East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.090920.180121

Vol. **11**, No. 4, pp. 674-685
November 2021

Lump and Interaction Solutions of the (2+1)-Dimensional bSK Equation

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Received 9 September 2020; Accepted (in revised version) 18 January 2021.

Abstract. In this paper, we study lump and interaction solutions of the (2+1)-dimensional bidirectional Sawada-Kotera equation. By using the Hirota bilinear method and considering a combined function with a positive quadratic function, we determine a few lump and interaction solutions of the bidirectional Sawada-Kotera equation. In order to ensure solvability of the corresponding solutions, five cases of necessary conditions for the parameters are explicitly presented. Dynamical properties of the resulting solutions are demonstrated, upon selecting appropriate values for the parameters.

AMS subject classifications: 35Q51, 35Q53, 35C99, 68W30, 74J35

Key words: Soliton, Hirota bilinear method, bidirectional Sawada-Kotera equation, lump solution, interaction solution.

1. Introduction

It is well known that the study on exact solutions of nonlinear equations plays a key role in nonlinear science. With the continuous advancement of research, solitons and important problems closely related to soliton theory have been found in many fields, such as physics, biology, medicine, oceanography, economics, and population problems. It is

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one of the most basic and important tasks in mathematics to find exact solutions of non-linear equations. In recent years, more and more scholars have paid attention to exact solutions, particularly to nonlinear differential equations. Indeed, many powerful methods have been developed, which include the inverse scattering approach [1], the Bäcklund transformation [29, 31], Darboux transformation [28], C-K direct method [3, 11], Hirota bilinear approach [7], the auxiliary equation method [15], and the long wave limit method. In the recent study of nonlinear problems, the Hirota bilinear method has become a popular and efficient method to generate lump solutions. Very recently, in soliton theory, lump solutions have been attracting more and more attention [13, 14]. Lump solutions are a kind of regular and rationally function solutions, localised in all directions in the space [19], which appear in many physical phenomena. Amazingly, lump and interaction solutions have been presented and analysed for many nonlinear equations of mathematical physics [2,6,8,9,12,16–18,20–23,26,27,30,32,34,36–40], even for linear wave equations — cf. [24], and with higher-order dispersion relations [25].

In this paper, we will study lump and interaction solutions and their dynamics for a (2+1)-dimensional bidirectional Sawada-Kotera (bSK) equation. The (2+1)-dimensional bSK equation reads [10]

$$\begin{aligned} &9u_y - \left(5u_{xt} + 15u^3 + 15uu_{xx} + u_{xxxx}\right)_x - 15uu_t \\ &\quad + 5(\partial x)^{-1}u_{tt} - 15u_x(\partial x)^{-1}u_t = 0, \end{aligned} \tag{1.1}$$

which is subordinate to the Kadomtsev-Petviashvili equation. Here, $(\partial x)^{-1} = (\partial/\partial x)^{-1}$ and u is a function of variables x, y and t. It was formulated as a bidirectional generalisation of the Sawada-Kotera (SK) equation [33]. Its bidirectionality and relationship with the SK equation was pointed out in [4]. In view of its connection with the SK equation, the bSK equation also belongs to the Kadomtsev-Petviashvili hierarchy of B-type (BKP hierarchy) [5], the first equation of which has been shown to possess lump solutions [35]. Therefore, it will be interesting to explore lump and interaction solutions to the (2+1)-dimensional bSK equation.

The structure of the rest of the paper is as follows. In Section 2, we present a Hirota bilinear form of the (2+1)-dimensional bSK equation by the Hirota bilinear method. Then, through analysis and symbolic computations with Maple, we compute exact solutions of the bSK equation. Some figures of the resulting lump and interaction solutions with appropriate choices of the involved parameters are made to exhibit energy distributions and dynamical properties. In Section 3, we give a brief conclusion to summarize our work.

