AN INTEGRABLE $\mathrm{SO}(3,\mathbb{R})$-COUNTERPART OF THE HEISENBERG SOLITON HIERARCHY

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An integrable counterpart of the Heisenberg soliton hierarchy is generated from a matrix spectral problem associated with $\mathrm{so}(3,\mathbb{R})$. Bi-Hamiltonian structures of the resulting counterpart soliton hierarchy are furnished by the trace identity, and all newly presented equations are shown to possess infinitely many commuting symmetries and conservation laws.

Keywords: zero curvature equation, trace identity, Hamiltonian structure, integrable equation, Heisenberg hierarchy.

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1. Introduction

Zero curvature equations are a basic tool to generate integrable soliton equations possessing Hamiltonian structures (see, e.g. [1–5]). Usually, one starts with matrix spectral problems (or Lax pairs) associated with given matrix loop algebras. The trace identity over semisimple Lie algebras [6, 7] and the variational identity over nonsemisimple Lie algebras [8, 9] provide powerful approaches for establishing Hamiltonian and quasi-Hamiltonian structures of soliton equations.

Soliton equations often come in hierarchies consisting of commuting equations (see, e.g. [6, 10]). Typical examples of soliton hierarchies which fit into the zero curvature formulation include the Korteweg–de Vries hierarchy [11], the Ablowitz–Kaup–Newell–Segur hierarchy [12], the Dirac hierarchy [13], the Kaup-Newell hierarchy [14], the Wadati–Konno–Ichikawa hierarchy [15] and the Boiti–Pempinelli–Tu hierarchy [16]. Those hierarchies only contain dependent variables less than or equal to three, and the case of more dependent variables is highly complicated, requiring considerable efforts in computations. Integrable couplings associated with nonsemisimple loop algebras present such examples, which can possess a large number of dependent variables.

Very recently, the three-dimensional special orthogonal Lie algebra so(3,$\mathbb{R}$) has been used in constructing soliton hierarchies (see, e.g. [17, 18]). This Lie algebra, being simple, can be realized through 3×3 skew-symmetric matrices, and thus, it has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.1}$$

which has the cyclic commutator relations:

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2. \tag{1.2}$$

Its derived algebra is the algebra itself, and thus, it is 3-dimensional, too. The only other such three-dimensional real Lie algebra is the special linear algebra sl(2,$\mathbb{R}$), and it has been widely used in studying soliton equations in soliton theory (see, e.g. [11–16]).

The matrix loop algebra we shall adopt below is

$$\widetilde{\mathrm{so}}(3, \mathbb{R}) = \Big\{ \sum_{i \geq 0} M_i \lambda^{n-i} \,\Big|\, n \in \mathbb{Z}, \text{ and } M_i \in \mathrm{so}(3, \mathbb{R}), \;\; i \geq 0 \Big\}, \tag{1.3}$$

where λ is the loop parameter. That is to say that $\widetilde{\mathrm{so}}(3, \mathbb{R})$ is the space of all Laurent series in λ with coefficients in $\mathrm{so}(3, \mathbb{R})$ and a finite regular part. A particular subalgebra of this loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$ is formed by linear combinations: $p_1(\lambda)e_1 + p_2(\lambda)e_2 + p_3(\lambda)e_3$, with p_1, p_2, p_3 being arbitrary polynomials in λ. Owing to the cyclic commutator relations (1.2), the loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$ provides a good structural basis for our study of soliton equations with Hamiltonian and quasi-Hamiltonian structures, and a few new soliton hierarchies have been already generated from $\widetilde{\mathrm{so}}(3, \mathbb{R})$, indeed (see, e.g. [17–22]).

In this paper, we would like to use the loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$ to introduce a counterpart matrix spectral problem for the Heisenberg spectral problem (see, e.g. [23–26]), and compute an integrable counterpart hierarchy of the Heisenberg soliton hierarchy (see, e.g. [3, 27]) by zero curvature equations. An application of the trace identity will engender bi-Hamiltonian structures for all members in the resulting counterpart soliton hierarchy, and thus the counterpart soliton hierarchy is Liouville integrable. The new counterpart hierarchy provides us with another interesting example of soliton hierarchies associated with the matrix loop algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$. A few concluding remarks and comments will be given at the end of the paper.

