Semidirect sums of Lie algebras and discrete integrable couplings ⊘

Wen-Xiu Ma; Xi-Xiang Xu; Yufeng Zhang

🔴 Check for updates

View Online   Export Citation

Semidirect sums of Lie algebras and discrete integrable couplings

Wen-Xiu Ma[a]
Department of Mathematics, University of South Florida, Tampa, Florida 33620-5700

Xi-Xiang Xu[b]
College of Science, Shandong University of Science and Technology, Qingdao 266510, People's Republic of China

Yufeng Zhang[c]
School of Mathematics, Liaoning Normal University, Dalian 116029, People's Republic of China

A relation between semidirect sums of Lie algebras and integrable couplings of lattice equations is established, and a practicable way to construct integrable couplings is further proposed. An application of the resulting general theory to the generalized Toda spectral problem yields two classes of integrable couplings for the generalized Toda hierarchy of lattice equations. The construction of integrable couplings using semidirect sums of Lie algebras provides a good source of information on complete classification of integrable lattice equations. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2194630]

I. INTRODUCTION

Integrable couplings have been receiving growing attention recently. A few ways to construct integrable couplings are presented by using perturbations,[1–3] enlarging spectral problems,[4,5] and creating new loop Lie algebras.[6,7]

The problem of integrable couplings can be expressed as follows:[2] *For a given integrable system, how can we construct a nontrivial system of differential equations which is still integrable and includes the original integrable system as a subsystem?* Obviously, a change of orders of equations in a system does not lose integrability of the system. Therefore, up to a permutation, an integrable coupling of a given integrable system $u_t = K(u)$ is given by a bigger and triangular system:

$$u_t = K(u), \quad v_t = S(u,v).$$

The vector-valued function S should satisfy the nontriviality condition $\partial S / \partial[u] \neq 0$, where $[u] = (u, D_x u, D_x^2 u, \dots)$ and $D_x^n u$ denotes a vector consisting of all derivatives of u of order n with respect to the space variable x. The above nontriviality condition means that the second subsystem involves the dependent variables of the first subsystem (i.e., the original system), and thus it guarantees that trivial diagonal systems with $S(u,v) = S(v)$ are not within our discussion.

A basic integrable coupling of an integrable system $u_t = K(u)$ is given by

---

[a] Electronic mail: mawx@math.usf.edu; Tel: (813)974-9563, Fax: (813)974-2700
[b] Electronic mail: xu_xixiang@sohu.com
[c] Electronic mail: zhang_yfshandong@163.com

$$u_t = K(u), \quad v_t = K'(u)[v], \tag{1.1}$$

which can be generated by a perturbation around a solution of the system $u_t = K(u)$.[1] In the above system and elsewhere throughout this paper, $P'(u)[v]$ denotes the Gateaux derivative of $P(u) \equiv P(u, D_x u, \dots)$ with respect to u in a direction v, i.e.,

$$P'(u)[v] = \frac{\partial}{\partial \varepsilon} P(u + \varepsilon v)|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} P(u + \varepsilon v, D_x u + \varepsilon D_x v, \dots)|_{\varepsilon=0}.$$

Obviously, the second subsystem $v_t = K'(u)[v]$ in the above integrable coupling (1.1) is linear with respect to v. Moreover, a symmetry $S(u)$ of the system $u_t = K(u)$ leads to a solution $(u, S(u))$ to the integrable coupling (1.1). However, the second component v of a solution (u, v) to the integrable coupling (1.1) is generally not a symmetry of the system $u_t = K(u)$. This is because v satisfies the linearized system $v_t = K'(u)[v]$ only for one solution, not for all solutions of the system $u_t = K(u)$. Therefore, the simple integrable coupling (1.1) is already a generalization of the symmetry problem. Another basic integrable coupling of an integrable system $u_t = K(u)$ reads as

$$u_t = K(u), \quad v_t = K'(u)[v] + K(u). \tag{1.2}$$

This system has a set of hereditary recursion operators[2]

$$\Phi(\beta_1, \beta_2) = \begin{pmatrix} \beta_1 \Phi(u) & 0 \\ \beta_1 \Phi'(u)[v] + \beta_2 \Phi(u) & \beta_1 \Phi(u) \end{pmatrix}$$

with two arbitrary constants β_1 and β_2, if the original system $u_t = K(u)$ has a hereditary recursion operator $\Phi(u)$. Therefore, integrable couplings possess richer integrable structures than the original integrable systems.

   The study of integrable couplings provides clues towards complete classification of integrable systems. Let us first observe classification of square matrix spectral problems through the Jordan blocks under similar transformations of matrices. Each triangular Jordan block corresponds to an undecomposable subsystem in a given integrable system. Now, note that an arbitrary Lie algebra has a semidirect sum structure of a solvable Lie algebra and a semisimple Lie algebra,[8] and we will see that semidirect sums of Lie algebras can result in integrable couplings. These imply that the study of integrable couplings through semidirect sums of Lie algebras is an inevitable step towards complete classification of integrable systems with an arbitrary number of components, from a point of view of Lie algebras.

   The study of integrable couplings also generates interesting mathematical structures such as Lax pairs with several spectral parameters,[9,10,2] integrable constrained flows with higher multiplicity,[11,12] local bi-Hamiltonian structures in higher dimensions[13] and hereditary recursion operators of higher order.[2,14] Very recently, we have proposed a relation between semidirect sums of Lie algebras and integrable couplings of continuous soliton equations, which provides an interesting and systematic approach to integrable couplings of continuous soliton equations.[15] In this paper, we would like to discuss the problem of discrete integrable couplings and develop a theory for constructing discrete integrable couplings by use of semidirect sums of Lie algebras.

   Throughout our discussion, we denote by E the shift operator, write

$$(E^m x)(n) = x^{(m)}(n) = x(m + n), \quad \text{where } x: \mathbb{Z} \to \mathbb{R}, \; m, n \in \mathbb{Z}, \tag{1.3}$$

and adopt an inverse of the difference operator $E - 1$ as follows:

$$(E - 1)^{-1} = \frac{1}{2} \left(\sum_{k=-\infty}^{-1} - \sum_{k=0}^{\infty} \right) E^k. \tag{1.4}$$