2. Lump and Interaction Solutions for the (2+1)-Dimensional bSK Equation

2.1. Bilinear form for the bSK equation

Under the bilinear transformation

$$u = 2\partial_x^2 \ln f, \tag{2.1}$$

where $f(x,y,t)$ is unknown real function, the Eq. (1.1) can be transformed into the bilinear forms as follows:

$$\begin{aligned}B_{bSK}(f) :=& \left(9D_xD_y - 5D_x^3D_t - D_x^6 + 5D_t^2\right)f\cdot f\\ =& 2\Big[9(f_{xy}f - f_xf_y) - 5(f_{xxxt}f - f_{xxx}f_t + 3f_{xx}f_{xt} - 3f_xf_{xxt})\\ &- \left(f_{xxxxxx}f - 6f_{xxxxx}f_x + 15f_{xxxx}f_{xx} - 10f_{xxx}^2\right) + 5\left(f_{tt}f - f_t^2\right)\Big],\end{aligned} \tag{2.2}$$

where D_t, D_x, D_y are the bilinear derivative operators [7] defined by

$$D_x^mD_y^nD_t^qf\cdot g = (\partial_x - \partial_{x'})^m(\partial_y - \partial_{y'})^n(\partial_t - \partial_{t'})^q \times f(x,y,t)g(x',y',t')|_{x'=x,y'=y,t'=t},$$

where m,n and q are the positive integers, f is the function of x,y and t, and g is the function of the formal variables x', y' and t'.

2.2. Lump and interaction solutions for bSK equation

In this section, we make $f(x,y,t)$ as a combination of positive quadratic function with trigonometric function and exponential functions (The positive polynomial part in f contributes to the lump, the trigono-metric function part, to the breather, and the exponential function part, to the soliton), that is

$$\begin{aligned}f(x,y,t) =& g^2 + h^2 + a_9\cos(a_{10}x + a_{11}y + a_{12}t + a_{13})\\ &+ \exp\left(-(a_{14}x + a_{15}y + a_{16}t + a_{17})\right)\\ &+ a_{18}\exp(a_{14}x + a_{15}y + a_{16}t + a_{17}),\end{aligned} \tag{2.3}$$

where two linear wave variables are defined by

$$\begin{aligned}g &= a_1x + a_2y + a_3t + a_4,\\ h &= a_5x + a_6y + a_7t + a_8,\end{aligned}$$

where a_i, $i = 1,\dots,18$ are real numbers to be confirmed later. Substituting (2.3) into (2.2), then we obtain a group of equations for the parameters. Next, equating all the coefficients of independent variables to zeros, we obtain a set of parameters. After analysis and calculations, we get a series of relationships between parameters. The solutions are as follows.

Case 1. In order to get lump solution of bSK equation, we set $a_i = 0, i \geq 10$. After calculation, we can get the lump solutions as follows:

$$\begin{aligned}u(x,y,t) =& \frac{4a_5^2}{(-Gy + a_3t + a_4)^2 + (a_5x - Hy + Dt + a_8)^2 + a_9}\\ &- \frac{8a_5^2(a_5x - Hy + Dt + a_8)^2}{\left((-Gy + a_3t + a_4)^2 + (a_5x - Hy + Dt + a_8)^2 + a_9\right)^2},\end{aligned} \tag{2.4}$$

and

$$u(x,y,t)=\frac{4(a_1^2+a_5^2)}{(a_1x-Ey+a_3t+a_4)^2+(a_5x-Fy+a_7t+a_8)^2+C} -\frac{2\big(2(a_1x-Ey+a_3t+a_4)a_1+2(a_5x-Fy+a_7t+a_8)a_5\big)^2}{\big((a_1x-Ey+a_3t+a_4)^2+(a_5x-Fy+a_7t+a_8)^2+C\big)^2}, \tag{2.5}$$