2. The Heisenberg soliton hierarchy

2.1. Solving the stationary zero curvature equation

The Heisenberg hierarchy [3] is associated with the following 2×2 matrix spectral problem [23, 24],

$$\phi_x = U\phi = U(u,\lambda)\phi, \qquad U = \begin{bmatrix} \lambda r & \lambda p \\ \lambda q & -\lambda r \end{bmatrix},$$
$$u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \qquad pq + r^2 = 1. \tag{2.1}$$

The corresponding stationary zero curvature equation

$$W_x = [U, W] \tag{2.2}$$

with

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \tag{2.3}$$

is equivalent to

$$\begin{cases} a_x = \lambda pc - \lambda qb, \\ b_x = 2\lambda rb - 2\lambda pa, \\ c_x = 2\lambda qa - 2\lambda rc. \end{cases} \tag{2.4}$$

Let a, b and c have the following Laurent expansions in λ:

$$a = \sum_{i\geq 0} a_i\lambda^{-i}, \qquad b = \sum_{i\geq 0} b_i\lambda^{-i}, \qquad c = \sum_{i\geq 0} c_i\lambda^{-i}, \tag{2.5}$$

and take the initial data

$$a_0 = r, \qquad b_0 = p, \qquad c_0 = q, \tag{2.6}$$

which are required by the equations on the first powers of λ in (2.4):

$$pc_0 - qb_0 = 0, \qquad rb_0 - pa_0 = 0, \qquad qa_0 - rc_0 = 0.$$

While determining the sequence of $\{a_i, b_i, c_i \,|\, i \geq 1\}$ by (2.4), we impose the condition that the constants of integration take the value of zero,

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \qquad i \geq 1, \tag{2.7}$$

so that the uniqueness can be guaranteed. Under (2.7), the system (2.4) equivalently generates

$$\begin{cases} a_{i+1} = -\frac{1}{2}\partial^{-1}\frac{q}{r}\partial b_{i+1} - \frac{1}{2}\partial^{-1}\frac{p}{r}\partial c_{i+1}, \\ b_{i,x} = 2rb_{i+1} - 2pa_{i+1}, \\ c_{i,x} = 2qa_{i+1} - 2rc_{i+1}, \end{cases} \quad i \geq 0, \tag{2.8}$$

the first of which is because we have

$$2ra_x = 2\lambda rpc - 2\lambda rqb = p(2\lambda qa - c_x) - q(b_x + 2\lambda pa) = -pc_x - qb_x, \tag{2.9}$$

from (2.4). The relationship in (2.8) clearly tells the inverse of a recursion relation for b_i and c_i,

$$\begin{bmatrix} b_{i,x} \\ c_{i,x} \end{bmatrix} = R \begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix}, \qquad R = \begin{bmatrix} 2r + p\partial^{-1}\frac{q}{r}\partial & p\partial^{-1}\frac{p}{r}\partial \\ -q\partial^{-1}\frac{q}{r}\partial & -2r - q\partial^{-1}\frac{p}{r}\partial \end{bmatrix}, \qquad i \geq 0.$$

Actually, we have the recursion relation

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = L \begin{bmatrix} b_{i,x} \\ c_{i,x} \end{bmatrix}, \qquad L = \begin{bmatrix} \frac{1}{2r} - \frac{1}{4}p\partial^{-1}q\partial\frac{1}{r} & \frac{1}{4}p\partial^{-1}p\partial\frac{1}{r} \\ -\frac{1}{4}q\partial^{-1}q\partial\frac{1}{r} & -\frac{1}{2r} + \frac{1}{4}q\partial^{-1}p\partial\frac{1}{r} \end{bmatrix}, \qquad i \geq 0. \tag{2.10}$$

We give a proof of (2.10) for the completeness of deriving the Heisenberg hierarchy. Through the second and third relations in (2.4), we can compute that

$$q_x b_x - p_x c_x = 2\lambda r(q_x b + p_x c) - 2\lambda(pq_x + p_x q)a = 2\lambda r(q_x b + p_x c + 2r_x a).$$

Then taking (2.9) into consideration, we obtain

$$\lambda(qb + pc + 2ra)_x = \frac{q_x}{2r}b_x - \frac{p_x}{2r}c_x,$$

and it thus follows that

$$qb_{i+1} + pc_{i+1} + 2ra_{i+1} = \frac{1}{2}\partial^{-1}\frac{q_x}{r}b_{i,x} - \frac{1}{2}\partial^{-1}\frac{p_x}{r}c_{i,x}, \qquad i \geq 0.$$

Together with (2.8), this leads to

$$\begin{aligned} a_{i+1} &= \frac{1}{4}r\partial^{-1}\frac{q_x}{r}b_{i,x} - \frac{1}{4}r\partial^{-1}\frac{p_x}{r}c_{i,x} - \frac{1}{4}qb_{i,x} + \frac{1}{4}pc_{i,x} \\ &= -\frac{1}{4}r\partial^{-1}q\partial\frac{1}{r}b_{i,x} + \frac{1}{4}r\partial^{-1}p\partial\frac{1}{r}c_{i,x}, \qquad i \geq 0, \end{aligned} \tag{2.11}$$

and finally to the recursion relation (2.10).

