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A Generalization of the Wadati-Konno-Ichikawa Soliton Hierarchy and its Liouville Integrability

Abstract: A new matrix spectral problem associated with $\mathrm{sl}(2,\mathbb{R})$, which generalizes the Wadati-Konno-Ichikawa spectral problem, is introduced, and the corresponding hierarchy of soliton equations is generated from the associated zero curvature equations. A bi-Hamiltonian structure of the resulting generalized soliton hierarchy is furnished by using the trace identity, and thus, every system in the generalized hierarchy is Liouville integrable.

Keywords: matrix spectral problem, zero curvature equation, Hamiltonian structure

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1 Introduction

Liouville integrability of partial differential equations is one of new and active research topics [1]. There are concrete examples of integrable equations which are generated from matrix spectral problems or Lax pairs associated with matrix Lie algebras (see, e.g., [2–4]). Existence of recursion operators [5] bring a hierarchy of integrable equations possessing the same integrable properties, called soliton hierarchies. Among celebrated soliotn hierarchies with dependent variables less than three are the Korteweg-de Vries hierarchy [6], the Ablowitz-Kaup-Newell-Segur hierarchy [7], the Dirac hierarchy [8], the Kaup-Newell hierarchy [9] and the Wadati-Konno-Ichikawa hierarchy [10]. They often possess bi-Hamiltonian structures [11], which show Liouville integrability. The associated Hamiltonian structures can be furnished by the variational identity when the underlying matrix Lie algebras are non-semisimple [12, 13], which reduces to the trace identity in the case of semisimple matrix Lie algebras [14].

Let us briefly describe the procedure for constructing soliton hierarchies by zero curvature equations (see, e.g., [14, 15]). The beginning is to take a matrix loop algebra $\tilde{\mathfrak{g}}$, associated with a matrix Lie algebra $\mathfrak{g}$ with the commutator

$$[A,B]=AB-BA,\quad A,B\in\mathfrak{g},\tag{1.1}$$

and introduce a spatial matrix spectral problem

$$\phi_x=U\phi,\quad U=U(u,\lambda)\in\tilde{\mathfrak{g}},\tag{1.2}$$

where u stands for a column vector of dependent variables, and λ, the spectral parameter (see, e.g., [16]). We then look for a solution

$$W=W(u,\lambda)=\sum_{i\geq 0}W_{0,i}\lambda^{-i},\quad W_{0,i}\in\mathfrak{g},\ i\geq 0,\tag{1.3}$$

to the stationary zero curvature equation

$$W_x=[U,W].\tag{1.4}$$

Further, we try to formulate the temporal matrix spectral problems

$$\phi_{t_m}=V^{[m]}\phi=V^{[m]}(u,\lambda)\phi,\quad m\geq 0,\tag{1.5}$$

by introducing the Lax matrices

$$V^{[m]}=V^{[m]}(u,\lambda)=(\lambda^mW)_++\Delta_m\in\tilde{\mathfrak{g}},\quad m\geq 0,$$

P_+ denoting the polynomial part of P in λ, such that the compatibility conditions of (1.2) and (1.5), i.e., the zero curvature equations

$$U_{t_m}-V_x^{[m]}+[U,V^{[m]}]=0,\quad m\geq 0,\tag{1.6}$$

will engender a hierarchy of soliton equations:

$$u_{t_m}=K_m(u),\quad m\geq 0.\tag{1.7}$$

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Such a soliton hierarchy usually possesses a recursion operator and Hamiltonian structures

$$u_{t_m} = K_m(u) = \Phi^m K_0 = J\frac{\delta \mathscr{H}_m}{\delta u}, \quad m \geq 0. \tag{1.8}$$

The recursion operator Φ is generated from the stationary zero curvature equation (1.4) and the Hamiltonian functionals $\mathscr{H}_m$'s can often be computed through the trace identity:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(\frac{\partial U}{\partial \lambda}W)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}(\frac{\partial U}{\partial u}W),$$
$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{1.9}$$

or generally, the variational identity:

$$\frac{\delta}{\delta u}\int \langle\frac{\partial U}{\partial \lambda}, W\rangle\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle\frac{\partial U}{\partial u}, W\rangle,$$
$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{1.10}$$

where $\langle\cdot,\cdot\rangle$ is a bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$, satisfying three conditions: the non-degenerate, symmetric and ad-invariant conditions.