Let G be a matrix Lie algebra with the standard Lie bracket $[A, B] = AB - BA$, and closed under matrix multiplication: $AB \in G$ for all $A, B \in G$. We assume that an integrable lattice equation (or system) of evolution type

$$u_t = K(u) = K(u, Eu, E^{-1}u, \ldots) \tag{1.5}$$

is associated with G, where $u = u(n, t)$ is a dependent variable. More precisely, there is a pair of square matrices U and V in G, called a Lax pair, so that the discrete spatial matrix spectral problem

$$E\phi = U\phi = U(u, \lambda)\phi \tag{1.6}$$

and the associated discrete temporal matrix spectral problem

$$\phi_t = V\phi = V(u, Eu, E^{-1}u, \ldots; \lambda)\phi, \tag{1.7}$$

where λ is a spectral parameter and ϕ is an eigenfunction, generate[16,17] the integrable lattice equation (1.5) through their isospectral (i.e., $\lambda_t = 0$) compatibility condition

$$U_t = (EV)U - UV, \tag{1.8}$$

which is called a discrete zero curvature equation. In other words, we have

$$U'(u)[K] = (EV)U - UV, \tag{1.9}$$

where $U'(u)[K]$ denotes the Gateaux derivative as above. In a nonisospectral case, e.g., $\lambda_t = f(\lambda)$, then we have

$$U'(u)[K] + fU_\lambda = (EV)U - UV, \tag{1.10}$$

where U_λ is a partial derivative of U with respect to λ. Based on (1.6) and (1.7), the lattice equation (1.5) can often be solved by the inverse scattering transform (for example, see Ref. 18). There are also a few interesting Lie algebraic structures hidden behind the equation (1.10) (see Ref. 17 for more information). An integrable hierarchy and its master symmetry hierarchy usually correspond to the isospectral case and the nonisospectral case $\lambda_t = \lambda^n$, $n \geqslant 0$, respectively. These two hierarchies constitute a semidirect sum of Lie algebras, each of which consists of symmetries in one hierarchy. The spatial matrix spectral problem (1.6) is our starting point in constructing discrete integrable couplings. The closure property of the Lie algebra G under matrix multiplication guarantees that $(EV)U - UV$ is still in G so that the discrete zero curvature equation (1.8) makes sense.

In what follows, we are going to establish a relation between semidirect sums of Lie algebras and integrable couplings of lattice equations and a technically practicable way to generate integrable couplings through semidirect sums of Lie algebras. The resulting general theory will be used to generate two classes of integrable couplings for the generalized Toda hierarchy presented in Ref. 19. It will also be indicated that the construction of integrable couplings using semidirect sums of Lie algebras provides a good source of information about classification of integrable lattice equations. A few concluding remarks will be given in the last section.

II. CONSTRUCTING INTEGRABLE COUPLINGS USING SEMIDIRECT SUMS OF LIE ALGEBRAS

A. Generating scheme

Assume that the lattice equation (1.5) has a Lax pair (U, V) in a matrix Lie algebra G closed under matrix multiplication.

To construct an integrable coupling of the lattice equation (1.5), we use semidirect sums of Lie algebras to enlarge the original Lie algebra G. Take another matrix Lie algebra G_c closed under matrix multiplication and then form a semidirect sum $\bar{G}$ of G and G_c:

$$\bar{G} = G \uplus G_c. \tag{2.1}$$

The notion of semidirect sums means that G and G_c satisfy

$$[G, G_c] \subseteq G_c, \tag{2.2}$$

where $[G, G_c] = \{[A, B] \,|\, A \in G, B \in G_c\}$. Obviously, G_c is an ideal Lie subalgebra of $\bar{G}$. The subscript c here indicates a contribution to the construction of couplings. We also require that the closure property between G and G_c under matrix multiplication,

$$GG_c, G_c G \subseteq G_c, \tag{2.3}$$

where $G_1 G_2 = \{AB \,|\, A \in G_1, B \in G_2\}$, to guarantee that a Lax pair from the semidirect sum $\bar{G}$ can generate a coupling system. Note that the two different binary operations were used in the above closure properties in (2.2) and (2.3).

Now choose a pair of new Lax matrices in the semidirect sum $\bar{G}$ of Lie algebras:

$$\bar{U} = U + U_c, \quad \bar{V} = V + V_c, \quad U_c, V_c \in G_c, \tag{2.4}$$

and make a pair of enlarged discrete spatial matrix spectral problems

$$E\bar{\phi} = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi},$$

$$\bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}(\bar{u}, E\bar{u}, E^{-1}\bar{u}, \dots; \lambda)\bar{\phi}, \tag{2.5}$$

where the matrix U_c in $\bar{U}$ introduces additional dependent variables and $\bar{u}$ consists of both the original dependent variables and the additional dependent variables. In addition, the matrix U_c could depend on the spectral parameter λ, and the matrix V_c in $\bar{V}$ really does almost in all cases. Based on the closure properties of G, G_c and between G and G_c, it is easy to see that

$$(E\bar{V})\bar{U} - \bar{U}\bar{V} = [(EV)U - UV] + \{[(EV)U_c - U_c V] + [(EV_c)U - UV_c] + [(EV_c)U_c - U_c V_c]\}$$
$$\in G \uplus G_c.$$

Therefore, under $u_t = K(u)$, the corresponding enlarged discrete zero curvature equation

$$\bar{U}_t = (E\bar{V})\bar{U} - \bar{U}\bar{V} \tag{2.6}$$

precisely presents

$$U_t = (EV)U - UV,$$

$$U_{c,t} = [(EV)U_c - U_c V] + [(EV_c)U - UV_c] + [(EV_c)U_c - U_c V_c]. \tag{2.7}$$

The first equation above is equivalent to the lattice equation (1.5), and hence, this is a coupling system for the lattice equation (1.5).

The whole construction process above shows that semidirect sums of a given Lie algebra G with new Lie algebras provide a great choice of candidates of integrable couplings for the lattice equation (1.5) generated from the Lie algebra G.

B. Realizations by particular semi-direct sums

To shed light on the above general scheme of constructing coupling systems, let us introduce the following particular class of semidirect sums of Lie algebras:

$$\bar{G} = G \uplus G_c, \quad G = \{\mathrm{diag}(\underbrace{A, \dots, A}_{\mu}, \underbrace{0, \dots, 0}_{\nu - \mu + 1})\},$$

$$G_c = \left\{ \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & B_{ij} & \\ & & 0 & & & & \\ & & & B_{\mu+1,\mu+1} & & & \\ & 0 & & & \ddots & & \\ & & & & & B_{\nu+1,\nu+1} \end{pmatrix} \right\}, \tag{2.8}$$

where A, B_{ii} are arbitrary square matrices, A is of the same order as U and the partitions of matrices in G and G_c are the same. Obviously, $B_{ij}, j \leq \mu$, are square but $B_{ij}, j \geq \mu+1$, may not; and all closure conditions of G, G_c and between G and G_c under matrix multiplication are satisfied.

