting Casorati determinant solution to the Toda lattice equation (2.1). Actually, if we have $M=P^{-1}\Lambda P$ for some invertible constant matrix P, then $\tilde{\Phi}_N=P\Phi_N$ satisfies

$$\tilde{\Phi}_N(n+1,t)+\tilde{\Phi}_N(n-1,t)=M\tilde{\Phi}_N(n,t),\quad (\tilde{\Phi}_N(n,t))_t=\tilde{\Phi}_N(n+\delta,t).$$

Obviously, the Casorati determinants generated from Φ_N and $\tilde{\Phi}_N$ have just a constant-factor difference, and thus the transformation (2.6) leads to the same Casorati determinant solutions from Φ_N and $\tilde{\Phi}_N$. Therefore as in the KdV case [21], we only need to consider the following two types of Jordan blocks of Λ:

$$\begin{bmatrix} \lambda_j & & & 0 \\ 1 & \lambda_j & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 1 & \lambda_j \end{bmatrix}_{k_j\times k_j}, \tag{2.19}$$

$$\begin{bmatrix} A_j & & & 0 \\ I_2 & A_j & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & I_2 & A_j \end{bmatrix}_{l_j\times l_j},\qquad A_j=\begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix},\qquad I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.20}$$

where λ_j, α_j and $\beta_j>0$ are all real constants, and k_j and l_j are positive integers. A Jordan block of the first type in (2.19) has the real eigenvalue λ_j with algebraic multiplicity k_j, and a Jordan block of the second type in (2.20) has the pair of complex eigenvalues $\lambda_{j,\pm}=\alpha_j\pm\beta_j i$ with algebraic multiplicity l_j.

The case of real eigenvalues greater than 2 and less than 2 corresponds to positons and negatons [22], respectively. The case of complex eigenvalues corresponds to complexitons [16]. In the following, we would like to show that the case of eigenvalues being 2 correspond to rational solutions and thus the corresponding Casoratian formulation will lead to a broad class of rational solutions to the Toda lattice equation.

2.2. Rational solutions

We first present a solution formula to solve the representative system of the system of differential-difference equations, (2.15) and (2.16), in the case of eigenvalues being 2.

Theorem 2.2. *Let $\delta=\pm 1$. Suppose that $f(n,t)$ is a given continuous function and satisfies $(f(n,t))_t=f(n+\delta,t)$. Then the system of differential-difference equations*

$$\phi(n+1,t)+\phi(n-1,t)=2\phi(n,t)+f(n,t),\quad (\phi(n,t))_t=\phi(n+\delta,t) \tag{2.21}$$

has the general solution

$$\phi(n,t)=\left[\alpha(n)t+\beta(n)+\int_0^t\int_0^s f(n+\delta,r)\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s\right]\mathrm{e}^t, \tag{2.22}$$

where $\alpha(n)$ and $\beta(n)$ are determined by

$$\begin{cases} \alpha(n+\delta)-\alpha(n)=f(n+\delta,0), \\ \beta(n+\delta)-\beta(n)=\alpha(n). \end{cases} \tag{2.23}$$

Proof. Let $\delta = 1$. Then we have $(f(n,t))_t = f(n+1,t)$ and thus we can compute that

$$\begin{aligned}\int_0^t \int_0^s f(n+2,r)\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s &= \int_0^t \int_0^s \mathrm{e}^{-r}\,\mathrm{d}f(n+1,r)\,\mathrm{d}s \\ &= \int_0^t \left[f(n+1,s)\,\mathrm{e}^{-s} - f(n+1,0) + \int_0^s f(n+1,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\right]\mathrm{d}s \\ &= f(n,t)\,\mathrm{e}^{-t} - f(n,0) + \int_0^t f(n,s)\,\mathrm{e}^{-s}\,\mathrm{d}s - f(n+1,0)t + \int_0^t \int_0^s f(n+1,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}\int_0^t \int_0^s f(n+l,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s &= -\int_0^t \int_0^s f(n+l,r)\,\mathrm{d}\mathrm{e}^{-r}\,\mathrm{d}s \\ &= -\int_0^t \left[f(n+l,s)\,\mathrm{e}^{-s} - f(n+l,0) - \int_0^s f(n+l+1,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\right]\mathrm{d}s \\ &= -\int_0^t f(n+l,s)\,\mathrm{e}^{-s}\,\mathrm{d}s + f(n+l,0)t + \int_0^t \int_0^s f(n+l+1,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s, \quad l = 0,1.\end{aligned}$$