We will make use of the three-dimensional special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$, consisting of trace-free 2×2 matrices. This Lie algebra is simple and has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{1.11}$$

whose commutator relations read

$$[e_1, e_2] = 2e_1, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1.$$

Its derived algebra is itself, and hence it is 3-dimensional as well. The matrix loop algebra we will adopt in what follows is

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \left\{\sum_{i\geq 0} M_i\lambda^{n-i} \,|\, M_i \in \mathrm{sl}(2,\mathbb{R}), i\geq 0 \text{ and } n \in \mathbb{Z}\right\}, \tag{1.12}$$

which is the space of all Laurent series in λ with coefficients in $\mathrm{sl}(2,\mathbb{R})$ and a finite regular part. The matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ contains matrices of the form

$$\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$$

with m, n, l being arbitrary integers. This loop algebra lays a foundation for our study of soliton equations, from which many well-known soliton hierarchies are generated (see, e.g., [6–10]).

In this paper, we would like to introduce a generalization of the Wadati-Konno-Ichikawa spectral problem, which is associated with the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$, and compute an associated hierarchy of bi-Hamiltonian soliton equations by zero curvature equations. The corresponding Hamiltonian structures will be furnished by applying the trace identity, and thus all soliton equations in the resulting generalized soliton hierarchy are Liouville integrable. The resulting hierarchy provides a new example of soliton hierarchies supplementing the existing ones associated with $\mathrm{sl}(2,\mathbb{R})$ in the literature (see, e.g., [17, 18] for examples). A few concluding remarks and interesting questions will round off the paper.

2 A generalized Wadati-Konno-Ichikawa soliton hierarchy

This section aims to present a generalized Wadati-Konno-Ichikawa soliton hierarchy associated with the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ defined in (1.12).

Let α be an arbitrarily given constant. We introduce a new spectral matrix U as

$$U = U(u,\lambda) = (\lambda + \alpha q)e_1 + \lambda p e_2 + \lambda q e_3$$
$$= \begin{bmatrix} \lambda + \alpha q & \lambda p \\ \lambda q & -\lambda - \alpha q \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{2.1}$$

to formulate a matrix spectral problem:

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.2}$$

where λ is the spectral parameter. A special case with $\alpha = 0$ reduces to the Wadati-Konno-Ichikawa matrix spectral problem in [10].

First, we solve the stationary zero curvature equation (1.4). The equation (1.4) becomes

$$\begin{cases} \lambda a_x + \alpha[(qa)_x + \lambda^{-1}c_{xx}] = \lambda(pc_x - qb_x), \\ \lambda(pa)_x + b_{xx} = 2\lambda b_x - 2\alpha(pc_x - qb_x), \\ \lambda(qa)_x + c_{xx} = -2\lambda c_x, \end{cases} \tag{2.3}$$

if W is particularly taken as follows:

$$W = aU + \alpha\lambda^{-1}c_x e_1 + b_x e_2 + c_x e_3$$
$$= \begin{bmatrix} \lambda a + \alpha q a + \alpha\lambda^{-1}c_x & \lambda p a + b_x \\ \lambda q a + c_x & -\lambda a - \alpha q a - \alpha\lambda^{-1}c_x \end{bmatrix}$$
$$\in \widetilde{\mathrm{sl}}(2,\mathbb{R}). \tag{2.4}$$

Obviously, the system (2.3) is equivalent to

$$\begin{cases} \lambda a_x - 2\alpha c_x = \lambda(pc_x - qb_x), \\ \lambda(pa)_x + b_{xx} = 2\lambda b_x - 2\alpha(pc_x - qb_x), \\ \lambda(qa)_x + c_{xx} = -2\lambda c_x. \end{cases} \quad (2.5)$$