Now, through (2.8) or (2.10) and (2.11), the first two sets of $\{a_i, b_i, c_i | i \geq 1\}$ can be worked out as follows:

$$\begin{aligned} a_1 &= -\frac{1}{4}p_x q + \frac{1}{4}pq_x, \qquad b_1 = -\frac{1}{4r}(pp_x q - p^2 q_x - 2p_x), \\ c_1 &= -\frac{1}{4r}(p_x q^2 - pqq_x + 2q_x); \\ a_2 &= -\frac{1}{32r^3}(3pp_x^2 q^3 - 6p^2 p_x q^2 q_x + 3p^3 qq_x^2 - 4pp_{xx}q^2 - 4p^2 qq_{xx} \\ &\quad - p_x^2 q^2 + 14pp_x qq_x - p^2 q_x^2 + 4p_{xx}q + 4pq_{xx} - 4p_x q_x), \\ b_2 &= \frac{1}{32r^2}(3pp_x^2 q^2 - 6p^2 p_x qq_x + 3p^3 q_x^2 - 8pp_{xx}q + 12pp_x q_x + 8p_{xx}), \end{aligned}$$

$$c_2 = \frac{1}{32r^2}(3p_x^2q^3 - 6pp_xq^2q_x + 3p^2qq_x^2 - 8pqq_{xx} + 12p_xqq_x + 8q_{xx}).$$

We point out that the localness of the first two sets of $\{a_i, b_i, c_i | i \geq 1\}$ is not an accident. All the functions a_i, b_i, c_i, $i \geq 1$, are differential functions, indeed. We explain this phenomenon as follows. First from the stationary zero curvature equation (2.2), i.e. $W_x = [U, W]$, we can compute

$$\frac{d}{dx}\operatorname{tr}(W^2) = 2\operatorname{tr}(WW_x) = 2\operatorname{tr}(W[U, W]) = 0,$$

and hence, the fact that $\operatorname{tr}(W^2) = 2(a^2 + bc)$ tells that

$$a^2 + bc = (a^2 + bc)|_{u=0} = 1,$$

where the initial data in (2.6) was used. Then, taking the Laurent expansions (2.5) into consideration, we see that

$$a_i = \frac{p}{4}c_{i-1,x} - \frac{q}{4}b_{i-1,x} - \frac{r}{2}\sum_{k+l=i,\, k,l\geq 1}(a_ka_l + b_kb_l), \qquad i \geq 1. \tag{2.12}$$

Finally, based on this recursion relation (2.12) and the last two recursion relations in (2.8), an application of the mathematical induction shows that all the functions a_i, b_i, c_i, $i \geq 1$, are differential functions in u, i.e. they are all local.

2.2. The Heisenberg hierarchy and its bi-Hamiltonian structure

It is, now, direct to see that the Lax operators

$$V^{[m]} = \lambda(\lambda^m W)_+ \equiv \sum_{i=0}^{m}\begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix}\lambda^{m+1-i}, \qquad m \geq 0, \tag{2.13}$$

guarantee that the corresponding zero curvature equations:

$$U_{t_m} - (V^{[m]})_x + [U, V^{[m]}] = 0, \qquad m \geq 0, \tag{2.14}$$

equivalently yield a hierarchy of soliton equations

$$u_{t_m} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix}, \qquad m \geq 0. \tag{2.15}$$

The third set of equations in (2.14) is

$$r_{t_m} = a_{m,x}, \quad m \geq 0,$$

which is automatically satisfied, due to (2.9). Every Heisenberg system in this soliton hierarchy is local, since all the functions a_i, b_i, c_i, $i \geq 0$, are differential equations.

The Hamiltonian structures of the Heisenberg hierarchy (2.15) can be furnished by the trace identity [6]

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(\frac{\partial U}{\partial u}W\right), \qquad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\operatorname{tr}(W^2)|, \tag{2.16}$$

or generally, the variational identity (see [8, 28]). A straightforward computation of

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} r & p \\ q & -r \end{bmatrix}, \qquad \frac{\partial U}{\partial p} = \begin{bmatrix} -\frac{1}{2}\frac{q}{r}\lambda & \lambda \\ 0 & \frac{1}{2}\frac{q}{r}\lambda \end{bmatrix}, \qquad \frac{\partial U}{\partial q} = \begin{bmatrix} -\frac{1}{2}\frac{p}{r}\lambda & 0 \\ \lambda & \frac{1}{2}\frac{p}{r}\lambda \end{bmatrix},$$

and

$$\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = qb + pc + 2ra,$$
$$\operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) = \lambda c - \frac{qa\lambda}{r},$$
$$\operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) = \lambda b - \frac{pa\lambda}{r},$$

and an application of (2.16) give rise to the following identity

$$\frac{\delta}{\delta u}\int (qb + pc + 2ra)\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda(c - qa/r) \\ \lambda(b - pa/r) \end{bmatrix}.$$

A balance of coefficients of λ^{-m} for each $m \geq 0$ in the equality obviously yields

$$\frac{\delta}{\delta u}\int (qb_m + pc_m + 2ra_m)\, dx = (\gamma - m + 1)\begin{bmatrix} c_m - \frac{q}{r}a_m \\ b_m - \frac{p}{r}a_m \end{bmatrix}, \qquad m \geq 0.$$