Further using those three equalities and (2.23), from (2.22) we can have

$$\begin{aligned}\mathrm{e}^{-t}[\phi(n+1,t) + \phi(n-1,t) - 2\phi(n,t)] &= \alpha(n+1)t + \beta(n+1) + f(n,t)\,\mathrm{e}^{-t} - f(n,0) + f(n+1,0)t + \alpha(n-1)t \\ &\quad + \beta(n-1) + f(n,0)t - 2\alpha(n)t - 2\beta(n) \\ &= [\alpha(n+1) + \alpha(n-1) - f(n+1,0) + f(n,0) - 2\alpha(n)]t \\ &\quad + [\beta(n+1) + \beta(n-1) - f(n,0) - 2\beta(n)] + f(n,t)\,\mathrm{e}^{-t} = f(n,t)\,\mathrm{e}^{-t},\end{aligned}$$

and

$$\begin{aligned}(\phi(n,t))_t &= \left[\alpha(n) + \int_0^t f(n+1,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\right]\mathrm{e}^t + \left[\alpha(n)t + \beta(n) + \int_0^t \int_0^s f(n+1,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s\right]\mathrm{e}^t \\ &= \left[\alpha(n) + \alpha(n)t + \beta(n) + f(n+1,0)t + \int_0^t \int_0^s f(n+2,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s\right]\mathrm{e}^t \\ &= \left[\alpha(n+1)t + \beta(n+1) + \int_0^t \int_0^s f(n+2,r)\,\mathrm{e}^{-r}\,\mathrm{d}r\,\mathrm{d}s\right]\mathrm{e}^t = \phi(n+1,t).\end{aligned}$$

Hence, the function $\phi(n,t)$ defined by (2.22) and (2.23) solves the system (2.21). On the other hand, the first equation in the system (2.21) is of the second order and the recurrence relation (2.23) has two arbitrary constants. Therefore, the solution defined by (2.22) and (2.23) is the general solution to the system (2.21).

If $\delta = -1$, we can transfer the problem on ϕ and f with $\delta = -1$ into the problem on $\bar{\phi}$ and $\bar{f}$ with $\delta = 1$, upon introducing

$$\bar{\phi}(n,t) = \phi(-n,t), \qquad \bar{f}(n,t) = f(-n,t).$$

Therefore, the second case is true as well. This completes the proof. □

This theorem provides a way to solve the triangular system of (2.15) and (2.16), whose coefficient matrix is block-diagonal with each block being of the following low-triangular form

$$\begin{bmatrix} 2 & & & 0 \\ * & 2 & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & 2 \end{bmatrix}_{k_j \times k_j}, \tag{2.24}$$

where the symbol $*$ denotes an arbitrary constant entry. Associated with each such low-triangular block, the solution process starts from $f(n,t) = 0$ and ends with $f(n,t)$ being a linear combination of the previous eigenfunctions $\phi_i(n,t)$. Note that the solution formula (2.22) with $f(n,t) = 0$ leads to

$$\phi(n,t) = [ct + cn + d]\mathrm{e}^t, \tag{2.25}$$

where $c = \alpha(0)$ and $d = \beta(0)$ are arbitrary constants. Then, we can see, again from the solution formula (2.22), that all components of the solution $(\phi_1, \ldots, \phi_N)$ are of the type

$$\phi_i(n,t) = e^t \psi_i(n,t), \quad 1 \leqslant i \leqslant N, \tag{2.26}$$

where $\psi_i(n,t)$ are polynomials in n and t. Such functions $\psi_i(n,t)$, $1 \leqslant i \leqslant N$, satisfy the system of (2.11) and (2.12) with $\xi(t) = -1$. Therefore, the system of (2.15) and (2.16), whose coefficient matrix is block-diagonal with each block being of the form (2.24), leads to a polynomial solution $\tau_n = \mathrm{Cas}(\psi_1, \ldots, \psi_N)$ to the bilinear Toda lattice equation (2.7).

We sum up the above result in the following theorem.

Theorem 2.3. *Let a set of eigenfunctions $\phi_1(n,t), \phi_1(n,t), \ldots, \phi_N(n,t)$ solve the triangular system of* (2.15) *and* (2.16), *whose coefficient matrix $\Lambda = (\lambda_{ij})$ is block-diagonal with each block being of the low-triangular form* (2.24). *Then the τ-function*

$$\tau_n = \mathrm{Cas}(\psi_1(n,t), \psi_2(n,t), \ldots, \psi_N(n,t)), \quad \psi_i(n,t) = e^{-t}\phi_i(n,t), \ 1 \leqslant i \leqslant N, \tag{2.27}$$

presents a polynomial solution to the bilinear Toda lattice equation (2.7) *and thus the transformation*

$$a_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad b_n = \frac{\tau_{n,t}\tau_{n+1} - \tau_n\tau_{n+1,t}}{\tau_n\tau_{n+1}} \tag{2.28}$$

gives a rational solution to the Toda lattice equation (2.1). *Moreover, the eigenfunctions required in the rational solution are recursively determined by the solution formula* (2.22).

The above two theorems show that the case of eigenvalues being 2 corresponds to rational solutions and provide a recursive procedure for constructing rational solutions to the Toda lattice equation (2.1). The procedure can be summarized as follows:

Step 1. Recursively using the solution formula (2.22), solve the triangular system of (2.15) and (2.16), whose coefficient matrix is of the Jordan form with each eigenvalue being 2 to obtain the set of eigenfunctions $(\phi_1, \ldots, \phi_N)$.

