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Dynamical Behavior and Wave Speed Perturbations in the (2 + 1) pKP Equation

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Abstract

The unidirectional propagation of long waves in certain nonlinear dispersive waves is explained by the (2 + 1) pKP equation, this equation admits infinite number of infinitesimals. We explored new Lie vectors thorough the commutative product properties. Using the Lie reduction stages and some assistant methods to solve the reduced ODEs, Exploiting a set of new solutions. Exploring a set of non-singular local multipliers; generating a set of local conservation laws for the studied equation. The nonlocally related (PDE) systems are found. Four nonlocally related systems are discussed reveal twenty-one interesting closed form solutions for this equation. We investigate new various solitons solutions as one soliton, many soliton waves move together, two and three Lump soliton solutions. Though three dimensions plots some selected solutions are plotted.

Keywords Commutative product · Nonlocal potential similarity transformation · Lie point symmetry · Integrating factors · Conservation law · (2 + 1) pKP equation

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1 Introduction

Partial derivatives are an excellent method for explaining the memory and genetic characteristics of various materials and processes. Today, Partial differential equation (PDE) plays an important role in number of fields such as mathematics and dynamic systems [1, 2]. Non-integer order derivatives and integrals have proved to be more useful in the formulating certain electrochemical problems than conventional models [3]. Partial differential equations (PDEs) provide a powerful tool for modeling a variety of potentially emerging phenomena with a wide range of applications across applied sciences, engineering and physics. It is the classical generalization of the calculation that involves operators of integration of non-integer orders and differentiation. They are valuable and efficient tools in both theoretical and technological fields to tackle the complexity and non-linearity of specific problems, including chemical science, fluid flow, fiber optics, signaling system, polymers, device recognition, elastic materials and so on. [4–5]. A great deal of attention has been paid to managing partial differential equations (PDEs). Specifically, fractional-order partial differential equations (PDEs) are increasingly described in visco-elasticity, mathematical biology, finance [6–8], airplane designing, traffic, population and particle chemistry [9–11]. Elasticity, plasma, fluid mechanics, optical fiber, and other disciplines of mathematics and physics involve wave phenomena. The (2 + 1) pKP equation, which addresses dissipation as well as nonlinearity and dispersion in the simulation of unidirectional plane waves, is defined as follows [12]. This paper study the (2 + 1) pKP Eqs. [13, 14]:

$$u_{xt} + \alpha u_x u_{xx} + \beta u_{xxxx} + \gamma u_{yy} = 0. \tag{1}$$

Nonlinearity, dispersive, and dissipative effects are all included in Eq. (1). Cracked rock, thermodynamics, acoustic waves in a harmonic crystal [15], acoustic-gravity waves in fluids, and other applications of the (2 + 1) pKP equation and their special cases can be found in engineering and science. The water wave model is notoriously difficult to solve analytically. So, several investigations have been done in recent years to determine the numerical solution of water wave model, (see, for example, [16, 17] and the references therein). Researchers introduced many different methods to develop an approximate analytical solution for the partial differential equation and systems, such as Painlevé analysis, Hirota bilinear method [18, 19], Darboux transformation by [20], Bäcklund transformation [21]. The (2 + 1) pKP equation was proposed to discuss the dynamic physical system.

Many scholars contributed by developing new analytical and numerical schemes to solve the boundary model [22, 23] implemented hirota bilinear method for the simulations of (2 + 1) pKP equation. The sine-Gordon expansion method has been implemented to investigate the exact solution of the Tzitzéica type equations in [24]. The dynamical behavior of mixed-type soliton solutions has been established in [25] to constitute analytical solution of the (2 + 1)-dimensional potential Kadomtsev-Petviashvili (pKP).

2 Exploring Novel Point Symmetries for the (2 + 1) pKP Equation

The point symmetries of the (2 + 1) pKP Eq. (1) are found as:

$$\begin{cases} X_1 = \left(\dfrac{xf_1'}{3} - \dfrac{y^2 f_1''}{6\gamma}\right)\dfrac{\partial}{\partial x} + \dfrac{2yf_1'}{3}\dfrac{\partial}{\partial y} + f_1(t)\dfrac{\partial}{\partial t} + \left(\dfrac{-u}{3}f_1' + \dfrac{x^2 f_1''}{6\alpha} - \dfrac{xy^2 f_1'''}{6\alpha\gamma} + \dfrac{y^4 f_1''''}{72\alpha\gamma^2}\right)\dfrac{\partial}{\partial u}, \\ X_2 = \dfrac{-yf_2'}{2\gamma}\dfrac{\partial}{\partial x} + f_2(t)\dfrac{\partial}{\partial y} + \left(\dfrac{-y^2 f_2''}{2\alpha\gamma} + \dfrac{y^3 f_2'''}{12\alpha\gamma^2}\right)\dfrac{\partial}{\partial u}, \\ X_3 = f_3(t)\dfrac{\partial}{\partial x} + \left(\dfrac{x}{3}f_3' - \dfrac{y^2 f_3''}{2\alpha\gamma}\right)\dfrac{\partial}{\partial u}, \\ X_4 = (f_4(t) + yf_5(t))\dfrac{\partial}{\partial u}. \end{cases} \tag{2}$$