Define

$$U_G := \mathrm{diag}(\underbrace{U, \ldots, U}_{\mu}, \underbrace{0, \ldots, 0}_{\nu-\mu+1}), \quad V_G := \mathrm{diag}(\underbrace{V, \ldots, V}_{\mu}, \underbrace{0, \ldots, 0}_{\nu-\mu+1}).$$

Note that U_G and V_G in G generate the same lattice equation as U and V, and thus for integrable couplings, the corresponding enlarged spectral matrices $\bar{U}$ and $\bar{V}$ in the semidirect sum $G \dotplus G_c$ can be chosen as

$$\bar{U} = U_G + U_{G,c} := \begin{pmatrix} U & & & & & & \\ & \ddots & & & & U_{ij} & \\ & & U & & & & \\ & & & U_{\mu+1,\mu+1} & & & \\ & 0 & & & \ddots & & \\ & & & & & U_{\nu+1,\nu+1} \end{pmatrix}$$

$$\bar{V} = V_G + V_{G,c} := \begin{pmatrix} V & & & & & & \\ & \ddots & & & & V_{ij} & \\ & & V & & & & \\ & & & V_{\mu+1,\mu+1} & & & \\ & 0 & & & \ddots & & \\ & & & & & V_{\nu+1,\nu+1} \end{pmatrix},$$

where the first two matrices U_G and V_G play the (U, V)-part and the second two matrices $U_{G,c}$ and $V_{G,c}$ play the (U_c, V_c)-part in the pair of $\bar{U}$ and $\bar{V}$ defined in (2.4). It is not difficult to see that the resulting coupling system (2.7) becomes

$$U_t = (EV)U - UV,$$

$$(U_{i+1,i+1})_t = (EV_{i+1,i+1})U_{i+1,i+1} - U_{i+1,i+1}V_{i+1,i+1}, \quad \mu \leq i \leq \nu, \tag{2.9}$$

$$U_{ij,t} = \sum_{k=i}^{j} [(EV_{ik})U_{kj} - U_{ik}V_{kj}], \quad 1 \leq i < j \leq \nu+1,$$

where $U_{ii} = U$ and $V_{ii} = V$, $1 \leq i \leq \mu$.

In particular, first, if we take

$$\bar{U} = \begin{pmatrix} U & U_{12} & \cdots & U_{1,\mu+1} \\ & \ddots & \ddots & \vdots \\ & & U & U_{\mu,\mu+1} \\ 0 & & & 0 \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} V & V_{12} & \cdots & V_{1,\mu+1} \\ & \ddots & \ddots & \vdots \\ & & V & V_{\mu,\mu+1} \\ 0 & & & 0 \end{pmatrix},$$

then the coupling system (2.9) becomes

$$U_t = (EV)U - UV,$$

$$U_{ij,t} = \sum_{k=i}^{j} [(EV_{ik})U_{kj} - U_{ik}V_{kj}], \quad 1 \leqslant i < j \leqslant \mu + 1,$$

(2.10)

where $U_{ii} = U$, $V_{ii} = V$, $1 \leqslant i \leqslant \mu$, and $U_{ii} = V_{ii} = 0$, $i = \mu + 1$. Second, if we take

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & \cdots & U_{a_\nu} \\ & U & \ddots & \vdots \\ & & \ddots & U_{a_1} \\ 0 & & & U \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} V & V_{a_1} & \cdots & V_{a_\nu} \\ & V & \ddots & \vdots \\ & & \ddots & V_{a_1} \\ 0 & & & V \end{pmatrix},$$

then the coupling system (2.9) becomes

$$U_t = (EV)U - UV,$$

$$U_{a_i,t} = \sum_{k+l=i, k,l \geqslant 0} [(EV_{a_k})U_{a_l} - U_{a_l}V_{a_k}], \quad 1 \leqslant i \leqslant \nu,$$

(2.11)

where $U_{a_0} = U$ and $V_{a_0} = V$.

We remark that here we have just presented one class of semidirect sums of Lie algebras, together with two specific examples. It is interesting to construct other possible realizations, especially those which could carry essential information for keeping integrable properties of the original lattice equations.

C. Linearly dependent case on the spectral parameter

Let us now assume that the spatial spectral matrix U depends linearly on the spectral parameter λ (see, for example, Refs. 17 and 19–21):

$$U = U(u,\lambda) = \lambda U_0 + U_1, \quad \frac{\partial U_0}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda} = 0. \tag{2.12}$$

Consider two specific examples of the enlarged spatial spectral matrices introduced in the last section,

$$\bar{U}_1 = \begin{pmatrix} U & U_a \\ 0 & 0 \end{pmatrix}, \quad \bar{U}_2 = \begin{pmatrix} U & U_a \\ 0 & U \end{pmatrix}, \quad \frac{\partial U_a}{\partial \lambda} = 0. \tag{2.13}$$

Note that the submatrices U_a in the above two enlarged spatial spectral matrices could be of different sizes. As in the continuous cases,[4,15] suppose that

$$\bar{W}_1 = \begin{pmatrix} W & W_a \\ 0 & 0 \end{pmatrix}, \quad \bar{W}_2 = \begin{pmatrix} W & W_a \\ 0 & W \end{pmatrix}$$

with

$$W = \sum_{i \geq 0} W_i \lambda^{-i}, \quad W_a = \sum_{i \geq -n_0} W_{a,i} \lambda^{-i}, \quad \frac{\partial W_i}{\partial \lambda} = 0, \quad \frac{\partial W_{a,i}}{\partial \lambda} = 0, \tag{2.14}$$

where $n_0 \geq 0$ is a proper integer, solve the corresponding enlarged discrete stationary zero curvature equations

$$(E\bar{W}_i)\bar{U}_i - \bar{U}_i\bar{W}_i = 0, \quad i = 1, 2, \tag{2.15}$$

respectively.

Then for each $m \geq 0$, choose

$$\bar{V}_1^{[m]} = \begin{pmatrix} V^{[m]} & V_a^{[m]} \\ 0 & 0 \end{pmatrix} = (\lambda^m \bar{W}_1)_+ + \bar{\Delta}_m, \quad \bar{\Delta}_m = \begin{pmatrix} \Delta_m & \Delta_{m,a} \\ 0 & 0 \end{pmatrix},$$

where Δ_m and $\Delta_{m,a}$ do not depend on λ and satisfy

$$(E\Delta_m)U_0 - U_0\Delta_m = 0, \quad U_0\Delta_{m,a} = 0, \tag{2.16}$$

and choose

$$\bar{V}_2^{[m]} = \begin{pmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{pmatrix} = (\lambda^m \bar{W}_2)_+ + \bar{\Delta}_m, \quad \bar{\Delta}_m = \begin{pmatrix} \Delta_m & \Delta_{m,a} \\ 0 & \Delta_m \end{pmatrix},$$

where Δ_m and $\Delta_{m,a}$ do not depend on λ and satisfy

$$(E\Delta_m)U_0 - U_0\Delta_m = 0, \quad (E\Delta_{m,a})U_0 - U_0\Delta_{m,a} = 0. \tag{2.17}$$