Step 2. Evaluate the τ-function $\tau_n = \mathrm{Cas}(\psi_1, \ldots, \psi_N)$ with $\psi_i = e^{-t}\phi_i$, $1 \leqslant i \leqslant N$, which presents a polynomial solution in the space and time variables, n and t, to the bilinear Toda lattice equation (2.7).

Step 3. Compute the dependent variable transformation

$$a_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad b_n = \frac{\tau_{n,t}\tau_{n+1} - \tau_n\tau_{n+1,t}}{\tau_n\tau_{n+1}}.$$

This provides a rational solution in the space and time variables, n and t, to the Toda lattice equation (2.1).

Associated with one Jordan block case, the first two eigenfunctions and the corresponding τ-functions yielding rational solutions are

$$\psi_1(n,t) = c_1 t + c_1 n + d_1, \tag{2.29}$$

$$\psi_2(n,t) = \tfrac{1}{6}c_1 t^3 + (\tfrac{1}{2}c_1 n + \tfrac{1}{2}c_1 + \tfrac{1}{2}d_1)t^2 + (\tfrac{1}{2}c_1 n^2 + \tfrac{1}{2}c_1 n + d_1 n + c_2)t + \tfrac{1}{6}c_1 n^3 + \tfrac{1}{2}d_1 n^2 - \tfrac{1}{6}c_1 n - \tfrac{1}{2}d_1 n + c_2 n + d_2, \tag{2.30}$$

$$\tau_n = \mathrm{Cas}(\psi_1) = c_1 t + c_1 n + d_1, \tag{2.31}$$

$$\begin{aligned}\tau_n &= \mathrm{Cas}(\psi_1, \psi_2) \\ &= \tfrac{1}{3}c_1^2 t^3 + (c_1^2 n + \tfrac{1}{2}c_1^2 + c_1 d_1)t^2 + (c_1^2 n^2 + c_1^2 n + 2c_1 d_1 n + c_1 d_1 + d_1^2)t + \tfrac{1}{3}c_1^2 n^3 + \tfrac{1}{2}c_1^2 n^2 + c_1 d_1 n^2 + \tfrac{1}{6}c_1^2 n \\ &\quad + c_1 d_1 n + d_1^2 n - c_1 d_2 + c_2 d_1,\end{aligned} \tag{2.32}$$

where c_i and d_i, $i = 1, 2$, are all arbitrary constants. They contain most of solutions presented in [7,8,11]. More general rational solutions can be generated from the case of more than one Jordan block.

We remark that the matrices defined by (2.24) are more general than their Jordan blocks, and thus they have broader applications. This is why we take the form (2.24) for diagonal blocks of the coefficient matrix of (2.15) and (2.16), although it presents the same solutions as the Jordan form does. In the following section, we will see some concrete applications of such a choice in (2.24).

3. Examples arising from the Taylor expansion

In this section, through the Taylor expansion, we would like to exhibit some specific rational solutions belonging to our general class, which will also explain why the Taylor expansion works well for constructing rational solutions.

It is known that soliton solutions are associated with the function of the type

$$\phi(n,t) = c\,e^{kn+t\,e^k} + d\,e^{-kn+t\,e^{-k}}, \tag{3.1}$$

where c and d are arbitrary constants and k is a modified spectral parameter. Actually, this function solves

$$\phi(n+1,t)+\phi(n-1,t) = 2\cosh(k)\phi(n,t), \quad (\phi(n,t))_t = \phi(n+1,t), \tag{3.2}$$

and so $\phi(n,t)$ satisfies the spectral problem (2.5) with an eigenvalue $\lambda = 2\cosh(k)$ under $a_n = 1$ and $b_n = 0$. Like the KdV case [9,21], we expand $\phi(n,t)$ and $2\cosh(k)$ with respect to k about $k = 0$:

$$\phi(n,t) = \sum_{i=0}^{\infty}\phi_i(n,t)k^i, \qquad 2\cosh(k) = \sum_{i=0}^{\infty}\frac{2}{(2i)!}k^{2i}, \tag{3.3}$$

and then we have

$$\phi_i(n+1,t)+\phi_i(n-1,t) = \sum_{j=0}^{[\frac{i}{2}]}\frac{2}{(2j)!}\phi_{i-2j}(n,t), \quad (\phi_i(n,t))_t = \phi_i(n+1,t), \ \ i \geqslant 0. \tag{3.4}$$

Now obviously, each set of functions $(\phi_0, \phi_1, \ldots, \phi_m)$ gives a solution to some special triangular system with a coefficient matrix of the form (2.24). It then follows from Theorem 2.3 that the corresponding Casorati determinant presents a rational solution to the Toda lattice equation (2.1).