The identity with $m = 1$ tells that $\gamma = 0$, and thus, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}_m = \begin{bmatrix} c_{m+1} - \frac{q}{r}a_{m+1} \\ b_{m+1} - \frac{p}{r}a_{m+1} \end{bmatrix}, \qquad m \geq 0, \tag{2.17}$$

with the Hamiltonian functionals being defined by

$$\mathcal{H}_0 = \int \frac{p_x q - p q_x}{2(1+r)}\, dx, \qquad \mathcal{H}_m = \int \left(-\frac{qb_{m+1} + pc_{m+1} + 2ra_{m+1}}{m}\right) dx, \qquad m \geq 1. \tag{2.18}$$

It then follows from (2.8) that

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix} = \begin{bmatrix} 2rb_{m+1} - 2pa_{m+1} \\ 2qa_{m+1} - 2rc_{m+1} \end{bmatrix} = J\begin{bmatrix} c_{m+1} - \frac{q}{r}a_{m+1} \\ b_{m+1} - \frac{p}{r}a_{m+1} \end{bmatrix}, \qquad m \geq 0. \tag{2.19}$$

where

$$J = J(u) = \begin{bmatrix} 0 & 2r \\ -2r & 0 \end{bmatrix}, \qquad pq + r^2 = 1. \tag{2.20}$$

A direct evaluation shows that J is a Hamiltonian operator, by noting

$$J'(u)[S] = \begin{bmatrix} 0 & -(qS^1 + pS^2)/r \\ (qS^1 + pS^2)/r & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} S^1 \\ S^2 \end{bmatrix}.$$

It follows now that the Heisenberg soliton hierarchy (2.15) has the bi-Hamiltonian structure

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \qquad m \geq 1, \tag{2.21}$$

where the Hamiltonian functionals $\mathcal{H}_m$'s and the first Hamiltonian operator J are respectively given by (2.18) and (2.20), and the second Hamiltonian operator M is defined by

$$M = \partial L J = \begin{bmatrix} -\frac{1}{2}\partial p \partial^{-1} p \partial & \partial - \frac{1}{2}\partial p \partial^{-1} q \partial \\ \partial - \frac{1}{2}\partial q \partial^{-1} p \partial & -\frac{1}{2}\partial q \partial^{-1} q \partial \end{bmatrix}. \tag{2.22}$$

It is straightforward to verify that the two operators J and M form a Hamiltonian pair and so

$$\Phi = MJ^{-1} = \begin{bmatrix} \frac{1}{2}\partial\frac{1}{r} - \frac{1}{4}\partial p \partial^{-1} q \partial \frac{1}{r} & \frac{1}{4}\partial p \partial^{-1} p \partial \frac{1}{r} \\ -\frac{1}{4}\partial q \partial^{-1} q \partial \frac{1}{r} & -\frac{1}{2}\partial\frac{1}{r} + \frac{1}{4}\partial q \partial^{-1} p \partial \frac{1}{r} \end{bmatrix} \tag{2.23}$$

presents a common hereditary recursion operator for the Heisenberg soliton hierarchy (2.15). The resulting functionals correspond to common conservation laws for each soliton system in the whole soliton hierarchy (2.15).

The first nonlinear system in the Heisenberg hierarchy (2.15) reads

$$\begin{cases} p_{t_1} = \frac{1}{8r^3}(-6pp_{xx}q + 2p^2 p_{xx} q^2 + pp_x^2 q^2 + 4pp_x q_x \\ \qquad -2p^2 p_x q q_x + 2p^2 q_{xx} - 2p^3 q q_{xx} + 4p_{xx} + p^3 q_x^2), \\ q_{t_1} = -\frac{1}{8r^3}(-2pp_{xx}q^3 + 2p^2 q^2 q_{xx} + p_x^2 q^3 - 2pp_x q^2 q_x \\ \qquad +p^2 q q_x^2 + 2p_{xx} q^2 - 6pq q_{xx} + 4p_x q q_x + 4q_{xx}). \end{cases} \tag{2.24}$$

This Heisenberg system has the bi-Hamiltonian structure

$$u_{t_1} = J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}, \tag{2.25}$$

where $\mathcal{H}_0$ is defined by (2.18) and $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \int \big[-\frac{1}{16r^2}(p^2q_x^2 - 2pp_xqq_x + p_x^2q^2 + 4p_xq_x)\big]\,dx. \tag{2.26}$$