The subscript $+$ above denotes to select the polynomial part in λ. Based on (2.10) and (2.11) and using (2.15), we can directly show that the enlarged discrete zero curvature equations

$$\bar{U}_{i,t_m} = (E\bar{V}_i^{[m]})\bar{U}_i - \bar{U}_i\bar{V}_i^{[m]}, \quad i = 1, 2,$$

namely,

$$U_{t_m} = (EV^{[m]})U - UV^{[m]},$$

$$U_{a,t_m} = (EV^{[m]})U_a - UV_a^{[m]},$$

and

$$U_{t_m} = (EV^{[m]})U - UV^{[m]},$$

$$U_{a,t_m} = (EV^{[m]})U_a + (EV_a^{[m]})U - UV_a^{[m]} - U_aV^{[m]},$$

present

$$U_{t_m} = (\Delta_m)_x + [U_0, W_{m+1}] - [U_1, \Delta_m],$$

$$U_{a,t_m} = U_0 W_{a,m+1} + (E\Delta_m)U_a - U_1\Delta_{m,a}, \tag{2.18}$$

and

$$U_{t_m} = (\Delta_m)_x + [U_0, W_{m+1}] - [U_1, \Delta_m], \tag{2.19a}$$

$$U_{a,t_m} = U_0 W_{a,m+1} - (E W_{a,m+1}) U_0 + (E\Delta_m) U_a - U_a \Delta_m + (E\Delta_{m,a}) U_1 - U_1 \Delta_{m,a}, \qquad (2.19b)$$

respectively.

We remark that these two enlarged hierarchies in (2.18) and (2.19) share the enlarged discrete spectral problems

$$E\bar{\phi} = \bar{U}_1 \bar{\phi}, \quad E\bar{\phi} = \bar{U}_2 \bar{\phi},$$

respectively. Thus, all lattice equations in each of the two enlarged hierarchies can possess infinitely many common conserved densities except the original ones (see Refs. 22–24 for a few concrete examples). Moreover, one can construct a specific nondegenerate bilinear form on $\bar{G}$ with the invariance property, to present Hamiltonian structures of the enlarged lattice equations by a generalized trace identity. The detailed analysis on those integrable properties will be left to a future presentation.

To sum up, each system of lattice equations in the hierarchy (2.18) or (2.19) can provide an integrable coupling for its first subsystem of lattice equations. In the next section, we will only discuss two examples of constructing enlarged lattice hierarchies, in the generalized Toda case presented in Ref. 19.

III. INTEGRABLE COUPLINGS OF THE GENERALIZED TODA HIERARCHY

A. The generalized Toda equations

Let us here recall the generalized Toda hierarchy.[19] The corresponding discrete spatial spectral problem reads

$$E\phi = U(u,\lambda)\phi, U(u,\lambda) = \begin{pmatrix} 0 & 1 \\ (\alpha\lambda + \beta)r & \lambda + s \end{pmatrix}, \quad u = \begin{pmatrix} r \\ s \end{pmatrix}, \qquad (3.1)$$

where λ is a spectral parameter, and α and β are two arbitrary constants satisfying $\alpha^2 + \beta^2 \neq 0$. When $\alpha=0$ and $\beta=-1$, (3.1) becomes the Toda spectral problem.[16]

Its stationary discrete zero curvature equation

$$(EW)U - UW = 0 \qquad (3.2)$$

has the solution

$$W = \begin{pmatrix} a & b \\ (\alpha\lambda + \beta)c & -a \end{pmatrix}, \qquad (3.3)$$

with

$$a = \sum_{i \geq 0} a_i \lambda^{-i}, \quad b = \sum_{i \geq 0} b_i \lambda^{-i}, \quad c = \sum_{i \geq 0} c_i \lambda^{-i},$$

where the coefficients are defined by the initial conditions

$$a_0 = -\tfrac{1}{2}, \quad b_0 = 0, \quad c_0 = 0,$$

and the recursion relation

$$c_{i+1} - r b_{i+1}^{(1)} = 0, \quad i \geq 0, \qquad (3.4a)$$

$$b_{i+1}^{(1)} + s b_i^{(1)} + (a_i^{(1)} + a_i) = 0, \quad i \geq 0, \qquad (3.4b)$$

$$(a_{i+1}^{(1)} - a_{i+1}) + s(a_i^{(1)} - a_i) + \alpha(rb_{i+1} - c_{i+1}^{(1)}) + \beta(rb_i - c_i^{(1)}) = 0, \quad i \geqslant 0, \tag{3.4c}$$

which are all difference polynomials in u with respect to the lattice variable n. Under the initial-value conditions

$$a_1|_{u=0} = c_1|_{u=0} = 0, \quad a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geqslant 2,$$

the recursion relation (3.4) uniquely determines the lattice functions a_i, b_i, and c_i, $i \geqslant 1$. The first few lattice functions are

$$a_1 = \alpha r, \quad b_1 = 1, \quad c_1 = r,$$

$$a_2 = -\alpha^2 r^{(1)} r - \alpha^2 r^2 - \alpha^2 r r^{(-1)} - \alpha r s - \alpha r s^{(-1)} + \beta r,$$

$$b_2 = -\alpha r - \alpha r^{(-1)} - s^{(-1)}, \quad c_2 = -rs - \alpha r^2 - \alpha r r^{(1)}.$$

As usual, choose that

$$V_m = \begin{pmatrix} (\lambda^m a)_+ & (\lambda^m b)_+ \\ (\alpha\lambda + \beta)(\lambda^m c)_+ & -(\lambda^m a)_+ \end{pmatrix}, \quad m \geqslant 0. \tag{3.5}$$

Then it follows from (3.4) that

$$(EV_m)U - UV_m = \begin{pmatrix} 0 & -b_{m+1}^{(1)} \\ (\alpha\lambda + \beta)c_{m+1} & \beta(c_m^{(1)} - rb_m) - s(a_m^{(1)} - a_m) \end{pmatrix}.$$

Take a modification

$$\Delta_m = \begin{pmatrix} b_{m+1} & 0 \\ 0 & 0 \end{pmatrix},$$

and define the temporal spectral matrices

$$V^{[m]} = V_m + \Delta_m, \quad m \geqslant 0. \tag{3.6}$$

Then, a direct calculation leads to the following matrix:

$$(EV^{[m]})U - UV^{[m]} = \begin{pmatrix} 0 & 0 \\ (\alpha\lambda + \beta)(c_{m+1} - rb_{m+1}) & \beta(c_m^{(1)} - rb_m) - s(a_m^{(1)} - a_m) \end{pmatrix}.$$

This is consistent with U_{t_m}, and thus, making the evolution laws

$$\phi_{t_m} = V^{[m]}\phi, \quad m \geqslant 0, \tag{3.7}$$

the compatibility conditions

$$U_{t_m} = (EV^{[m]})U - UV^{[m]}, \quad m \geqslant 0,$$

of the discrete spatial spectral problem (3.1) and the associated discrete temporal spectral problems (3.7) give rise to the following hierarchy of lattice equations:

$$\begin{aligned} r_{t_m} &= c_{m+1} - rb_{m+1}, \\ s_{t_m} &= -\alpha(c_{m+1}^{(1)} - rb_{m+1}) + (a_{m+1}^{(1)} - a_{m+1}), \end{aligned} \quad m \geqslant 0. \tag{3.8}$$

This generalized Toda hierarchy is Liouville integrable,[19] and its Hamiltonian structure leads to infinitely many conservation laws and symmetries for every system in the hierarchy.