There are an infinite number of possibilities for these vectors due to the existence of the arbitrary functions $f_i(t),\ i = 1 \ldots 5$. We derive optimal values for these functions first, by evaluating the commutative product of these infinitesimals as listed in Table 1.

where;

$$a_1 = \frac{-yf_1{}'f_2{}'}{6\gamma} - \frac{yf_1 f_2''}{2} + \frac{yf_2 f_1''}{3\gamma},\ a_2 = f_1 f_2' - \frac{2}{3}f_2 f'_1,$$

$$a_3 = \frac{-2yf_1{}'f_2''}{3\alpha\gamma} + \frac{y^3 f_1{}'f_2'''}{6\alpha\gamma^2} - \frac{y^2 f_1 f_2'''}{2\alpha\gamma} + \frac{y^3 f_1 f_2''''}{12\alpha\gamma^2} - \frac{y^2 f_1{}'f_2''}{6\alpha\gamma} + \frac{y^3 f_1{}'f_2'''}{36\alpha\gamma^2} + \frac{xyf_2{}'f_1''}{6\alpha\gamma} - \frac{y^3 f_2{}'f_1'''}{6\alpha\gamma^2} - \frac{4y^3 f_2 f_1''''}{72\alpha\gamma^2},$$

$$a_4 = f_1 f_3' - \frac{f_3 f_1'}{3},$$

$$\begin{aligned} a_5 &= \frac{xf_1' f_3'}{3\alpha} - \frac{y^2 f_1'' f_3'}{6\alpha\gamma} - \frac{2y^2 f_1' f_3''}{3\alpha\gamma} + \frac{xf_1 f_3''}{\alpha} - \frac{y^2 f_1 f_3'''}{2\alpha\gamma} - \frac{xf_3 f_1''}{3\alpha} + \frac{y^2 f_3 f_1'''}{2\alpha\gamma} + \frac{xf_1' f_3'}{3\alpha} - \frac{y^2 f_1' f_3''}{6\alpha\gamma}, \\ a_6 &= yf_5 f_1' + f_1 f_4' + \frac{1}{3}f_4 f_1' + yf_1 f_5' \end{aligned} \tag{3}$$

$$a_7 = \frac{-y}{3\alpha\gamma}f_2' f_3' - \frac{2y}{3\alpha\gamma}f_2 f_3''.$$

Table 1 Commutator table

	X_1	X_2	X_3	X_4
X_1	0	$a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial u}$	$a_4\frac{\partial}{\partial x} + a_5\frac{\partial}{\partial u}$	$a_6\frac{\partial}{\partial u}$
X_2	$-a_1\frac{\partial}{\partial x} - a_2\frac{\partial}{\partial y} - a_3\frac{\partial}{\partial u}$	0	$a_7\frac{\partial}{\partial u}$	$f_2 f_5\frac{\partial}{\partial u}$
X_3	$-a_4\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u}$	$-\frac{f_5'}{t}\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} - a_5\frac{\partial}{\partial u}$	0	0
X_4	$-a_6\frac{\partial}{\partial u}$	$-f_2 f_5\frac{\partial}{\partial u}$	0	0

Simplifying Table 1 by setting the values for a's, generates a nonlinear system of ODEs;

$$f_2 = f_1 f'_2 - \frac{2}{3} f'_1 f_2,\ f_3 = f_1 f'_3 - \frac{1}{3} f'_1 f_3,$$
$$f'_2 = \frac{1}{3} f'_2 f'_1 + f_1 f''_2 - \frac{2}{3} f_2 f''_1,$$
$$f_4 + y f_5 = f_2 f_5, \tag{4}$$
$$f_4 + y f_5 = y f'_1 f_5 + y f'_5 f_1 + \frac{1}{3} f'_1 f_4,$$

Solving this system of differential equations manually and the assumption of some values, results in;

$$f_1 = 3t$$

$$f_5 = \frac{1}{18}\frac{y(9t^{2/3} - 4)}{\alpha\gamma t^{5/3}},\ f_4 = \frac{1}{18}\frac{y(9t^{5/3} - 9yt^{2/3} - 4t + 4y)}{\alpha\gamma t^{5/3}} \tag{5}$$

$f_3 = \frac{1}{2}t + t^{\frac{1}{3}},\ f_2 = t$

Substituting from (5) in (2), we explore the four unknown Lie infinitesimals then simplifies Table 1 to an optimized form described in Table 2.