3. An integrable so(3,$\mathbb{R}$)-counterpart

3.1. The Heisenberg type hierarchy associated with so(3,$\mathbb{R}$)

To generate an integrable counterpart of the Heisenberg hierarchy (2.15), associated with so(3,$\mathbb{R}$), we introduce a new 3×3 matrix spectral problem,

$$\phi_x = U\phi = U(u,\lambda)\phi, \qquad u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{3.1}$$

where the spectral matrix U is chosen as

$$U = \lambda re_1 + \lambda pe_2 + \lambda qe_3 = \begin{bmatrix} 0 & -\lambda q & -\lambda r \\ \lambda q & 0 & -\lambda p \\ \lambda r & \lambda p & 0 \end{bmatrix}, \qquad p^2+q^2+r^2 = 1. \tag{3.2}$$

We similarly follow a standard procedure using the zero curvature formulation (see [6, 10]), to present an integrable hierarchy. First, we solve the stationary zero curvature equation

$$W_x = [U, W], \qquad W \in \widetilde{\mathrm{so}}(3,\mathbb{R}). \tag{3.3}$$

If we assume W to be

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix}, \tag{3.4}$$

then Eq. (3.3) becomes

$$\begin{cases} a_x = \lambda pc - \lambda qb, \\ b_x = -\lambda rc + \lambda qa, \\ c_x = \lambda rb - \lambda pa. \end{cases} \tag{3.5}$$

Note that the second and third relations in (2.4) and (3.5) are slightly different, which generate different soliton hierarchies with different recursion operators.

Let a, b and c be the following Laurent expansions in λ:

$$a = \sum_{i\ge 0} a_i\lambda^{-i}, \qquad b = \sum_{i\ge 0} b_i\lambda^{-i}, \qquad c = \sum_{i\ge 0} c_i\lambda^{-i}, \tag{3.6}$$

and take the initial data

$$a_0 = r, \qquad b_0 = p, \qquad c_0 = q, \tag{3.7}$$

which are required by the equations on the first powers of λ in (3.5):

$$pc_0 - qb_0 = 0, \qquad -rc_0 + qa_0 = 0, \qquad rb_0 - pa_0 = 0.$$

While determining the sequence of $\{a_i, b_i, c_i \,|\, i \geq 1\}$ by (3.5), we similarly impose the condition that the constants of integration take the value of zero,

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \qquad i \geq 1, \tag{3.8}$$

to guarantee the uniqueness.

Under (3.8), the system (3.5) leads equivalently to

$$\begin{cases} a_{i+1} = -\partial^{-1}\dfrac{p}{r}\partial b_{i+1} - \partial^{-1}\dfrac{q}{r}\partial c_{i+1}, \\ b_{i,x} = -rc_{i+1} + qa_{i+1}, \\ c_{i,x} = rb_{i+1} - pa_{i+1}, \end{cases} \qquad i \geq 0, \tag{3.9}$$

the first of which is because from (3.5) we have

$$ra_x = \lambda rpc - \lambda rqb = p(-b_x + \lambda qa) - q(c_x + \lambda pa) = -pb_x - qc_x. \tag{3.10}$$

The relationship in (3.9) tells the inverse of recursion relations for b_i and c_i,

$$\begin{bmatrix} b_{i,x} \\ c_{i,x} \end{bmatrix} = R \begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix}, \qquad R = \begin{bmatrix} -q\partial^{-1}\dfrac{p}{r}\partial & -r - q\partial^{-1}\dfrac{q}{r}\partial \\ r + p\partial^{-1}\dfrac{p}{r}\partial & p\partial^{-1}\dfrac{q}{r}\partial \end{bmatrix}, \qquad i \geq 0.$$

Actually, we have the following recursion relation

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = L \begin{bmatrix} b_{i,x} \\ c_{i,x} \end{bmatrix}, \qquad L = \begin{bmatrix} p\partial^{-1}q\partial\dfrac{1}{r} & \dfrac{1}{r} - p\partial^{-1}p\partial\dfrac{1}{r} \\ -\dfrac{1}{r} + q\partial^{-1}q\partial\dfrac{1}{r} & -q\partial^{-1}p\partial\dfrac{1}{r} \end{bmatrix}, \qquad i \geq 0. \tag{3.11}$$

The proof for (3.11) is straightforward. First, the second and third relations in (3.5) yield

$$q_x b_x - p_x c_x = -\lambda r(p_x b + q_x c) + \lambda(pp_x + qq_x)a = -\lambda r(p_x b + q_x c + r_x a).$$

Then, by (3.10), we arrive at

$$\lambda(pb + qc + ra)_x = \frac{p_x}{r}c_x - \frac{q_x}{r}b_x,$$

and this tells us that

$$pb_{i+1} + qc_{i+1} + ra_{i+1} = \partial^{-1}\frac{p_x}{r}c_{i,x} - \partial^{-1}\frac{q_x}{r}b_{i,x}, \qquad i \geq 0.$$

Together with (3.9) we obtain

$$a_{i+1} = r\partial^{-1}\frac{p_x}{r}c_{i,x} - r\partial^{-1}\frac{q_x}{r}b_{i,x} + qb_{i,x} - pc_{i,x}$$
$$= -r\partial^{-1}p\partial\frac{1}{r}c_{i,x} + r\partial^{-1}q\partial\frac{1}{r}b_{i,x}, \qquad i \geq 0, \tag{3.12}$$

and further the recursion relation (3.11). In (3.12), the operator relation $f\partial = -f_x + \partial f$ was used for simplification.