where

$$\begin{aligned} &G=\frac{10a_3^3a_9}{27a_5^4},\quad H=\frac{5a_3^2\big(-9a_5^6+a_3^2a_9^2\big)}{81a_5^7},\quad D=\frac{a_3^2a_9}{3a_5^3},\\ &C=\frac{1}{(a_1a_7-a_3a_5)^2}\Big(3a_1^5a_3+3a_1^4a_5a_7+6a_1^3a_3a_5^2+4a_1^2a_3a_4a_7a_8\\ &\qquad\qquad +6a_1^2a_5^3a_7-2a_1a_3^2a_4a_5a_8\\ &\qquad\qquad +3a_1a_3a_5^4+2a_1a_4a_5a_7^2a_8+3a_5^5a_7\Big),\\ &E=\frac{5\big(a_1a_3^2-a_1a_7^2+2a_3a_5a_7\big)}{9a_1^2+9a_5^2},\quad F=\frac{5\big(2a_1a_3a_7-a_3^2a_5+a_5a_7^2\big)}{9a_1^2+9a_5^2}, \end{aligned}$$

and $a_1a_7-a_3a_5\neq 0$. Fig. 1 shows 3D plots and contour plot of these two solutions at $t=0$, respectively.

Case 2. In this case

$$\begin{aligned} &a_2=0,\quad a_3=0,\quad a_6=0,\quad a_7=0,\quad a_{10}=0,\\ &a_{11}=0,\quad a_{12}=0,\quad a_{15}=-a_{14}^5,\quad a_{16}=-a_{14}^3,\quad a_{18}=0,\\ &a_i=a_i,\quad i=1,4,5,8,9,13,14,17. \end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.3), we have

$$\begin{aligned} f(x,y,t)=&(a_1x+a_4)^2+(a_5x+a_8)^2+a_9\cos(a_{13})\\ &+\exp\big(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\big). \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.1), we can get the form of u

$$u(x,y,t)=\frac{2\big(2a_1^2+2a_5^2+a_{14}^2\exp\big(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\big)\big)}{(a_1x+a_4)^2+(a_5x+a_8)^2+a_9\cos(a_{13})+\exp\big(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\big)} -\frac{2\big(2(a_1x+a_4)a_1+2(a_5x+a_8)a_5-a_{14}\exp\big(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\big)\big)^2}{\big((a_1x+a_4)^2+(a_5x+a_8)^2+a_9\cos(a_{13})+\exp\big(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\big)\big)^2}.$$

In Fig. 2, we give two different plots to show the change of u. That is a soliton with a lump wave.

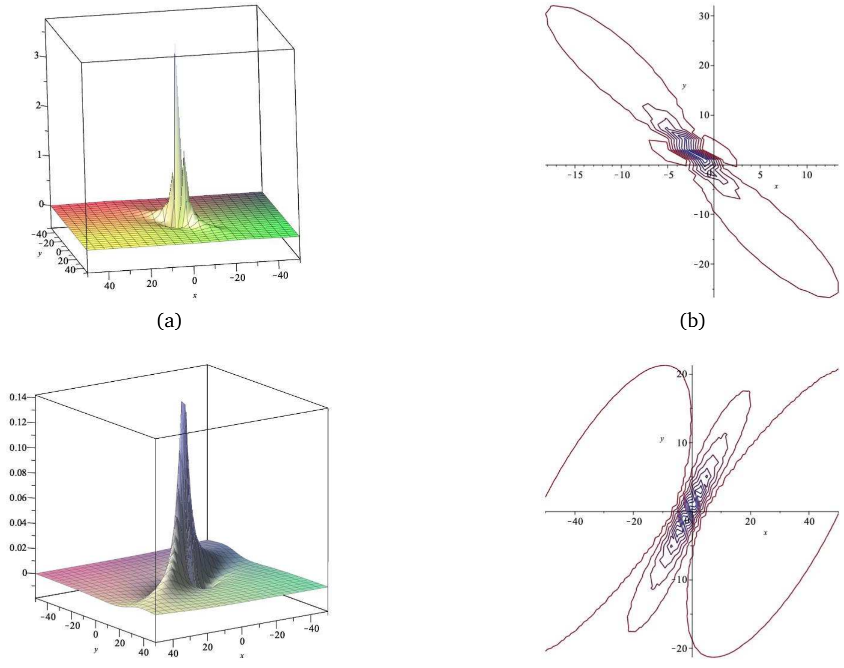


Figure 1: Spatiotemporal structure of solution (2.4) with the parameter selections $a_1 = 1$, $a_4 = 1$, $a_5 = 1$, $a_8 = 1$, $a_9 = 1$; (a) $t = 0$; (b) contour plots (top) $t = 0$. Spatiotemporal structure of solution (2.5) with the parameter selections $a_1 = 1$, $a_4 = 1$, $a_5 = 1$, $a_8 = 1$, $a_9 = 1$, $a_3 = 1$, $a_9 = 1$, (c) $t = 0$; (d) contour plots (top) $t = 0$.