To see what kind of eigenfunctions we can have from the above function $\phi(n,t)$, let us expand $e^{kn+t\,e^k}$ as

$$e^{kn+t\,e^k} = e^t\sum_{i=0}^{\infty}\frac{1}{i!}n^ik^i\sum_{j=0}^{\infty}\beta_j(t)k^j = e^t\sum_{l=0}^{\infty}\left[\sum_{j=0}^{l}\frac{n^{l-j}}{(l-j)!}\beta_j(t)\right]k^l, \tag{3.5}$$

where the functions $\beta_j(t)$ are determined by

$$\sum_{j=0}^{\infty}\beta_j(t)k^j = \sum_{p=0}^{\infty}\frac{t^p}{p!}\left(\sum_{q=1}^{\infty}\frac{1}{q!}k^q\right)^p,$$

as in [17]. More concretely, we have

$$\beta_0(t) = 1, \quad \beta_j(t) = \sum_{p=1}^{j}\frac{1}{p!}t^p\sum_{\substack{q_1+\cdots+q_p=j\\ q_1,\ldots,q_p\geqslant 1}}\prod_{r=1}^{p}\frac{1}{q_r!}, \ \ j \geqslant 1, \tag{3.6}$$

of which the first few functions are

$$\beta_1(t) = t, \qquad \beta_2(t) = \tfrac{1}{2}(t+t^2), \qquad \beta_3(t) = \tfrac{1}{6}t+\tfrac{1}{2}t^2+\tfrac{1}{6}t^3,$$
$$\beta_4(t) = \tfrac{1}{24}t+\tfrac{7}{24}t^2+\tfrac{1}{4}t^3+\tfrac{1}{24}t^4, \qquad \beta_5(t) = \tfrac{1}{120}t+\tfrac{1}{8}t^2+\tfrac{5}{24}t^3+\tfrac{1}{12}t^4+\tfrac{1}{120}t^5,$$
$$\beta_6(t) = \tfrac{1}{720}t+\tfrac{31}{720}t^2+\tfrac{1}{8}t^3+\tfrac{13}{144}t^4+\tfrac{1}{48}t^5+\tfrac{1}{720}t^6.$$

On the other hand, we can similarly have

$$e^{-kn+t\,e^{-k}} = e^t\sum_{l=0}^{\infty}\left[\sum_{j=0}^{l}\frac{n^{l-j}}{(l-j)!}\beta_j(t)\right](-k)^l. \tag{3.7}$$

It then follows that

$$\phi_i(n,t) = e^t\sum_{j=0}^{i}[c+(-1)^id]\frac{n^{i-j}}{(i-j)!}\beta_j(t), \quad i \geqslant 0. \tag{3.8}$$

If we use the case of $\delta = -1$, we will have

$$\phi_i(n,t) = e^t \sum_{j=0}^{i} [(-1)^i c + d](-1)^j \frac{n^{i-j}}{(i-j)!} \beta_j(t), \quad i \geqslant 0, \tag{3.9}$$

which can be generated from another eigenfunction

$$\phi(n,t) = c\, e^{-kn+t\, e^k} + d\, e^{kn+t\, e^{-k}}$$

associated with soliton solutions.

Therefore, if we take two specific choices of $c = 1$ but $d = 0$ and $c = 0$ but $d = 1$, then we have

$$\phi(n,t)|_{c=1,d=0} = e^t \sum_{i=0}^{\infty} \psi_i(n,t) k^i, \qquad \psi_i(n,t) = \sum_{j=0}^{i} \frac{n^{i-j}}{(i-j)!} \beta_j(t), \tag{3.10}$$

$$\phi(n,t)|_{c=0,d=1} = e^t \sum_{i=0}^{\infty} \psi_i(n,t) k^i, \qquad \psi_i(n,t) = \sum_{j=0}^{i} (-1)^i \frac{n^{i-j}}{(i-j)!} \beta_j(t). \tag{3.11}$$

The first series of solutions $\{\psi_i(n,t)\}_{i=0}^{\infty}$ was stated in [17].

Let us now choose $c = d = 1$ and $c = -d = 1$ and then we obtain the following two specific Taylor expansions:

$$\phi(n,t) = e^{kn+t\, e^k} + e^{-kn+t\, e^{-k}} = \sum_{i=0}^{\infty} \phi_i(n,t) k^{2i}, \tag{3.12}$$

$$\phi(n,t) = e^{kn+t\, e^k} - e^{-kn+t\, e^{-k}} = \sum_{i=0}^{\infty} \phi_i(n,t) k^{2i+1}. \tag{3.13}$$

The resulting two series of functions $\{\phi_i(n,t)\}_{i=0}^{\infty}$ satisfy

$$\phi_i(n+1,t) + \phi_i(n-1,t) = \sum_{j=0}^{i} \frac{2}{(2j)!} \phi_{i-j}(n,t), \quad (\phi_i(n,t))_t = \phi_i(n+1,t), \;\; i \geqslant 0. \tag{3.14}$$

Thus, each set of functions $(\phi_0, \phi_1, \ldots, \phi_m)$ from one of both series presents a solution to certain special triangular system with a coefficient matrix of the form (2.24). This gives another two series of eigenfunctions

$$\phi(n,t)|_{c=d=1} = e^t \sum_{i=0}^{\infty} \psi_i(n,t) k^{2i}, \qquad \psi_i(n,t) = \sum_{j=0}^{2i} \frac{2n^{2i-j}}{(2i-j)!} \beta_j(t), \tag{3.15}$$

$$\phi(n,t)|_{c=-d=1} = e^t \sum_{i=0}^{\infty} \psi_i(n,t) k^{2i+1}, \qquad \psi_i(n,t) = \sum_{j=0}^{2i+1} \frac{2n^{2i+1-j}}{(2i+1-j)!} \beta_j(t), \tag{3.16}$$

which were computed in [17].