Upon letting

$$a = \sum_{i\geq 0} a_i\lambda^{-i}, \quad b = \sum_{i\geq 0} b_i\lambda^{-i}, \quad c = \sum_{i\geq 0} c_i\lambda^{-i}, \quad (2.6)$$

and taking the initial values

$$a_0 = \frac{1}{\sqrt{pq+1}}, \quad b_0 = \frac{p}{2\sqrt{pq+1}}, \quad c_0 = -\frac{q}{2\sqrt{pq+1}}, \quad (2.7)$$

which are required by the equations on the first powers of λ in (2.5):

$$a_{0,x} = pc_{0,x} - qb_{0,x}, \quad (pa_0)_x = 2b_{0,x}, \quad (qa_0)_x = -2c_{0,x},$$

the system (2.5) equivalently yields

$$\begin{bmatrix} -c_{i+1} \\ b_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} -c_i \\ b_i \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \quad i \geq 0, \quad (2.8)$$

where

$$\begin{cases} \Psi_{11} = -\frac{1}{2}\partial + \frac{1}{4}\bar{q}\partial^{-1}\bar{p}\partial^2 - \alpha\bar{q}\partial^{-1}\frac{1}{\sqrt{pq+1}}\partial \\ \qquad + \frac{1}{2}\alpha\bar{q}\partial^{-1}p\bar{q}\partial, \\ \Psi_{12} = -\frac{1}{4}\bar{q}\partial^{-1}\bar{q}\partial^2 + \frac{1}{2}\alpha\bar{q}\partial^{-1}q\bar{q}\partial, \\ \Psi_{21} = \frac{1}{4}\bar{p}\partial^{-1}\bar{p}\partial^2 - \alpha\bar{p}\partial^{-1}\frac{1}{\sqrt{pq+1}}\partial \\ \qquad + \frac{1}{2}\alpha\bar{p}\partial^{-1}p\bar{q}\partial - \alpha\partial^{-1}p\partial, \\ \Psi_{22} = \frac{1}{2}\partial - \frac{1}{4}\bar{p}\partial^{-1}\bar{q}\partial^2 + \frac{1}{2}\alpha\bar{p}\partial^{-1}q\bar{q}\partial - \alpha\partial^{-1}q\partial, \end{cases} \quad (2.9)$$

with

$$\bar{p} = \frac{p}{\sqrt{pq+1}}, \quad \bar{q} = \frac{q}{\sqrt{pq+1}}, \quad (2.10)$$

and

$$a_{i+1,x} = pc_{i+1,x} - qb_{i+1,x} + 2\alpha c_{i,x}, \quad i \geq 0. \quad (2.11)$$

Note that all vectors $(-c_i, b_i)^T$, $i \geq 0$, are gradient (see the next section for a proof), and so, the adjoint operator of Ψ, $\Phi = \Psi^\dagger$, will provide a candidate of hereditary operators (see [19] for definition).

Let us show how to derive the recursion relation (2.8). First, from the second and third equations of (2.5), we have

$$\begin{aligned} qb_{i,x} - pc_{i,x} &= \frac{1}{2}q(pa_i)_x + \frac{1}{2}qb_{i-1,xx} + \alpha q(pc_{i-1,x} - qb_{i-1,x}) \\ &\quad + \frac{1}{2}p(qa_i)_x + \frac{1}{2}pc_{i-1,xx} \\ &= \frac{1}{2}(pq)_x a_i + pqa_{i,x} + \frac{1}{2}qb_{i-1,xx} + \frac{1}{2}pc_{i-1,xx} \\ &\quad + \alpha q(pc_{i-1,x} - qb_{i-1,x}), \quad i \geq 1. \end{aligned}$$