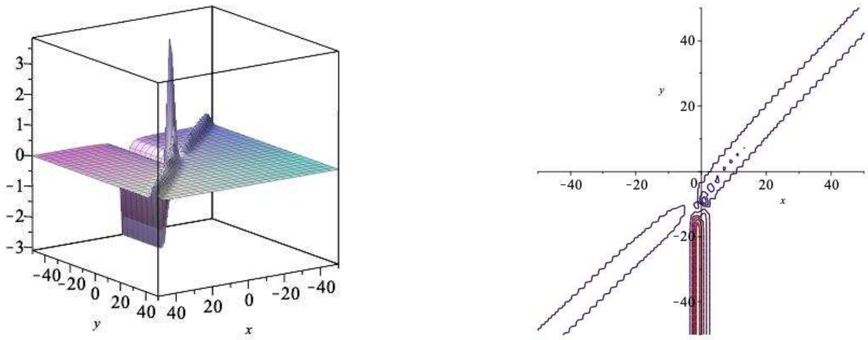
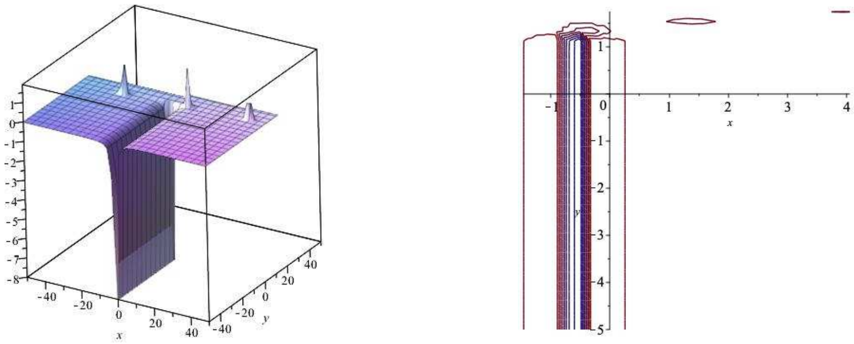


Figure 2: Spatiotemporal structure of solution u with the parameter selections $a_1 = 2$, $a_4 = 1$, $a_5 = 2$, $a_8 = 1$, $a_9 = 2$, $a_{13} = 1$, $a_{14} = 1$, $a_{17} = 2$; (a) $t = 15$; (b) contour plots (top) $t = 15$.

Furthermore, in next case, we show a special form in Case 2.

$$\begin{aligned} &a_2 = 0, \quad a_3 = 0, \quad && a_6 = 0, \quad && a_7 = 0, \\ &a_9 = 0, \quad a_{15} = -a_{14}^5, \quad && a_{16} = -a_{14}^3, \quad && a_{18} = 0, \\ &a_i = a_i, \quad i = 1,4,5,8,14,17. \end{aligned} \tag{2.8}$$



(a) (b)

Figure 3: Spatiotemporal structure of solution u with the parameter selections $a_1=1$, $a_4=1$, $a_5=2$, $a_8=1$, $a_{14}=2$, $a_{17}=3$, (a) $t=-5$; (b) contour plots (top) $t=-5$.

Substituting (2.8) into (2.3), we get the concrete form of the function f

$$f(x,y,t)=(a_1x+a_4)^2+(a_5x+a_8)^2+\exp\left(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\right). \tag{2.9}$$

Substituting (2.9) into (2.1), we obtain the exact solution of the Eq. (1.1). Then we have

$$\begin{aligned}u(x,y,t)=&\frac{2\left(2a_1^2+2a_5^2+a_{14}^2\exp\left(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\right)\right)}{(a_1x+a_4)^2+(a_5x+a_8)^2+\exp\left(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\right)}\\&-\frac{2\left(2(a_1x+a_4)a_1+2(a_5x+a_8)a_5-a_{14}\exp(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17})\right)^2}{\left((a_1x+a_4)^2+(a_5x+a_8)^2+\exp\left(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\right)\right)^2}.\end{aligned}$$

When the values of these parameters are changed, the structure of the lump solutions will also change accordingly. Here, the spatial variation of u is shown in Fig. 3. That are breathers with a rogue wave.