The change of n into $-n$ leads to the other four series of eigenfunctions $\{\bar\psi_i(n,t)\}_{i=0}^{\infty}$:

$$\bar\phi(n,t)|_{c=1,d=0} = e^t \sum_{i=0}^{\infty} \bar\psi_i(n,t) k^i, \qquad \bar\psi_i(n,t) = \sum_{j=0}^{i} \frac{(-n)^{i-j}}{(i-j)!} \beta_j(t), \tag{3.17}$$

$$\bar\phi(n,t)|_{c=0,d=1} = e^t \sum_{i=0}^{\infty} \bar\psi_i(n,t) k^i, \qquad \bar\psi_i(n,t) = \sum_{j=0}^{i} (-1)^j \frac{n^{i-j}}{(i-j)!} \beta_j(t), \tag{3.18}$$

$$\bar\phi(n,t)|_{c=d=1} = e^t \sum_{i=0}^{\infty} \bar\psi_i(n,t) k^{2i}, \qquad \bar\psi_i(n,t) = \sum_{j=0}^{2i} \frac{2(-n)^{2i-j}}{(2i-j)!} \beta_j(t), \tag{3.19}$$

$$\bar\phi(n,t)|_{c=-d=1} = e^t \sum_{i=0}^{\infty} \bar\psi_i(n,t) k^{2i+1}, \qquad \bar\psi_i(n,t) = \sum_{j=0}^{2i+1} \frac{2(-n)^{2i+1-j}}{(2i+1-j)!} \beta_j(t). \tag{3.20}$$

Unfortunately, if we take the series of eigenfunctions defined by (3.8) [or (3.9)], especially one of the first two series of eigenfunctions (3.10) and (3.11) [or (3.17) and (3.18)], the corresponding τ-function $\tau_n = \mathrm{Cas}(\psi_0, \psi_1, \ldots, \psi_N)$ only leads to a constant solution. This is shown in the following theorem.

Theorem 3.1. *If* $\phi_i(n,t)$, $i \geqslant 0$, *are defined by* (3.8) [*or* (3.9)] *and* $\psi_i(n,t) = \mathrm{e}^{-t}\phi_i(n,t)$, $i \geqslant 0$, *then the* τ-*function* $\tau_n = \mathrm{Cas}(\psi_0, \psi_1, \ldots, \psi_N)$ *is a constant.*

Proof. The case of $\delta = -1$ corresponds to τ_{-n}. Thus, it is sufficient to prove the case of $\delta = 1$.

Let $\delta = 1$. Assume that $\phi_s(n,t)$ is defined by (3.1) with $c = 1$ and $d = 0$. Then we have

$$\mathrm{e}^k\phi_s(n,t) = \mathrm{e}^t\sum_{i=0}^{\infty}\psi_{s,i}(n+1,t)k^i,$$

which leads to

$$\psi_{s,i}(n+1,t) = \sum_{\substack{p+q=i\\ p,q\geqslant 0}}\frac{1}{p!}\psi_{s,q}(n,t), \quad i \geqslant 0, \tag{3.21}$$

where $\psi_{s,i}$, $i \geqslant 0$, are defined by (3.10) and the subscript s is just for avoiding confusion. Note that the general eigenfunctions $\phi_i(n,t)$ are scalar multiples of the above special eigenfunctions $\phi_{s,i}(n,t)$. Recursively applying the equality (3.21) to the Casorati determinant $\mathrm{Cas}(\psi_0, \psi_1, \ldots, \psi_N)$ from the last row to the first row, we have

$$\tau_n = \mathrm{Cas}(\psi_0,\psi_1,\ldots,\psi_N) = \begin{vmatrix} \psi_0(n,t) & \psi_0(n+1,t) & \cdots & \psi_0(n+N,t)\\ \psi_1(n,t) & \psi_1(n+1,t) & \cdots & \psi_1(n+N,t)\\ \cdots & \cdots & \cdots & \cdots\\ \psi_N(n,t) & \psi_N(n+1,t) & \cdots & \psi_N(n+N,t) \end{vmatrix}$$

$$= C_N(c,d)\begin{vmatrix} \psi_{s,0}(n,t) & \psi_{s,0}(n+1,t) & \cdots & \psi_{s,0}(n+N,t)\\ \psi_{s,1}(n,t) & \psi_{s,1}(n+1,t) & \cdots & \psi_{s,1}(n+N,t)\\ \cdots & \cdots & \cdots & \cdots\\ \psi_{s,N}(n+1,t) & \psi_{s,N}(n+2,t) & \cdots & \psi_{s,N}(n+N+1,t) \end{vmatrix}$$