Then from the first equation of (2.5), we have

$$\begin{aligned} \sqrt{pq+1}(\sqrt{pq+1}\,a_i)_x &= (pq+1)a_{i,x} + \frac{1}{2}(pq)_x a_i \\ &= -\frac{1}{2}qb_{i-1,xx} - \frac{1}{2}pc_{i-1,xx} \\ &\quad - \alpha q(pc_{i-1,x} - qb_{i-1,x}) \\ &\quad + 2\alpha c_{i-1,x}, \quad i \geq 1, \end{aligned}$$

and thus

$$\begin{aligned} a_i = \frac{1}{\sqrt{pq+1}}[&-\frac{1}{2}\partial^{-1}\bar{q}\partial^2 b_{i-1} - \frac{1}{2}\partial^{-1}\bar{p}\partial^2 c_{i-1} \\ &+ 2\alpha\partial^{-1}\frac{1}{\sqrt{pq+1}}\partial c_{i-1} \\ &- \alpha\partial^{-1}\bar{q}(p\partial c_{i-1} - q\partial b_{i-1})], \quad i \geq 1. \end{aligned}$$

Based on this, the recursion relation (2.8) finally follows directly from the second and third equations of (2.5).

While using the above recursion relations (2.8) and (2.11), we impose the following conditions on constants of integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \quad (2.12)$$

to determine the sequence of $\{a_i, b_i, c_i \,|\, i \geq 1\}$ uniquely. This way, the first set can be computed as follows:

$$\begin{aligned} a_1 &= \frac{pq_x - qp_x}{4(pq+1)^{3/2}} - \frac{\alpha q}{(pq+1)^{3/2}}, \\ b_1 &= \frac{p_x}{4(pq+1)^{3/2}} - \frac{\alpha pq}{2(pq+1)^{3/2}} + \frac{\alpha}{\sqrt{pq+1}}, \\ c_1 &= \frac{q_x}{4(pq+1)^{3/2}} + \frac{\alpha q^2}{2(pq+1)^{3/2}}. \end{aligned}$$

We point out that the localness of the sequence of $\{a_i, b_i, c_i \,|\, i \geq 1\}$ can be shown by the mathematical induction, based on (2.5) and

$$\begin{aligned} &(1+pq)a^2\lambda^4 + (qab_x + pac_x + 2\alpha qa^2)\lambda^3 \\ &\quad + (b_xc_x + 2\alpha ac_x + \alpha^2q^2a^2)\lambda^2 + 2\alpha^2 qac_x\lambda + \alpha^2c_x^2 = \lambda^4. \end{aligned}$$

The left hand side of the above equality is $\frac{1}{2}\lambda^2\operatorname{tr}(W^2)$, where W is defined by (2.4). This quantity doesn't depend on x, due to the stationary zero curvature equation (1.4); and thus, it is equal to λ^4, based on the initial data (2.7) and the zero constants of integration (2.12).

Now, observing the recursion relations in (2.8) and (2.11) and the structure of the spectral matrix U in (2.1), we introduce

$$W^{[m]} = \lambda[(\lambda^m a)_+ U + \alpha\lambda^{-1}(\lambda^m c_x)_+ e_1 + (\lambda^m b_x)_+ e_2 + (\lambda^m c_x)_+ e_3], \quad m \geq 0, \tag{2.13}$$

and consequently,

$$W_x^{[m]} - [U, W^{[m]}] = \begin{bmatrix} \alpha(-2\lambda c_{m,x} + c_{m,xx}) & \lambda b_{m,xx} + 2\alpha\lambda(pc_{m,x} - qb_{m,x}) \\ \lambda c_{m,xx} & -\alpha(-2\lambda c_{m,x} + c_{m,xx}) \end{bmatrix}.$$

This matrix is not of the same form as the Gateaux derivative operator U'. To make a matrix of the same form as U', we take a modification choice

$$\Delta_m = 2\alpha c_m U, \quad m \geq 0, \tag{2.14}$$

and then we find that the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \tag{2.15}$$

with the Lax matrices being defined by

$$V^{[m]} = W^{[m]} + \Delta_m, \quad m \geq 0, \tag{2.16}$$

engender a hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,xx} + 2\alpha(pc_{m,x} - qb_{m,x}) + 2\alpha(pc_m)_x \\ c_{m,xx} + 2\alpha(qc_m)_x \end{bmatrix}, \quad m \geq 0, \tag{2.17}$$

which are all local, because of the localness of the sequence of $\{a_i, b_i, c_i \,|\, i \geq 1\}$. The first nonlinear system in the soliton hierarchy (2.17) reads