Through the above recursion relations in (3.9) or (3.11) and (3.12), the first two sets of $\{a_i, b_i, c_i \,|\, i \geq 1\}$ can be computed as follows:

$$a_1 = p_x q - pq_x, \qquad b_1 = \frac{1}{r}(-p^2 q_x + pp_x q + q_x), \qquad c_1 = \frac{1}{r}(p_x q^2 - pqq_x - p_x);$$

$$a_2 = -\frac{1}{2r^3}(3p^4q_x^2 + 3p_x^2q^4 + 2p^3p_{xx} + 2q^3q_{xx} - 6pq^3p_xq_x + 3p^2q^2p_x^2$$
$$- 3q^2q_x^2 + 2pp_{xx}q^2 + 3p^2q^2q_x^2 - 6p^3p_xqq_x + 2p^2qq_{xx} - 2qq_{xx}$$
$$+ 2pp_xqq_x - 3p^2p_x^2 - 2pp_{xx} - 4p^2q_x^2 - 4p_xq^2 + p_x^2 + q_x^2),$$

$$b_2 = \frac{1}{2r^2}(3pp_x^2q^2 + 2p_{xx}q^2 - 6p^2p_xqq_x - 3pp_x^2 - 3pq_x^2 + 2p^2p_{xx} + 3p^3q_x^2 - 2p_{xx}),$$

$$c_2 = \frac{1}{2r^2}(3q^3p_x^2 - 6pp_xq^2q_x + 2q^2q_{xx} + 3p^2qq_x^2 - 3qq_x^2 - 3p_x^2q - 2q_{xx} + 2p^2q_{xx}).$$

The localness of the first two sets of $\{a_i, b_i, c_i \,|\, i \geq 1\}$ is not an accident. All the functions a_i, b_i, c_i, $i \geq 1$, are local, indeed.

PROPOSITION 3.1. *Let* $p^2 + q^2 + r^2 = 1$. *Assume that* a_i, b_i, c_i, $i \geq 1$, *be defined by* (3.9) *from the initail data* (3.7) *under the condition* (3.8). *Then the functions* a_i, b_i, c_i, $i \geq 1$, *are all differential functions in* p *and* q.

Proof: The verification of the localness property per se is straightforward and easy. First from the stationary zero curvature equation (3.3), i.e. $W_x = [U, W]$, we can similarly compute

$$\frac{d}{dx}\,\mathrm{tr}(W^2) = 2\,\mathrm{tr}(WW_x) = 2\,\mathrm{tr}(W[U, W]) = 0,$$

and so, due to $\mathrm{tr}(W^2) = -2(a^2 + b^2 + c^2)$, we have

$$a^2 + b^2 + c^2 = (a^2 + b^2 + c^2)|_{u=0} = 1,$$

the last step of which follows from the initial data in (3.7). This implies that

$$a_i = -pc_{i-1,x} + qb_{i-1,x} - \frac{r}{2}\sum_{k+l=i,\,k,l\geq 1}(a_ka_l + b_kb_l + c_kc_l), \qquad i \geq 1. \tag{3.13}$$

By using this recursion relation (3.13) and the last two recursion relations in (3.9), i.e.

$$\begin{cases} b_i = \dfrac{1}{r}(c_{i-1,x} + pa_i), \\ c_i = \dfrac{1}{r}(-b_{i-1,x} + qa_i), \end{cases} \qquad i \geq 1,$$

application of the mathematical induction finally shows that all the functions $a_i, b_i, c_i,\ \ i \geq 1$, are local, i.e. they are all differential functions in p and q. The proof is finished. □

As usual, let us introduce

$$V^{[m]} = \lambda(\lambda^m W)_+ \equiv \sum_{i=0}^{m}(a_i e_1 + b_i e_2 + c_i e_3)\lambda^{m+1-i} \in \widetilde{\mathrm{so}}(3, \mathbb{R}), \qquad m \geq 0. \quad (3.14)$$

Then, based on (3.5), we can see that the corresponding zero curvature equations

$$U_{t_m} - (V^{[m]})_x + [U, V^{[m]}] = 0, \ \ m \geq 0, \quad (3.15)$$

equivalently engender a hierarchy of soliton equations:

$$u_{t_m} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix}, \qquad m \geq 0. \quad (3.16)$$

Note that the third set of equations in (3.15) is

$$r_{t_m} = a_{m,x}, \qquad m \geq 0,$$

which is, thanks to (3.10), automatically satisfied:

$$r_{t_m} = -\frac{p}{r}p_{t_m} - \frac{q}{r}q_{t_m} = -\frac{p}{r}b_{m,x} - \frac{q}{r}c_{m,x} = a_{m,x}, \qquad m \geq 0. \quad (3.17)$$