$$= \cdots = C_N(c,d)\begin{vmatrix} \psi_{s,0}(n+1,t) & \psi_{s,0}(n+2,t) & \cdots & \psi_{s,0}(n+N+1,t)\\ \psi_{s,1}(n+1,t) & \psi_{s,1}(n+2,t) & \cdots & \psi_{s,1}(n+N+1,t)\\ \cdots & \cdots & \cdots & \cdots\\ \psi_{s,N}(n+1,t) & \psi_{s,N}(n+2,t) & \cdots & \psi_{s,N}(n+N+1,t) \end{vmatrix} = \tau_{n+1},$$

where $C_N(c,d) = (c^2-d^2)^{(N+1)/2}$ when N is odd and $C_N(c,d) = (c^2-d^2)^{N/2}(c+d)$ when N is even. This implies that τ_n does not depend on the variable n.

Furthermore, we can have

$$\tau_n = \begin{vmatrix} \psi_0(0,t) & \psi_0(1,t) & \cdots & \psi_0(N,t)\\ \psi_1(0,t) & \psi_1(1,t) & \cdots & \psi_1(N,t)\\ \cdots & \cdots & \cdots & \cdots\\ \psi_N(0,t) & \psi_N(1,t) & \cdots & \psi_N(N,t) \end{vmatrix} = C_N(c,d)\begin{vmatrix} \beta_0(t) & \beta_1(t) & \cdots & \beta_N(t)\\ \cdots & \cdots & \cdots & \cdots\\ \beta_i(t) & \sum_{j=0}^{i}\frac{1^{i-j}}{(i-j)!}\beta_j(t) & \cdots & \sum_{j=0}^{i}\frac{N^{i-j}}{(i-j)!}\beta_j(t)\\ \cdots & \cdots & \cdots & \cdots\\ \beta_N(t) & \sum_{j=0}^{N}\frac{1^{N-j}}{(N-j)!}\beta_j(t) & \cdots & \sum_{j=0}^{N}\frac{N^{N-j}}{(N-j)!}\beta_j(t) \end{vmatrix}. \tag{3.22}$$

Let us choose a solution $(x_0, x_1, \ldots, x_N)$ to the linear system

$$\begin{cases} x_0 + x_1 + \cdots + x_N = 0,\\ \frac{1}{1!}x_1 + \cdots + \frac{N}{1!}x_N = 0,\\ \frac{1}{2!}x_1 + \cdots + \frac{N^2}{2!}x_N = 0,\\ \cdots\\ \frac{1}{N!}x_1 + \cdots + \frac{N^N}{N!}x_N = 1. \end{cases} \tag{3.23}$$

Using the property of the Vandermonde determinant, it is easy to see that this system (3.23) has a unique solution with $x_N \neq 0$. Now for the last determinant in (3.22), let us multiply the first column by x_0, the second column by $x_1, \ldots,$ the last column by x_N, and then add them up to form a new last column. This gives

$$\tau_n = \frac{C_N(c,d)}{x_N}\begin{vmatrix} \beta_0(t) & \beta_1(t) & \cdots & \beta_{N-1}(t) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_i(t) & \sum_{j=0}^{i}\frac{1^{i-j}}{(i-j)!}\beta_j(t) & \cdots & \sum_{j=0}^{i}\frac{(N-1)^{i-j}}{(i-j)!}\beta_j(t) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_N(t) & \sum_{j=0}^{N}\frac{1^{N-j}}{(N-j)!}\beta_j(t) & \cdots & \sum_{j=0}^{N}\frac{(N-1)^{N-j}}{(N-j)!}\beta_j(t) & \beta_0(t) \end{vmatrix}$$

$$= \frac{C_N(c,d)}{x_N}\begin{vmatrix} \beta_0(t) & \beta_1(t) & \cdots & \beta_{N-1}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \beta_i(t) & \sum_{j=0}^{i}\frac{1^{i-j}}{(i-j)!}\beta_j(t) & \cdots & \sum_{j=0}^{i}\frac{(N-1)^{i-j}}{(i-j)!}\beta_j(t) \\ \cdots & \cdots & \cdots & \cdots \\ \beta_N(t) & \sum_{j=0}^{N}\frac{1^{N-j}}{(N-j)!}\beta_j(t) & \cdots & \sum_{j=0}^{N}\frac{(N-1)^{N-j}}{(N-j)!}\beta_j(t) \end{vmatrix},$$

where a cofactor expansion along the last column has been made and $\beta_0(t) = 1$ has been used. It then follows from the mathematical induction that τ_n does not depend on the variable t either. This completes the proof. □

Using the other two series of eigenfunctions in (3.15) and (3.16) [or (3.19) and (3.20)], we can have two classes of specific rational solutions to the Toda lattice equation (2.1) by Theorem 2.3. We state this result in the following theorem.

Theorem 3.2. *If* $\psi_i(n,t)$, $i \geqslant 0$, *are defined by one of* (3.15) *and* (3.16) [*or* (3.19) *and* (3.20)], *then the* τ-*function* $\tau_n = \mathrm{Cas}(\psi_0,\psi_1,\ldots,\psi_N)$ *presents a polynomial solution to the bilinear Toda lattice equation* (2.7) *and thus the transformation* (2.6) *determines a rational solution to the Toda lattice equation* (2.1).