Case 3. In this case

$$\begin{aligned}&a_1=\frac{2}{3}\frac{a_7}{a_{14}^2}, \quad a_2=\frac{5}{6}a_{14}^2a_7, \quad a_3=-\frac{3}{2}a_{14}^2a_5, \quad a_6=\frac{5}{4}a_{14}^4a_5,\\&a_9=0, \qquad a_{15}=\frac{1}{4}a_{14}^5, \quad a_{16}=\frac{1}{2}a_{14}^3, \qquad a_{18}=0,\\&a_{14}\neq 0, \qquad a_i=a_i, \qquad i=4,5,7,8,14,17.\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.3), we get the quadratic function f

$$\begin{aligned}f(x,y,t)=&\left(\frac{2}{3}\frac{a_7x}{a_{14}^2}+\frac{5}{6}a_{14}^2a_7y-\frac{3}{2}a_{14}^2a_5t+a_4\right)^2+\left(a_5x+\frac{5}{4}a_{14}^4a_5y+a_7t+a_8\right)^2\\&+\exp\left(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\right).\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.1), we obtain the value of u

$$u(x,y,t)=\frac{2\left(\frac{8}{9}\frac{a_7^2}{a_{14}^4}+2a_5^2+a_{14}^2\exp\left(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\right)\right)}{k_1^2+k_2^2+\exp\left(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\right)}-\frac{2\left(\frac{4}{3}\frac{k_1a_7}{a_{14}^2}+2k_2a_5-a_{14}\exp\left(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\right)\right)^2}{\left(k_1^2+k_2^2+\exp\left(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\right)\right)^2},$$

where two linear variables are defined by

$$\begin{aligned}k_1&=\frac{2}{3}\frac{a_7x}{a_{14}^2}+\frac{5}{6}a_{14}^2a_7y-\frac{3}{2}a_{14}^2a_5t+a_4,\\ k_2&=a_5x+\frac{5}{4}a_{14}^4a_5y+a_7t+a_8.\end{aligned}$$

Here, we put two plots to present the change of lump solution — cf. Fig. 4. That is a soliton with a lump wave.