Every system in this counterpart soliton hierarchy is local, since all the functions $a_i, b_i, c_i,\ i \geq 0$, are differential functions. The first nonlinear system in the counterpart hierarchy (3.16) reads

$$\begin{cases} p_{t_1} = -\dfrac{1}{r^3}(-qq_x^2 - 2pp_xq^2q_x + p^2qq_x^2 - q_{xx} + 2p^2q_{xx} + q^2q_{xx} \\ \qquad -p_x^2q + q^3p_x^2 - pp_{xx}q + p^3p_{xx}q + pp_{xx}q^3 - p^4q_{xx} - p^2q^2q_{xx}), \\ q_{t_1} = -\dfrac{1}{r^3}(-p^3qq_{xx} - p^3q_x^2 + p^2p_{xx}q^2 + 2p^2p_xqq_x - pq^3q_{xx} - pp_x^2q^2 \\ \qquad +p_{xx}q^4 - p^2p_{xx} + pqq_{xx} + pp_x^2 + pq_x^2 - 2p_{xx}q^2 + p_{xx}). \end{cases} \quad (3.18)$$

3.2. Bi-Hamiltonian structure and Liouville integrability

We shall show that all systems in the counterpart soliton hierarchy (3.16) are Liouville integrable. Let us first establish a bi-Hamiltonian structure for the counterpart hierarchy (3.16).

We shall use the trace identity (2.16), i.e.

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\ \mathrm{tr}\left(\frac{\partial U}{\partial u}W\right),\ \ \gamma=-\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\,\mathrm{tr}(W^2)|.$$

It is direct to find that

$$\frac{\partial U}{\partial \lambda}=\begin{bmatrix}0 & -q & -r\\ q & 0 & -p\\ r & p & 0\end{bmatrix},\quad \frac{\partial U}{\partial p}=\begin{bmatrix}0 & 0 & \frac{p}{r}\lambda\\ 0 & 0 & -\lambda\\ -\frac{p}{r}\lambda & \lambda & 0\end{bmatrix},\quad \frac{\partial U}{\partial q}=\begin{bmatrix}0 & -\lambda & \frac{q}{r}\lambda\\ \lambda & 0 & 0\\ -\frac{q}{r}\lambda & 0 & 0\end{bmatrix},$$

and so, we obtain

$$\begin{aligned}\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right)&=-2qc-2pb-2ra,\\ \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right)&=-2\lambda b+\frac{2pa\lambda}{r},\\ \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right)&=-2\lambda c+\frac{2qa\lambda}{r}.\end{aligned}$$

Plugging these quantities into the above trace identity gives rise to

$$\frac{\delta}{\delta u}\int(-qc-pb-ra)\,dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix}\lambda(pa/r-b)\\ \lambda(qa/r-c)\end{bmatrix}.$$

A balance of coefficients of λ^{-m} for each $m\geq 0$ then yields

$$\frac{\delta}{\delta u}\int(-qc_m-pb_m-ra_m)\,dx=(\gamma-m+1)\begin{bmatrix}\frac{p}{r}a_m-b_m\\ \frac{q}{r}a_m-c_m\end{bmatrix},\qquad m\geq 0.$$

This identity with $m=1$ tells us that $\gamma=0$, and hence, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}_m=\begin{bmatrix}\frac{p}{r}a_{m+1}-b_{m+1}\\ \frac{q}{r}a_{m+1}-c_{m+1}\end{bmatrix},\qquad m\geq 0, \tag{3.19}$$

where the Hamiltonian functionals are defined by

$$\mathcal{H}_0=\int\frac{p_xq-pq_x}{1+r}\,dx,\qquad \mathcal{H}_m=\int\frac{qc_{m+1}+pb_{m+1}+ra_{m+1}}{m}\,dx,\qquad m\geq 1. \tag{3.20}$$

The functional $\mathcal{H}_0$ was directly computed from the vector

$$\left(\frac{pa_1}{r}-b_1,\frac{qa_1}{r}-c_1\right)^T=\left(-\frac{q_x}{r},\frac{p_x}{r}\right)^T.$$

Now, it follows from (3.9) that

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix} = \begin{bmatrix} -rc_{m+1} + qa_{m+1} \\ rb_{m+1} - pa_{m+1} \end{bmatrix} = J \begin{bmatrix} \frac{p}{r}a_{m+1} - b_{m+1} \\ \frac{q}{r}a_{m+1} - c_{m+1} \end{bmatrix}, \qquad m \geq 0, \tag{3.21}$$

where

$$J = J(u) = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}, \qquad p^2 + q^2 + r^2 = 1. \tag{3.22}$$