Obviously, the first nonlinear lattice equation in the hierarchy is

$$r_{t_1} = r(s^{(-1)} - s) + \alpha r(r^{(-1)} - r^{(1)}),$$

$$s_{t_1} = \alpha s(r - r^{(1)}) + \beta(r^{(1)} - r). \tag{3.9}$$

When $\alpha=0$ and $\beta=-1$, (3.9) becomes the Toda lattice equation,[25]

$$r_{t_1} = r(s^{(-1)} - s), \quad s_{t_1} = r - r^{(1)}, \tag{3.10}$$

and when $\alpha=1$ and $\beta=0$, (3.9) becomes the following lattice equation presented in Ref. 26:

$$r_{t_1} = r(s^{(-1)} - s) + r(r^{(-1)} - r^{(1)}), \quad s_{t_1} = s(r - r^{(1)}). \tag{3.11}$$

The lattice equation (3.11) is linearly independent of the Toda lattice equation (3.10). There exist a voluminous literature on the Toda lattice equation, and its generalizations and solution structures (for example, see Refs. 27–32).

B. Integrable couplings from specific semidirect sums

The generalized Toda spectral problem (3.1) linearly depends on the spectral parameter λ, and thus we can write

$$U = \begin{pmatrix} 0 & 1 \\ (\alpha\lambda + \beta)r & \lambda + s \end{pmatrix} = U_0\lambda + U_1, \quad U_0 = \begin{pmatrix} 0 & 0 \\ \alpha r & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 1 \\ \beta r & s \end{pmatrix}. \tag{3.12}$$

We will also see that there is a difference between the two cases of $\alpha=0$ and $\alpha\neq 0$ in computing integrable couplings.

Let us first consider the semidirect sum of Lie algebras of 3×3 matrices,

$$G \dotplus G_c, \quad G = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathbb{C}[\lambda,\lambda^{-1}] \otimes M_{2\times 2} \right\}, \quad G_c = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \middle| B \in \mathbb{C}[\lambda,\lambda^{-1}] \otimes M_{2\times 1} \right\},$$

where $\mathbb{C}[\lambda,\lambda^{-1}] \otimes M_{m\times n} = \mathrm{span}\{\lambda^k A \,|\, k \in \mathbb{Z}, A \in M_{m\times n}\}$. In this case, G_c is an Abelian ideal of $G \dotplus G_c$. We define the corresponding enlarged spatial spectral matrix as

$$\bar{U} = \bar{U}(\bar{u},\lambda) = \begin{pmatrix} U & U_a \\ 0 & 0 \end{pmatrix} \in G \dotplus G_c, \quad U_a = U_a(v) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{3.13}$$

where v_1 and v_2 are new dependent variables and

$$v = (v_1,v_2)^T, \quad \bar{u} = (u^T, v^T)^T = (r,s,v_1,v_2)^T.$$

Upon setting

$$\bar{W} = \begin{pmatrix} W & W_a \\ 0 & 0 \end{pmatrix}, \quad W_a = W_a(\bar{u},\lambda) = \begin{pmatrix} e \\ f \end{pmatrix},$$

where W is a solution to $(EW)U - UW = 0$, defined by (3.3), the corresponding enlarged discrete stationary zero curvature equation $(E\bar{W})\bar{U} - \bar{U}\bar{W} = 0$ becomes

$$(EW)U_a - UW_a = 0, \tag{3.14}$$

which is equivalent to

$$W_a = WU^{-1}U_a,$$

namely,

$$f = a^{(1)}v_1 + b^{(1)}v_2, \tag{3.15a}$$

$$(\alpha\lambda + \beta)re = (\alpha\lambda + \beta)c^{(1)}v_1 - a^{(1)}v_2 - (\lambda + s)f. \tag{3.15b}$$

This system determines a solution for e and f as follows:

$$e = \sum_{i \geqslant -n_0} e_i \lambda^{-i}, \quad f = \sum_{i \geqslant 0} f_i \lambda^i,$$

where $n_0 = 1$ if $\alpha = 0$ and $n_0 = 0$ if $\alpha \neq 0$ [see (2.14) for introduction of n_0]. Now define the enlarged temporal spectral matrix as

$$\bar{V}^{[m]} = \begin{pmatrix} V^{[m]} & V_a^{[m]} \\ 0 & 0 \end{pmatrix}, \quad V_a^{[m]} = (\lambda^m W_a)_+ + \Delta_{m,a}, \quad m \geqslant 0, \tag{3.16}$$

where $V^{[m]}$ is defined as in (3.6). To satisfy (2.16), choose $\Delta_{m,a}$ as

$$\Delta_{m,a} = \begin{pmatrix} h_m \\ -\alpha r h_m \end{pmatrix}, \quad h_m - \text{arbitrary}, \quad m \geqslant 0. \tag{3.17}$$

Then based on (3.15), we can compute that

$$(EV^{[m]})U_a - UV_a^{[m]} = (EV_m)U_a - U(\lambda^m W_a)_+ + (E\Delta_m)U_a - U_1\Delta_{m,a} = \begin{pmatrix} 0 \\ -\alpha c_{m+1}^{(1)}v_1 + \alpha r e_{m+1} + f_{m+1} \end{pmatrix}$$

$$+ \begin{pmatrix} b_{m+1}^{(1)}v_1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\alpha r h_m \\ \beta r h_m - \alpha r s h_m \end{pmatrix}$$

$$= \begin{pmatrix} b_{m+1}^{(1)}v_1 + \alpha r h_m \\ -\alpha c_{m+1}^{(1)}v_1 + \alpha r e_{m+1} + f_{m+1} - \beta r h_m + \alpha r s h_m \end{pmatrix}, \quad m \geqslant 0.$$

Therefore, the mth enlarged discrete zero curvature equation

$$\bar{U}_{t_m} = (E\bar{V}^{[m]})\bar{U} - \bar{U}\bar{V}^{[m]}$$

leads to

$$v_{t_m} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{t_m} = S_m(u,v) = \begin{pmatrix} b_{m+1}^{(1)}v_1 + \alpha r h_m \\ -\alpha c_{m+1}^{(1)}v_1 + \alpha r e_{m+1} + f_{m+1} - \beta r h_m + \alpha r s h_m \end{pmatrix}, \tag{3.18}$$

together with the mth generalized Toda equation in (3.8). Therefore, we obtain a hierarchy of coupling systems defined by (2.18),

$$\bar{u}_{t_m} = \begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = \bar{K}_m(u) = \begin{pmatrix} K_m(u) \\ S_m(u,v) \end{pmatrix}, \quad m \geqslant 0 \tag{3.19}$$

for the generalized Toda hierarchy (3.8).