Let us now list the first few examples associated with the third and fourth series of eigenfunctions $\{\psi_i(n,t)\}_{i=0}^{\infty}$. Associated with the third series, we have

$$\mathrm{Cas}(\psi_0) = 2, \qquad \mathrm{Cas}(\psi_0,\psi_1) = 4n + 4t + 2, \tag{3.24}$$

$$\mathrm{Cas}(\psi_0,\psi_1,\psi_2) = \tfrac{8}{3}t^3 + (8n+8)t^2 + (8n^2+16n+6)t + \tfrac{8}{3}n^3 + 8n^2 + \tfrac{22}{3}n + 2, \tag{3.25}$$

$$\begin{aligned}\mathrm{Cas}(\psi_0,\psi_1,\psi_2,\psi_3) = {} & \tfrac{16}{45}t^6 + (\tfrac{32}{15}n + \tfrac{16}{5})t^5 + (\tfrac{16}{3}n^2 + 16n + \tfrac{32}{3})t^4 + (\tfrac{64}{9}n^3 + 32n^2 + \tfrac{392}{9}n + \tfrac{52}{3})t^3 \\ & + (\tfrac{16}{3}n^4 + 32n^3 + \tfrac{200}{3}n^2 + 56n + 15)t^2 + (\tfrac{32}{15}n^5 + 16n^4 + \tfrac{136}{3}n^3 + 60n^2 + \tfrac{548}{15}n + 8)t \\ & + \tfrac{16}{45}n^6 + \tfrac{16}{5}n^5 + \tfrac{104}{9}n^4 + \tfrac{64}{3}n^3 + \tfrac{949}{45}n^2 + \tfrac{157}{15}n + 2,\end{aligned} \tag{3.26}$$

and associated with the fourth series, we obtain

$$\mathrm{Cas}(\psi_0) = 2n + 2t, \qquad \mathrm{Cas}(\psi_0,\psi_1) = \tfrac{4}{3}t^3 + (4n+2)t^2 + (4n^2+4n)t + \tfrac{4}{3}n^3 + 2n^2 + \tfrac{2}{3}n, \tag{3.27}$$

$$\begin{aligned}\mathrm{Cas}(\psi_0,\psi_1,\psi_2) = {} & \tfrac{8}{45}t^6 + (\tfrac{16}{15}n + \tfrac{16}{15})t^5 + (\tfrac{8}{3}n^2 + \tfrac{16}{3}n + 2)t^4 + (\tfrac{32}{9}n^3 + \tfrac{32}{3}n^2 + \tfrac{76}{9}n + \tfrac{4}{3})t^3 + (\tfrac{8}{3}n^4 + \tfrac{32}{3}n^3 + \tfrac{40}{3}n^2 + \tfrac{16}{3}n)t^2 \\ & + (\tfrac{16}{15}n^5 + \tfrac{16}{3}n^4 + \tfrac{28}{3}n^3 + \tfrac{20}{3}n^2 + \tfrac{8}{5}n)t + \tfrac{8}{45}n^6 + \tfrac{16}{15}n^5 + \tfrac{22}{9}n^4 + \tfrac{8}{3}n^3 + \tfrac{62}{45}n^2 + \tfrac{4}{15}n.\end{aligned} \tag{3.28}$$

These are polynomial solutions to the bilinear Toda lattice equation (2.7) and they yield four rational solutions to the Toda lattice equation (2.1) in the case of $\delta = 1$. The change of n into $-n$ will lead to the other four rational solutions in the case of $\delta = -1$.

4. Concluding remarks

A recursive procedure for constructing rational solutions has been presented through the Casoratian formulation, together with the general solution to the system yielding eigenfunctions required in rational solutions. This allows us to construct a broad class of rational solutions to the Toda lattice equation, without hard computation of long wave limits in soliton solutions. Interestingly, only the eigenfunctions corresponding to the eigenvalue 2 of the Lax pair lead to Casorati determinant rational solutions. All rational solutions arising from the Taylor expansions of the generating functions of soliton solutions have been analyzed and only the Taylor expansions containing even or odd powers of the modified spectral parameter lead to non-constant rational solutions. Two series of eigenfunctions yielding non-constant rational solutions have been explicitly given. The resulting solution space contains all rational solutions presented in [17] and many other existing rational solutions [7,8].

Like the KdV case [23], there also exist mixed rational-soliton type solutions. Such solutions can be viewed as a kind of interaction solutions [21] and they can be easily generated through the Casoratian formulation in Section 2, by taking two subsets of eigenfunctions associated with the Jordan blocks of the first type with eigenvalues being 2 and greater than 2. Other kind of mixed rational-positon solutions and rational-complexiton solutions can similarly be constructed. They are interaction solutions between rational solutions and positons [24] and between rational solutions and complexitons [16]. More general interaction solutions among various different kinds of basic solutions can also be generated directly using the Casoratian formulation.