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} \left(\frac{p}{2\sqrt{pq+1}}\right)_{xx} - \alpha\frac{(pq)_x}{(pq+1)^{3/2}} + \alpha\left(\frac{pq}{\sqrt{pq+1}}\right)_x \\ -\left(\frac{q}{2\sqrt{pq+1}}\right)_{xx} - \alpha\left(\frac{q^2}{\sqrt{pq+1}}\right)_x \end{bmatrix}. \tag{2.18}$$

3 Hamiltonian structures and Liouville integrability

3.1 Application of the trace identity

To furnish Hamiltonian structures, we use the trace identity (1.9) (or generally, the variational identity (1.10)). It is direct to compute

$$\frac{\partial U}{\partial\lambda} = \begin{bmatrix} 1 & p \\ q & -1 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} \alpha & 0 \\ \lambda & -\alpha \end{bmatrix},$$

and thus, we obtain

$$\begin{aligned} \operatorname{tr}(W\frac{\partial U}{\partial\lambda}) &= 2\lambda(pq+1)a + pc_x + qb_x + 2\alpha(qa + \lambda^{-1}c_x) \\ &= 2\lambda(pq+1)a + pc_x + qb_x - 4\alpha c, \\ \operatorname{tr}(W\frac{\partial U}{\partial p}) &= \lambda(\lambda qa + c_x) = -2\lambda^2 c, \\ \operatorname{tr}(W\frac{\partial U}{\partial q}) &= 2\alpha(\lambda a + \alpha qa + \alpha\lambda^{-1}c_x) + \lambda^2 pa + \lambda b_x = 2\lambda^2 b. \end{aligned}$$

Then, an application of the trace identity (1.9) to this case tells

$$\frac{\delta}{\delta u}\int[\lambda(pq+1)a + \frac{1}{2}pb_x + \frac{1}{2}qc_x - 2\alpha c]\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix} -\lambda^2 c \\ \lambda^2 b \end{bmatrix}.$$

Upon balancing coefficients of all powers of λ in the equality, we get

$$\frac{\delta}{\delta u}\int(pq+1)a_0\,dx = (\gamma+2)\begin{bmatrix} -c_0 \\ b_0 \end{bmatrix},$$

and

$$\frac{\delta}{\delta u}\int[(pq+1)a_m + \frac{1}{2}pb_{m-1,x} + \frac{1}{2}qc_{m-1,x} - 2\alpha c_{m-1}]\,dx = (\gamma - m + 2)\begin{bmatrix} -c_m \\ b_m \end{bmatrix}, \quad m \geq 1.$$

The first identity yields $\gamma = -1$, and so, we arrive at

$$\frac{\delta}{\delta u}\mathscr{H}_m = \begin{bmatrix} -c_m \\ b_m \end{bmatrix}, \quad m \geq 0, \tag{3.1}$$

where

$$\mathscr{H}_0 = \int \sqrt{pq+1}\, dx, \tag{3.2a}$$

$$\mathscr{H}_1 = \int [\frac{qp_x - pq_x}{4\sqrt{pq+1}(\sqrt{pq+1}+1)} + \frac{\alpha q}{\sqrt{pq+1}}]\, dx, \tag{3.2b}$$

$$\mathscr{H}_{m+1} = \int [-\frac{2(pq+1)a_{m+1} + pb_{m,x} + qc_{m,x} - 4\alpha c_m}{2m}]\, dx, \quad m \geq 1. \tag{3.3}$$

The functional $\mathscr{H}_1$ above was determined directly from $(-c_1, b_1)^T$.

Now, it follows that the soliton hierarchy (2.17) has the Hamiltonian structures:

$$u_{t_m} = K_m = J \begin{bmatrix} -c_m \\ b_m \end{bmatrix} = J \frac{\delta \mathscr{H}_m}{\delta u}, \quad m \geq 0, \tag{3.4}$$

where the Hamiltonian operator is defined by

$$J = \begin{bmatrix} -2\alpha(p\partial + \partial p) & \partial^2 - 2\alpha q \partial \\ -\partial^2 - 2\alpha \partial q & 0 \end{bmatrix}, \tag{3.5}$$

and the Hamiltonian functionals $\mathscr{H}_m$'s are given by (3.2) and (3.3).