Let us second consider the semidirect sum of Lie algebras of 4×4 matrices,

$$G \uplus G_c, \quad G = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \Big| A \in \mathbb{C}[\lambda, \lambda^{-1}] \otimes M_{2\times 2} \right\}, \quad G_c = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \Big| B \in \mathbb{C}[\lambda, \lambda^{-1}] \otimes M_{2\times 2} \right\}.$$

In this case, G_c is an Abelian ideal of $G \uplus G_c$, too. We define the corresponding enlarged spatial spectral matrix as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{pmatrix} U & U_a \\ 0 & U \end{pmatrix} \in G \uplus G_c, \quad U_a = U_a(v) = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, \tag{3.20}$$

where v_i, $1 \leqslant i \leqslant 4$, are new dependent variables and

$$v = (v_1, v_2, v_3, v_4)^T, \quad \bar{u} = (u^T, v^T)^T = (r, s, v_1, v_2, v_3, v_4)^T.$$

If we set

$$\bar{W} = \begin{pmatrix} W & W_a \\ 0 & W \end{pmatrix}, \quad W_a = W_a(\bar{u}, \lambda) = \begin{pmatrix} e & f \\ g & -e \end{pmatrix},$$

where W is a solution to $(EW)U - UW = 0$, defined by (3.3), then the corresponding enlarged discrete stationary zero curvature equation $(E\bar{W})\bar{U} - \bar{U}\bar{W} = 0$ becomes

$$(EW)U_a + (EW_a)U - UW_a - U_aW = 0, \tag{3.21}$$

which is equivalent to

$$(e^{(1)} + e) + (\lambda + s)f^{(1)} + (a^{(1)} + a)v_2 + b^{(1)}v_4 - bv_1 = 0,$$

$$-(\alpha\lambda + \beta)r(e^{(1)} + e) - (\lambda + s)g + (\alpha\lambda + \beta)(c^{(1)}v_1 - cv_4) - (a^{(1)} + a)v_3 = 0, \tag{3.22}$$

$$g^{(1)} - (\lambda + s)(e^{(1)} - e) - (\alpha\lambda + \beta)(rf - c^{(1)}v_2) - (a^{(1)} - a)v_4 - bv_3 = 0.$$

This system can determine a solution for e, f, and g as follows:

$$e = \sum_{i \geq 0} e_i \lambda^{-i}, \quad f = \sum_{i \geq 0} f_i \lambda^{-i}, \quad g = \sum_{i \geq 0} g_i \lambda^{-i}.$$

Now, we define the enlarged temporal spectral matrix as

$$\bar{V}^{[m]} = \begin{pmatrix} V^{[m]} & V_a^{[m]} \\ 0 & V^{[m]} \end{pmatrix}, \quad V_a^{[m]} = (\lambda^m W_a)_+ + \Delta_{m,a}, \quad m \geq 0, \tag{3.23}$$

where $V^{[m]}$ is defined as in (3.6). To satisfy (2.17), choose $\Delta_{m,a}$ as

$$\Delta_{m,a} = \begin{pmatrix} h_m & 0 \\ -\alpha r h_m & 0 \end{pmatrix}, \quad h_m - \text{arbitrary}, \quad m \geq 0. \tag{3.24}$$

Then based on (3.22), we can compute that

$$(EV^{[m]})U_a + (EV_a^{[m]})U - UV_a^{[m]} - U_aV^{[m]}$$

$$= [(EV_m)U_a + (E(\lambda^m W_a)_+)U - U(\lambda^m W_a)_+ - U_aV_m]$$

$$\quad + [(E\Delta_m)U_a - U_a\Delta_m] + [(E\Delta_{m,a})U_1 - U_1\Delta_{m,a}]$$

$$= \begin{pmatrix} -\alpha c_{m+1}v_2 + \alpha r f_{m+1}^{(1)} & f_{m+1}^{(1)} \\ \alpha c_{m+1}^{(1)}v_1 - \alpha c_{m+1}v_4 - \alpha r(e_{m+1}^{(1)} + e_{m+1}) - g_{m+1} & \alpha c_{m+1}^{(1)}v_2 - \alpha r f_{m+1} - (e_{m+1}^{(1)} - e_{m+1}) \end{pmatrix}$$

$$\quad + \begin{pmatrix} (b_{m+1}^{(1)} - b_{m+1})v_1 & b_{m+1}^{(1)}v_2 \\ -b_{m+1}v_3 & 0 \end{pmatrix} + \begin{pmatrix} \alpha r h_m & h_m^{(1)} \\ r(\alpha s - \beta)h_m & -\alpha r^{(1)}h_m^{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha c_{m+1}v_2 + \alpha r f_{m+1}^{(1)} + (b_{m+1}^{(1)} - b_{m+1})v_1 + \alpha r h_m, \\ \alpha c_{m+1}^{(1)}v_1 - \alpha c_{m+1}v_4 - \alpha r(e_{m+1}^{(1)} + e_{m+1}) - g_{m+1} - b_{m+1}v_3 + r(\alpha s - \beta)h_m, \\ f_{m+1}^{(1)} + b_{m+1}^{(1)}v_2 + h_m^{(1)} \\ \alpha c_{m+1}^{(1)}v_2 - \alpha r f_{m+1} - (e_{m+1}^{(1)} - e_{m+1}) - \alpha r^{(1)}h_m^{(1)} \end{pmatrix}.$$

Then the mth enlarged discrete zero curvature equation

$$\bar{U}_{t_m} = (E\bar{V}^{[m]})\bar{U} - \bar{U}\bar{V}^{[m]}$$

leads to

$$
v_{t_m} = (v_1, v_2, v_3, v_4)^T_{t_m} = T_m(u,v)
$$
$$
= \begin{pmatrix}
-\alpha c_{m+1} v_2 + \alpha r f^{(1)}_{m+1} + (b^{(1)}_{m+1} - b_{m+1}) v_1 + \alpha r h_m \\
f^{(1)}_{m+1} + b^{(1)}_{m+1} v_2 + h^{(1)}_m \\
\alpha c^{(1)}_{m+1} v_1 - \alpha c_{m+1} v_4 - \alpha r(e^{(1)}_{m+1} + e_{m+1}) - g_{m+1} - b_{m+1} v_3 + r(\alpha s - \beta)h_m \\
\alpha c^{(1)}_{m+1} v_2 - \alpha r f_{m+1} - (e^{(1)}_{m+1} - e_{m+1}) - \alpha r^{(1)} h^{(1)}_m
\end{pmatrix}, \quad (3.25)
$$

together with the mth generalized Toda equation in (3.8). Therefore, we obtain a hierarchy of coupling systems defined by (2.19),

$$\bar{u}_{t_m} = \begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = \bar{K}_m(u) = \begin{pmatrix} K_m(u) \\ T_m(u,v) \end{pmatrix}, \quad m \geq 0 \qquad (3.26)$$

for the generalized Toda hierarchy (3.8).