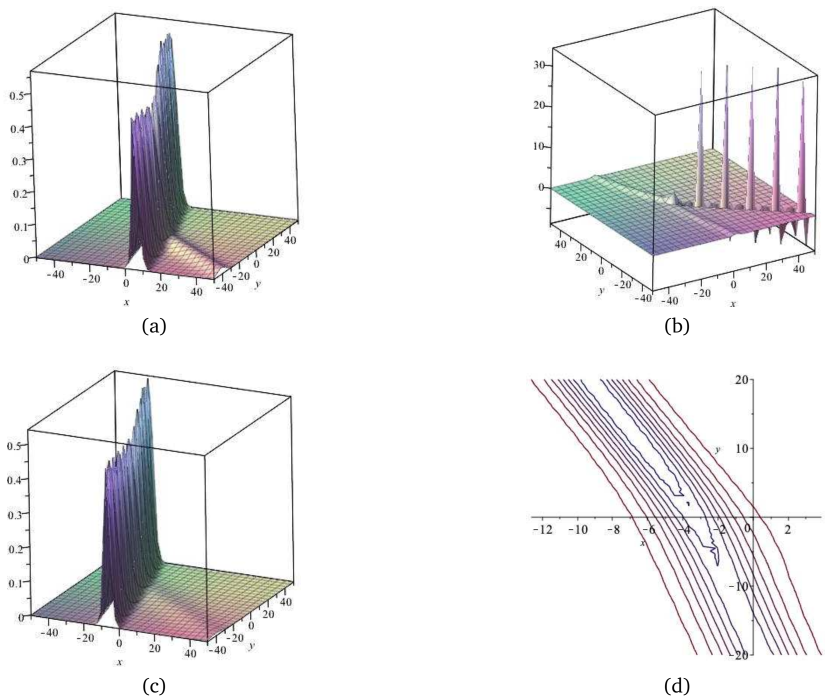


Figure 4: Spatiotemporal structure of solution u with the parameter selections: $a_4=2$, $a_5=1$, $a_7=2$, $a_8=2$, $a_{14}=1$, $a_{17}=2$, (a) $t=-10$; (b) $t=0$; (c) $t=15$, (d) contour plots (top) $t=-10$.

Case 4. In this case

$$\begin{aligned}
&a_1=-\frac{2}{3}\frac{a_7}{a_{14}^2}, && a_2=-\frac{5}{6}a_{14}^2a_7, && a_3=\frac{3}{2}a_{14}^2a_5, && a_6=\frac{5}{4}a_{14}^4a_5,\\
&a_9=0, && a_{15}=\frac{1}{4}a_{14}^5, && a_{16}=\frac{1}{2}a_{14}^3, && a_{18}=0,\\
&a_{14}\neq 0, && a_i=a_i, && i=4,5,7,8,14,17.
\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.3), we get the form of the function f

$$\begin{aligned}
f(x,y,t)=&\Big(-\frac{2}{3}\frac{a_7x}{a_{14}^2}-\frac{5}{6}a_{14}^2a_7y+\frac{3}{2}a_{14}^2a_5t+a_4\Big)^2+\Big(a_5x+\frac{5}{4}a_{14}^4a_5y+a_7t+a_8\Big)^2\\
&+\exp\Big(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\Big).
\end{aligned} \tag{2.13}$$

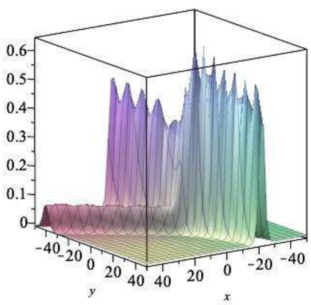
Substituting (2.13) into (2.1), we have

$$\begin{aligned}
u(x,y,t)=&\frac{2\Big(\frac{8}{9}\frac{a_7^2}{a_{14}^4}+2a_5^2+a_{14}^2\exp\big(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\big)\Big)}{l_1^2+l_2^2+\exp\big(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\big)}\\
&-\frac{2\Big(-\frac{4}{3}\frac{l_1a_7}{a_{14}^2}+2l_2a_5-a_{14}\exp\big(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\big)\Big)^2}{\Big(l_1^2+l_2^2+\exp\big(-a_{14}x-\frac{1}{4}a_{14}^5y-\frac{1}{2}a_{14}^3t-a_{17}\big)\Big)^2},
\end{aligned}$$

where two linear variables are defined by

$$\begin{aligned}
l_1&=-\frac{2}{3}\frac{a_7x}{a_{14}^2}-\frac{5}{6}a_{14}^2a_7y+\frac{3}{2}a_{14}^2a_5t+a_4,\\
l_2&=a_5x+\frac{5}{4}a_{14}^4a_5y+a_7t+a_8.
\end{aligned}$$

In Fig. 5, we can get the change of u. That is a soliton with a lump wave.


(a)

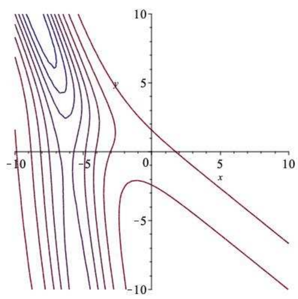

(b)

Figure 5: Spatiotemporal structure of solution u with the parameter selections $a_4=1$, $a_5=3$, $a_7=2$, $a_8=2$, $a_{14}=1$, $a_{17}=2$, (a) $t=-5$; (b) contour plots (top) $t=-5$.