The presented functionals $\mathscr{H}_m$, $m \geq 1$, can generate infinitely many conservation laws with commuting conserved densities for each soliton system in the whole generalized soliton hierarchy (2.17). We point out that differential polynomial type conservation laws can be directly computed by computer algebra systems (see, e.g., [20]) or from certain Riccati equation inherited from the underlying matrix spectral problem (see, e.g., [21–24]).

3.2 Recursion operator and Hamiltonian pair

It is direct to show that J defined by (3.5) and

$$M = \Phi J \tag{3.6}$$

where

$$\Phi = \Psi^\dagger = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \quad i \geq 0, \tag{3.7}$$

with the entries of Φ being defined by

$$\begin{cases} \Phi_{11} = \frac{1}{2}\partial - \frac{1}{4}\partial^2 \bar{p} \partial^{-1} \bar{q} - \alpha \partial \frac{1}{\sqrt{pq+1}} \partial^{-1} \bar{q} \\ \qquad + \frac{1}{2}\alpha \partial p \bar{q} \partial^{-1} \bar{q}, \\ \Phi_{12} = -\frac{1}{4}\partial^2 \bar{p} \partial^{-1} \bar{p} - \alpha \partial \frac{1}{\sqrt{pq+1}} \partial^{-1} \bar{p} \\ \qquad + \frac{1}{2}\alpha \partial p \bar{q} \partial^{-1} \bar{p} - \alpha \partial p \partial^{-1}, \\ \Phi_{21} = \frac{1}{4}\partial^2 \bar{q} \partial^{-1} \bar{q} + \frac{1}{2}\alpha \partial q \bar{q} \partial^{-1} \bar{q}, \\ \Phi_{22} = -\frac{1}{2}\partial + \frac{1}{4}\partial^2 \bar{q} \partial^{-1} \bar{p} + \frac{1}{2}\alpha \partial q \bar{q} \partial^{-1} \bar{p} - \alpha \partial q \partial^{-1}, \end{cases} \tag{3.8}$$

constitute a Hamiltonian pair (see [11, 25] for details), i.e., any linear combination N of J and M is skew-symmetric,

$$\int K^T (NS)\, dx = -\int S^T (NK)\, dx \tag{3.9}$$

for all vector fields K and S, and satisfies the Jacobi identity

$$\begin{aligned} &\int K^T N'(u)[NS] T\, dx + \text{cycle}(K, S, T) \\ &= \int K^T N'(u)[NS] T\, dx + \int S^T N'(u)[NT] K\, dx \\ &+ \int T^T N'(u)[NK] S\, dx = 0 \end{aligned} \tag{3.10}$$

for all vector fields K, S and T. It follows then that the operator $\Phi = \Psi^\dagger$ is hereditary (see [19] for definition), i.e., it satisfies

$$\begin{aligned} &\Phi'(u)[\Phi K] S - \Phi \Phi'(u)[K] S \\ &= \Phi'(u)[\Phi S] K - \Phi \Phi'(u)[S] K \end{aligned} \tag{3.11}$$

for all vector fields K and S.

We know that the Lie derivative $L_K \Phi$ is defined by

$$(L_K \Phi) S = \Phi[K, S] - [K, \Phi S],$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields:

$$[K, S] = K'(u)[S] - S'(u)[K]. \tag{3.12}$$

On one hand, the condition (3.11) is equivalent to

$$L_{\Phi K} \Phi = \Phi L_K \Phi \tag{3.13}$$

where K is an arbitrary vector field. On the other hand, an operator $\Phi = \Phi(u, u_x, \ldots)$ which doesn't depend on t explicitly is a recursion operator of a given evolution equation $u_t = K = K(u)$ if and only if $L_K \Phi = 0$. Therefore, due to

$$L_{K_0} \Phi = 0, \tag{3.14}$$

where K_0 is defined as in (2.18), we see that

$$L_{K_m}\Phi = L_{\Phi K_{m-1}}\Phi = \Phi L_{K_{m-1}}\Phi = 0, \quad m \ge 1, \quad (3.15)$$

where the K_m's are defined by (2.17). This implies that the operator $\Phi = \Psi^\dagger$, defined by (3.7) and (3.8), is a common hereditary recursion operator for the whole generalized soliton hierarchy (2.17). We remark that various recursion operators can be found through Lax representations or by computer algebra systems for partial differential equations (see, e.g., [26, 27]), and that there exist direct computer algorithms for constructing symmetries of differential and/or differential-difference equations (see, e.g., [28]).