C. Illustrative examples

We now work out two concrete examples as follows, one in each of the two above cases.

Case of $\alpha=1$ and $\beta=0$: Let us first compute an example of the hierarchy (3.19). Assume that $\alpha=1$ and $\beta=0$ for convenience, which corresponds to the lattice hierarchy presented in Ref. 26. In this case, we have $n_0=0$ in (2.14).

It directly follows from (3.15) that

$$f_i = a^{(1)}_i v_1 + b^{(1)}_i v_2,$$

$$r e_{i+1} = c^{(1)}_{i+1} v_1 - a^{(1)}_i v_2 - f_{i+1} - s f_i,$$

where $i \geq 0$ and $r e_0 = c^{(1)}_0 v_1 - f_0$. We can then obtain that

$$f_0 = -\tfrac{1}{2} v_1, \quad f_1 = r^{(1)} v_1 + v_2,$$

$$f_2 = -[r^{(2)} r^{(1)} + (r^{(1)})^2 + r^{(1)} r + r^{(1)} s^{(1)} + r^{(1)} s] v_1 - (r^{(1)} + r + s) v_2;$$

$$r e_0 = \tfrac{1}{2} v_1, \quad r e_1 = \tfrac{1}{2} s v_1 - \tfrac{1}{2} v_2, \quad r e_2 = r^{(1)} r v_1 + r v_2.$$

If we choose

$$h_1 = 2\xi f_0 f_1 = -\xi v_1(r^{(1)} v_1 + v_2), \quad \xi = \text{const},$$

then the vector-valued function S_1 defined by (3.18) becomes

$$S_1(u,v) = \begin{pmatrix} -(r^{(1)} + r + s) v_1 - \xi r v_1 (r^{(1)} v_1 + v_2) \\ -r^{(1)} s v_1 - (r^{(1)} + s) v_2 - \xi r s v_1 (r^{(1)} v_1 + v_2) \end{pmatrix}.$$

Therefore, the integrable coupling of the generalized Toda lattice equation (3.11), defined by (3.19), reads as

$$r_{t_1} = r(s^{(-1)} - s) + r(r^{(-1)} - r^{(1)}),$$

$$s_{t_1} = s(r - r^{(1)}),$$

$$v_{1,t_1} = -(r^{(1)} + r + s)v_1 - \xi r v_1(r^{(1)}v_1 + v_2),$$

$$(3.27)$$

$$v_{2,t_1} = -r^{(1)}sv_1 - (r^{(1)} + s)v_2 - \xi r s v_1(r^{(1)}v_1 + v_2),$$

the second subsystem of which is nonlinear with respect to both subsets of dependent variables when $\xi \neq 0$.

Case of $\alpha = 0$ and $\beta = 1$: Let us second compute an example of the hierarchy (3.26). Assume that $\alpha = 0$ and $\beta = 1$ for convenience, which corresponds to the Toda lattice hierarchy. In this case, we have $n_0 = 1$ in (2.14).

We take the initial set of functions as follows:

$$f_0 = g_0 = 0, \quad e_0 = -\tfrac{1}{2}.$$

Obviously from (3.22), we can have

$$f_{i+1}^{(1)} = -sf_i^{(1)} - (e_i^{(1)} + e_i) - (a_i^{(1)} + a_i)v_2 - b_i^{(1)}v_4 + b_i v_1,$$

$$g_{i+1} = -sg_i - r(e_i^{(1)} + e_i) - (a_i^{(1)} + a_i)v_3 + c_i^{(1)}v_1 - c_i v_4,$$

$$e_{i+1}^{(1)} - e_{i+1} = -s(e_i^{(1)} - e_i) + g_i^{(1)} - rf_i - (a_i^{(1)} - a_i)v_4 + c_i^{(1)}v_2 - b_i v_3,$$

where $i \geq 0$. It then follows that

$$f_1 = 1 + v_2^{(-1)}, \quad g_1 = r + v_3, \quad e_1 = v_3 + r(1 + v_2^{(-1)}),$$

$$f_2 = -(s^{(-1)} + r)(1 + v_2^{(-1)}) - v_3 - v_3^{(-1)} - r^{(-1)}(1 + v_2^{(-2)}) - v_4^{(-1)} + v_1^{(-1)},$$

$$g_2 = -s(r + v_3) - r[v_3^{(1)} + v_3 + r^{(1)}(1 + v_2) + r(1 + v_2^{(-1)})] + r^{(1)}v_1 - r v_4,$$

$$e_2^{(1)} - e_2 = (1 - s)(r^{(1)} - r + v_3^{(1)} - v_3 + r^{(1)}v_2 - r v_2^{(-1)}).$$

We can use the inverse formula (1.4) to compute e_2 here, but as we will see, this is not necessary for computing the corresponding integrable coupling.

Now if we choose

$$h_1 = \eta v_1 e_1 = \eta v_1[v_3 + r(1 + v_2^{(-1)})], \quad \eta = \text{const},$$

then the vector-valued function T_1 defined by (3.25) becomes

$$T_1(u,v) = \begin{pmatrix} -(s - s^{(-1)})v_1, \\ -sv_2 - (s + r^{(1)})(1 + v_2) - r(1 + v_2^{(-1)}) + v_1 - v_3^{(1)} \\ - v_3 - v_4 + \eta v_1^{(1)}[v_3^{(1)} + r^{(1)}(1 + v_2)], \\ (s^{(-1)} + s)v_3 + rs + r[v_3^{(1)} + v_3 + r^{(1)}(1 + v_2) + r(1 + v_2^{(-1)})] \\ - r^{(1)}v_1 + r v_4 - \eta r v_1[v_3 + r(1 + v_2^{(-1)})], \\ (s - 1)(r^{(1)} - r + v_3^{(1)} - v_3 + r^{(1)}v_2 - r v_2^{(-1)}) \end{pmatrix}.$$