We remark that the generating functions of positon solutions,

$$\phi(n,t) = c\,\mathrm{e}^{t\cos k}\cos(kn + \delta t\sin k) + d\,\mathrm{e}^{t\cos k}\sin(kn + \delta t\sin k),$$

where c, d and k are arbitrary constants, do not yield new rational solutions. They bring the same set of rational solutions as the generating functions of soliton solutions do. It is interesting to research for other special functions whose Taylor expansions can lead to determinant rational solutions to the Toda lattice equation. Moreover, it is worth studying whether the rational solutions expressed by the Casorati determinant can always be viewed as some kind of long wave limits of soliton solutions.

There are also rational solutions to continuous and difference Painlevé equations, which are connected with Schur functions and q-Schur functions [25,26]. It would be important to find relations between the general polynomial solution and special polynomial theories for the bilinear Toda lattice equation. More generally, what kind of rational solutions one can have for the bilinear Toda lattice equation?

Acknowledgements

The first author was supported in part by the University of South Florida Internal Awards Program under grant no. 1249-936RO.

References

[1] Airault H, McKean HP, Moser J. Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem. Commun Pure Appl Math 1977;30:95–148.
[2] Adler M, Moser J. On a class of polynomials connected with the Korteweg–de Vries equation. Commun Math Phys 1978;61:1–30.
[3] Strampp W, Oevel W, Steeb W-H. Similarity, Bäcklund transformations and rational solutions. Lett Math Phys 1983;7:445–52.
[4] Gesztesy F, Schweiger W. Rational KP and mKP-solutions in Wronskian form. Rep Math Phys 1991;30:205–22.
[5] Ablowitz MJ, Satsuma J. Solitons and rational solutions of nonlinear evolution equations. J Math Phys 1978;19:2180–6.
[6] Toda M. Vibration of a chain with nonlinear interaction. J Phys Soc Jpn 1967;22:431–6.
[7] Cârstea AS, Grecu D. On a class of rational and mixed soliton-rational solutions of the Toda lattice. Prog Theoret Phys 1996;96:29–36.
[8] Narita K. Solutions for the Mikhailov–Shabat–Yamilov difference-differential equations and generalized solutions for the Volterra and the Toda lattice equations. Prog Theoret Phys 1998;99:337–48.
[9] Nimmo JJC, Freeman NC. Rational solutions of the Korteweg–de Vries equation in Wronskian form. Phys Lett A 1983;96:443–6.
[10] Weiss J. Modified equations, rational solutions, and the Painlevé property for the Kadomtsev–Petviashvili and Hirota–Satsuma equations. J Math Phys 1985;26:2174–80.
[11] Hu XB, Clarkson PA. Rational solutions of a differential-difference KdV equation, the Toda equation and the discrete KdV equation. J Phys A: Math Gen 1995;28:5009–16.
[12] Medina E, Marín MJ. Rational solutions of the KP equation through symmetry reductions: coherent structures and other solutions. Inverse Problems 2001;17:985–98.
[13] Nimmo JJC. Soliton solution of three differential-difference equations in Wronskian form. Phys Lett A 1983;99:281–6.
[14] Tamizhmani T, Kanaga Vel S, Tamizhmani KM. Wronskian and rational solutions of the differential-difference KP equation. J Phys A: Math Gen 1998;31:7627–33.
[15] Ma WX. Complexiton solutions to the Korteweg–de Vries equation. Phys Lett A 2002;301:35–44.
[16] Ma WX, Maruno K. Complexiton solutions of the Toda lattice equation. 2003, preprint.
[17] Wu H, Zhang DJ. Mixed rational-soliton solutions of two differential-difference equations in Casorati determinant form. J Phys A: Math Gen 2003;36:4867–73.
[18] Toda M. Nonlinear waves and solitons. Dordrecht: Kluwer Academic Publishers; 1989.
[19] Hirota R, Satsuma J. A variety of nonlinear network equations generated from the Bäcklund transformation for the Toda lattice. Prog Theoret Phys Suppl 1976;(59):64–100.
[20] Matveev VB, Salle MA. Darboux transformations and solitons. Berlin: Springer-Verlag; 1991.

[21] Ma WX, You Y. Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions. 2003, preprint.
[22] Maruno K, Ma WX, Oikawa M. Generalized Casorati determinant and positon-negaton type solutions of the Toda lattice equation. 2003, preprint.
[23] Ablowitz MJ, Cornille H. On solutions of the Korteweg–de Vries equation. Phys Lett A 1979;72:277–80.
[24] Stahlhofen AA, Matveev VB. Positons for the Toda lattice and related spectral problems. J Phys A: Math Gen 1995;28:1957–65.
[25] Kajiwara K, Noumi M, Yamada Y. q-Painlevé systems arising from q-KP hierarchy. Lett Math Phys 2002;62:259–68.
[26] Clarkson PA, Mansfield EL. The second Painlevé equation, its hierarchy and associated special polynomials. Nonlinearity 2003;16:R1–R26.