3.3 Liouville integrability

It now follows that except the first one, all members in the soliton hierarchy (2.17) are bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta \mathscr{H}_m}{\delta u} = M\frac{\delta \mathscr{H}_{m-1}}{\delta u}, \quad m \ge 1. \quad (3.16)$$

Therefore, noting distinct differential orders of K_m, $m \ge 1$, those bi-Hamiltonian structures show that the generalized soliton hierarchy (2.17) is Liouville integrable [29]. In particular, it possesses infinitely many commuting conserved functionals and symmetries:

$$\{\mathscr{H}_k, \mathscr{H}_l\}_J = \int \left(\frac{\delta \mathscr{H}_k}{\delta u}\right)^T J \frac{\delta \mathscr{H}_l}{\delta u}\,dx = 0, \quad k,l \ge 0, \quad (3.17)$$

$$\{\mathscr{H}_k, \mathscr{H}_l\}_M = \int \left(\frac{\delta \mathscr{H}_k}{\delta u}\right)^T M \frac{\delta \mathscr{H}_l}{\delta u}\,dx = 0, \quad k,l \ge 0, \quad (3.18)$$

and

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \quad k,l \ge 0. \quad (3.19)$$

4 Concluding remarks

Based on the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$, we introduced a generalization of the Wadati-Konno-Ichikawa spectral problem and generated a hierarchy of soliton equations from the associated zero curvature equations. The resulting generalized soliton hierarchy has been shown to be bi-Hamiltonian, which guarantees its Liouville integrability.

Recently, on one hand, the special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$ has been used to generate new soliton hierarchies [30, 31, 32]. Among typical discussed spectral matrices in $\mathrm{so}(3,\mathbb{R})$ are the following three:

$$U = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix},$$

which correspond to the AKNS spectral matrix, the Kaup-Newell spectral matrix and the Wadati-Konno-Ichikawa spectral matrix associated with $\mathrm{sl}(2,\mathbb{R})$, respectively. There are also many higher-order matrix spectral problems yielding soliton hierarchies (see, e.g., [33–39]).

On the other hand, there has been a growing interest in generating soliton hierarchies of integrable couplings [40] from matrix spectral problems associated with non-semisimple matrix loop algebras [41]. Non-semisimple matrix loop algebras are a diverse and rich field, which increases our motivations to study multi-component integrable systems [42]. Bi-integrable couplings and tri-integrable couplings do show various structures on recursion operators in block matrix form [13, 42]. It is very interesting to explore more algebraic and geometric mathematical structures on integrable couplings. Very recently, the irreducible representations of matrix algebras have been used to generate matrix loop algebras which lead to integrable couplings [16].

There are many interesting questions on integrable couplings. Let K' be the Gateaux derivative operator of $K = K(u)$. Does the bi-integrable coupling

$$u_t = K(u), \quad v_t = K'(u)[v], \quad w_t = K'(u)[w],$$

possess any Hamiltonian structure, when $u_t = K$ is Hamiltonian? How can we solve a Cauchy problem of the perturbation system

$$u_t = K(u), \quad v_t = K'(u)[v]?$$

The KdV case with $K(u) = 6uu_x + u_{xxx}$ gives

$$u_t = 6uu_x + u_{xxx}, \quad v_t = 6(uv)_x + v_{xxx}.$$

For u given, how to solve the second equation above for v? One can obtain special solutions to the perturbation system, among which are symmetries of $u_t = K(u)$. We expect to have a general theory to solve the linearized equations of given nonlinear equations.

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