Therefore, the integrable coupling of the Toda lattice equation (3.10), defined by (3.26), reads as

$$r_{t_1} = r(s^{(-1)} - s), \quad s_{t_1} = r^{(1)} - r,$$

$$v_{1,t_1} = -(s - s^{(-1)})v_1,$$

$$v_{2,t_1} = -sv_2 - (s + r^{(1)})(1 + v_2) - r(1 + v_2^{(-1)}) + v_1 - v_3^{(1)} - v_3 - v_4 + \eta v_1^{(1)}[v_3^{(1)} + r^{(1)}(1 + v_2)],$$

$$\text{(3.28)}$$

$$v_{3,t_1} = (s^{(-1)} + s)v_3 + rs + r[v_3^{(1)} + v_3 + r^{(1)}(1 + v_2) + r(1 + v_2^{(-1)})]$$

$$- r^{(1)}v_1 + rv_4 - \eta r v_1[v_3 + r(1 + v_2^{(-1)})],$$

$$v_{4,t_1} = (s - 1)(r^{(1)} - r + v_3^{(1)} - v_3 + r^{(1)}v_2 - rv_2^{(-1)}),$$

the second subsystem of which is nonlinear with respect to both subsets of dependent variables when $\eta \neq 0$.

IV. CONCLUSIONS AND REMARKS

A feasible approach to construct integrable couplings of discrete soliton equations has been proposed by taking advantage of semidirect sums of Lie algebras, and the resulting theory has been applied to the generalized Toda hierarchy of lattice equations to generate integrable couplings for the hierarchy. The key point in our generating scheme is to establish a relation between semidirect sums of Lie algebras and integrable couplings of discrete soliton equations. The underlying discrete matrix spectral problems are generated from semidirect sums of Lie algebras, and the discrete Lax spectral matrices associated with given soliton equations play the nonideal part in the semidirect sums.

In our analysis of the two specific semidirect sums, we have seen that there is always an arbitrary modified term $\Delta_{n,a}$. This indicates that higher order matrix spectral problems have more degrees of freedom in generating integrable systems. On the other hand, in all additional spectral submatrices such as U_{a_i}, one can take their dependence on the spectral parameter into consideration, and this will bring much more diverse integrable couplings.

There are also other questions about integrable properties of the resulting enlarged lattice equations, even in the case where additional spectral submatrices are independent of the spectral parameter. For example, can we solve the enlarged lattice equations by the inverse scattering transform? The class of Lie algebras in (2.8) provides a few realizations of semidirect sums of Lie algebras. Other possible realizations are still interesting, especially those which could carry significant information about integrable properties. Reductions of the presented cases of semidirect sums, which keep the uniqueness property of discrete spectral problems (see Ref. 17), could be good examples.

We would especially like to emphasize that we have been considering the problem of integrable couplings and the key is semidirect sums of Lie algebras. The initial Lie algebras G associated with given integrable systems in our construction can be simple (e.g., see Refs. 29 and 30), but semidirect sums of Lie algebras $\bar{G}$ are normally nonsimple (see Ref. 15). Our examples in Sec. II are all nonsimple, since the Killing forms on those semidirect sums of Lie algebras $\bar{G}$ are degenerate. However, there still exist specific nondegenerate bilinear forms on the Lie algebras $\bar{G}$, with nice invariance properties, and their corresponding generalized trace identities, which present Hamiltonian structures of the enlarged lattice equations.

To conclude, semidirect sums of Lie algebras provide a good source of matrix spectral problems for generating integrable systems, and thus the study of integrable couplings using semidirect sums of Lie algebras will enhance our understanding of classification of integrable systems. We are expecting to see more research on related topics.

ACKNOWLEDGMENTS

[1] W. X. Ma and B. Fuchssteiner, Chaos, Solitons Fractals **7**, 1227 (1996).
[2] W. X. Ma, Methods Appl. Anal. **7**, 21 (2000).
[3] W. X. Ma and B. Fuchssteiner, Phys. Lett. A **213**, 49 (1996).
[4] W. X. Ma, Phys. Lett. A **316**, 72 (2003).
[5] W. X. Ma, J. Math. Phys. **46**, 033507 (2005).
[6] F. G. Guo and Y. Zhang, J. Math. Phys. **44**, 5793 (2003).
[7] Y. Zhang, Chaos, Solitons Fractals **21**, 305 (2004).
[8] L. Frappat, A. Sciarrino, and P. Sorba, *Dictionary on Lie Algebras and Superalgebras* (Academic, San Diego, CA, 2000).
[9] S. Yu. Sakovich, J. Nonlinear Math. Phys. **5**, 230 (1998).
[10] S. Yu. Sakovich, J. Nonlinear Math. Phys. **6**, 255 (1999).
[11] W. X. Ma and R. G. Zhou, Physica A **296**, 60 (2001).
[12] W. X. Ma and R. G. Zhou, Chaos, Solitons Fractals **13**, 1451 (2002).
[13] W. X. Ma, J. Math. Phys. **43**, 1408 (2002).
[14] E. G. Fan and Y. Zhang, Chaos, Solitons Fractals **25**, 425 (2005).
[15] W. X. Ma, X. X. Xu, and Y. Zhang, Phys. Lett. A **351**, 125 (2006).
[16] G. Z. Tu, J. Phys. A **23**, 3903 (1990).
[17] W. X. Ma and B. Fuchssteiner, J. Math. Phys. **40**, 2400 (1999).
[18] M. J. Ablowitz and J. F. Ladik, J. Math. Phys. **16**, 598 (1975).
[19] W. X. Ma and X. X. Xu, J. Phys. A **37**, 1323 (2004).
[20] H. W. Tam and Y. Zhang, Chaos, Solitons Fractals **23**, 151 (2005).
[21] H. X. Yang, X. X. Xu, and H. Y. Ding, Chaos, Solitons Fractals **26**, 1091 (2005).
[22] T. Tsuchida, H. Ujino, and M. Wadati, J. Math. Phys. **39**, 4785 (1998).
[23] T. Tsuchida, H. Ujino, and M. Wadati, J. Phys. A **32**, 2239 (1999).
[24] D. J. Zhang and D. Y. Chen, Chaos, Solitons Fractals **14**, 573 (2002).
[25] M. Toda, *Theory of Nonlinear Lattices*, 2nd ed., Springer Series in Solid-State Sciences 20 (Springer, Berlin, 1989).
[26] W. X. Ma and X. X. Xu, Int. J. Theor. Phys. **43**, 219 (2004).
[27] S. V. Manakov, Sov. Phys. JETP **40**, 269 (1975).
[28] H. Flaschka, Phys. Rev. B **9**, 1924 (1974).
[29] O. I. Bogoyavlensky, Commun. Math. Phys. **51**, 201 (1976).
[30] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71**, 313 (1981).
[31] W. X. Ma and Y. You, Chaos, Solitons Fractals **22**, 395 (2004).
[32] W. X. Ma and K. Maruno, Physica A **343**, 219 (2004).