Noting that the Gateaux derivative of J reads

$$J'(u)[S] = \begin{bmatrix} 0 & -\frac{pS^1 + qS^2}{r} \\ \frac{pS^1 + qS^2}{r} & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} S^1 \\ S^2 \end{bmatrix}, \tag{3.23}$$

a straightforward argument shows that J is a Hamiltonian operator. Again based on (3.11), it now follows that the counterpart soliton hierarchy (3.16) has the bi-Hamiltonian structure

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1, \tag{3.24}$$

where the Hamiltonian functionals $\mathcal{H}_m$ and the first Hamiltonian operator J are respectively given by (3.20) and (3.22), and the second Hamiltonian operator M is defined by

$$M = \partial L J = \begin{bmatrix} -\partial + \partial p \partial^{-1} p \partial & \partial p \partial^{-1} q \partial \\ \partial q \partial^{-1} p \partial & -\partial + \partial q \partial^{-1} q \partial \end{bmatrix}. \tag{3.25}$$

A Maple computation can show that J and M constitute a Hamiltonian pair and so, the operator

$$\Phi = MJ^{-1} = \begin{bmatrix} \partial p \partial^{-1} q \partial \frac{1}{r} & \partial \frac{1}{r} - \partial p \partial^{-1} p \partial \frac{1}{r} \\ -\partial \frac{1}{r} + \partial q \partial^{-1} q \partial \frac{1}{r} & -\partial q \partial^{-1} p \partial \frac{1}{r} \end{bmatrix} \tag{3.26}$$

is a common hereditary recursion operator for the counterpart soliton hierarchy (3.16). The first nonlinear system (3.18) in the hierarchy (3.16) has the bi-Hamiltonian structure

$$u_{t_1} = K_1 = J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}, \tag{3.27}$$

where $\mathcal{H}_0$ is defined by (3.20) and $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \int \frac{1}{2r^2}(p^2 q_x^2 - 2pp_x q q_x + p_x^2 q^2 - p_x^2 - q_x^2)\, dx. \tag{3.28}$$

The resulting functionals correspond to common conservation laws for each soliton system in the counterpart soliton hierarchy (3.16). We point out that such differential polynomial conservation laws can also be generated either directly by computer algebra codes (see, e.g. [29]) or from some Riccati equation obtained from the underlying matrix spectral problem (see, e.g. [30–32]).

The bi-Hamiltonian structures in (3.24) show that the counterpart soliton hierarchy (3.16) is Liouville integrable, i.e. it possesses infinitely many conserved functionals and symmetries which form Abelian algebras:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0, \qquad k, l \geq 0,$$

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T M \frac{\delta \mathcal{H}_l}{\delta u}\, dx = 0, \qquad k, l \geq 0,$$

and

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \qquad k, l \geq 0.$$

These commuting relations are also consequences of the Virasoro algebras of Lax operators. We refer the interested readers to [33–35] for a detailed and systematical study on algebraic structures of Lax operators and zero curvature equations.

4. Concluding remarks

Starting with the special orthogonal Lie algebra $\widetilde{\mathrm{so}}(3, \mathbb{R})$, we introduced a counterpart matrix spectral problem of the Heisenberg spectral problem by using the same linear combination of basis matrices, and generated an integrable so(3,$\mathbb{R}$)-counterpart of the Heisenberg soliton hierarchy. All members in the resulting counterpart soliton hierarchy are local and bi-Hamiltonian. So, they are Liouville integrable PDEs, and particularly possess infinitely many commuting symmetries and conserved densities.

We remark that the following three typical matrix spectral matrices associated with $\widetilde{\mathrm{so}}(3, \mathbb{R})$:

$$U(u, \lambda) = \lambda e_1 + p e_2 + q e_3,$$

$$U(u, \lambda) = \lambda^2 e_1 + \lambda p e_2 + \lambda q e_3,$$

$$U(u, \lambda) = \lambda e_1 + \lambda p e_2 + \lambda q e_3,$$

where $u = (p, q)^T$ includes two dependent variables, have been considered of late [17–19]. Our example above provides a new spectral problem associated with so(3,$\mathbb{R}$), fitting into the zero curvature formulation. We hope that more examples of such soliton hierarchies, even with dependent variables more than two, can be presented in future research.

Given an initial matrix loop algebra, one normally needs only a considerable investment of time and computational dexterity to formulate spectral problems and construct the corresponding integrable hierarchies of soliton equations. Higher-order matrix spectral problems can engender soliton hierarchies consisting of multi-component integrable systems (see, e.g. [36–41]). Integrable couplings (see, e.g. [42–45]) associated with enlarged matrix loop algebras [46, 47] provide such specific examples of soliton hierarchies. They possess triangular forms [28] and their conserved densities can be generated by applying the variational identity [8, 28]. Nevertheless, there exist nonsemisimple loop algebras [48, 49], which do not possess any ad-invariant, symmetric and nondegenerate bilinear form, and so, one needs to develop new tools, besides the variational identity, to compute Hamiltonian structures of soliton equations.

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