Case 5. In this case

$$
\begin{aligned}
&a_2=0, \quad && a_3=0, \quad && a_6=0, \quad a_7=0, \\
&a_9=0, \quad && a_{15}=-a_{14}^5, \quad && a_{16}=-a_{14}^3, \\
&a_{14}\neq 0, \quad && a_i=a_i, \quad && i=1,4,5,8,14,17,18.
\end{aligned} \tag{2.14}
$$

Substituting (2.14) into (2.3), we get the form of the function f

$$
\begin{aligned}
f(x,y,t)=&(a_1x+a_4)^2+(a_5x+a_8)^2+\exp\left(a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}\right) \\
&+a_{18}\exp\left(-a_{14}^5y-a_{14}^3t+a_{14}x+a_{17}\right).
\end{aligned} \tag{2.15}
$$

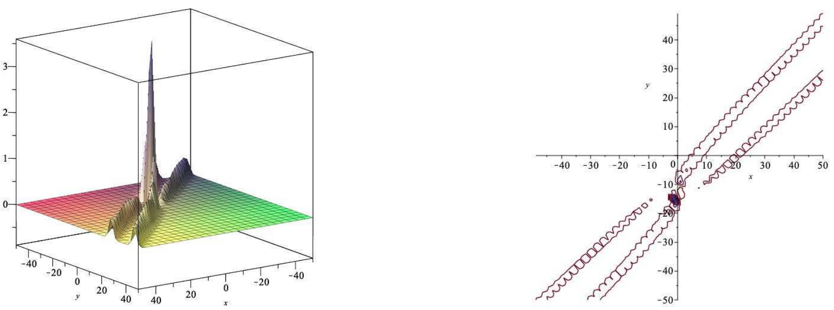
Substituting (2.15) into (2.1), we have

$$
\begin{aligned}
u(x,y,t)=&\frac{2\left(2a_1^2+2a_5^2+a_{14}^2\exp(A)\right)+a_{18}a_{14}^2\exp(B)}{(a_1x+a_4)^2+(a_5x+a_8)^2+\exp(A)+a_{18}\exp(B)} \\
&-\frac{2\left(2(a_1x+a_4)a_1+2(a_5x+a_8)a_5-a_{14}\exp(A)+a_{18}a_{14}\exp(B)\right)^2}{\left((a_1x+a_4)^2+(a_5x+a_8)^2+\exp(A)+a_{18}\exp(B)\right)^2},
\end{aligned}
$$

where two linear variables are defined by

$$
\begin{aligned}
A&=a_{14}^5y+a_{14}^3t-a_{14}x-a_{17}, \\
B&=-a_{14}^5y-a_{14}^3t+a_{14}x+a_{17}.
\end{aligned}
$$

In Fig. 6, three-dimensional plots and contour plots of the lump solutions u are made via Maple plot tools, to shed light on the characteristic of the lump solutions. That is a soliton with a lump wave.



(a) (b)

Figure 6: Spatiotemporal structure of solution u with the parameter selections $a_1=2, a_4=1,\ a_5=2,\ a_9=2,\ a_8=1,\ a_{13}=1,\ a_{14}=1, a_{17}-2, a_{18}=2$, (a) $t=15$; (b) contour plots (top) $t=15$.

3. Conclusions

In this paper, we have studied the (2+1)-dimensional bidirectional Sawada-Kotera equation. Based on the Hirota bilinear method and symbolic computations, we have transformed the (2+1)-dimensional bSK equation into a Hirota bilinear form. Through choosing a combed function with a quadratic function and symbolic computation with Maple, five classes of lump and interaction solutions to the (2+1)-dimensional bSK equation have been obtained. We have also made different plots with particular choices of the involved parameters to show energy distributions and dynamical properties of the resulting lump and interaction solutions. It is hoped that our results could be helpful in generating exact wave solutions to nonlinear dispersive wave equations in higher dimensions and recognizing propagation and interaction characteristics of nonlinear waves.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (Project Nos. 11371086, 11671258, 11975145), by the Fund of Science and Technology Commission of Shanghai Municipality (Project No. 13ZR1400100), by the Fund of Donghua University, Institute for Non-Linear Sciences and by the Fundamental Research Funds for the Central Universities with contract number 2232021G-13.

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