Theorem 1 If we start from the commutative product between the known vectors and through algebraic calculations, we can say the Lie infinitesimals for the (2 + 1)-dimensional potential Kadomtsev-Petviashvili (pKP) are;

$$\begin{cases} X_1 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \\ X_2 = \frac{-y}{2\gamma}\frac{\partial}{\partial x} + t\frac{\partial}{\partial y}, \\ X_3 = \left(\frac{1}{2}t + t^{\frac{1}{3}}\right)\frac{\partial}{\partial x} + \left(\frac{x}{3}\left(\frac{1}{2} + \frac{1}{3}t^{\frac{-2}{3}}\right) + \frac{y^2}{9\alpha\gamma t^{\frac{5}{3}}}\right)\frac{\partial}{\partial u}, \\ X_4 = \left(\frac{1}{18}\frac{y(9t^{5/3} - 9yt^{2/3} - 4t + 4y)}{\alpha\gamma t^{5/3}} + \frac{1}{18}\frac{y^2(9t^{2/3} - 4)}{\alpha\gamma t^{5/3}}\right)\frac{\partial}{\partial u}. \end{cases} \tag{6}$$

3 Reduction the Independent, Dependent Variables and Generating new Solutions for the (2 + 1) pKP Equation

Suppose **that**, $\alpha = -3$, $\beta = \gamma = 1$ and substitute in (1) and the infinitesimals in Eq. (6).

Table 2 Commutator table after optimization

	X_1	X_2	X_3	X_4
X_1	0	X_2	X_3	X_4
X_2	$-X_2$	0	X_4	X_4
X_3	$-X_3$	$-X_4$	0	0
X_4	$-X_4$	$-X_4$	0	0

3.1 Case I

Using the transformation $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 2\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$

So, the characteristic equation will be;

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dt}{2} = \frac{du}{1} \tag{7}$$

Solving this equation leads to;

$$u(t, x, y) = F(r, s) + x \tag{8}$$

where $F(r, s)$ is a new independent variable and;

$$r = -x + y, s = -2x + t \tag{9}$$

Using the new similarity and the dependent variables, Eq. (1) will be reduced to;

$$\begin{aligned} &F_{rrrr} + 8F_{rrrs} + 24F_{rrss} + 32F_{rsss} + 16F_{ssss} + 3F_{rr}F_r + 6F_sF_{rr} - 2F_{rr} \\ &+ 12F_rF_{rs} + 24F_sF_{rs} - 13F_{rs} + 12F_rF_{ss} + 24F_sF_{ss} - 14F_{ss} = 0 \end{aligned} \tag{10}$$

The reduced Eq. (10) has four Lie vectors; we can select a combination between them as follow;

$$V = \frac{\partial}{\partial r} + \frac{\partial}{\partial s} + \frac{\partial}{\partial F}. \tag{11}$$

Using this vector, so, Eq. (10) can be reduced to the following nonlinear ODE;

$$\theta_{\eta\eta\eta\eta} + 3\theta_{\eta\eta}\theta_{\eta} = 0 \tag{12}$$

Where

$$\eta = -r + s, \theta(\eta) = F(r, s) - r \tag{13}$$

This equation has no analytical solution, so we try to solve it using the integrating factors as follow;

- **Integrating factors**.

Integrate (12) once with respect to η, set the integration constant equal to zero;

$$\theta_{\eta\eta\eta} = -\frac{3}{2}\theta_{\eta}^{2} \tag{14}$$

Secondly, multiply Eq. (12) by (θ_η) then integrate once with respect toη,

$$\theta_{\eta\eta\eta} = \frac{1}{2\theta_\eta}(-2\theta_\eta^3 + \theta_{\eta\eta}^2) \tag{15}$$

Equating Eq. (14) to Eq. (15), results in;

$$\theta_\eta^3 + \theta_{\eta\eta}^2 = 0 \tag{16}$$

By solving this equation;

$$\theta(\eta) = \frac{4}{\eta + \boldsymbol{c}_1} + \boldsymbol{c}_2,$$

Then back substitution using (13), (9) and (8), we could obtain an invariant solution of the pKP equation of the form;

$$u(t, x, y) = \left(\frac{4}{-x - y + t + \boldsymbol{c}_1} + \boldsymbol{c}_2\right) + \boldsymbol{y} \tag{17}$$

The result is depicted in Fig. 1.

The profile plots show one soliton solution which that by increasing the time the amplitude is decreasing and the wave moving without and distortion in it.

3.2 Case II

Using the transformation $\boldsymbol{\xi}$ $= x + y + t$, $u(x, y, t) = u(\xi)$in Eq. (1);

So, the Eq. (1) will be;

$$u_{\xi\xi} + \alpha u_\xi u_{\xi\xi} + \beta u_{\xi\xi\xi\xi} + \gamma u_{\xi\xi} = 0 \tag{18}$$

Solving this equation using **Riccati equation method and follow the algorithm in** [26];

Assume

$$u(\xi) = \varphi'(\xi) = a\varphi(\xi)^2 + b\varphi(\xi) + c \tag{19}$$

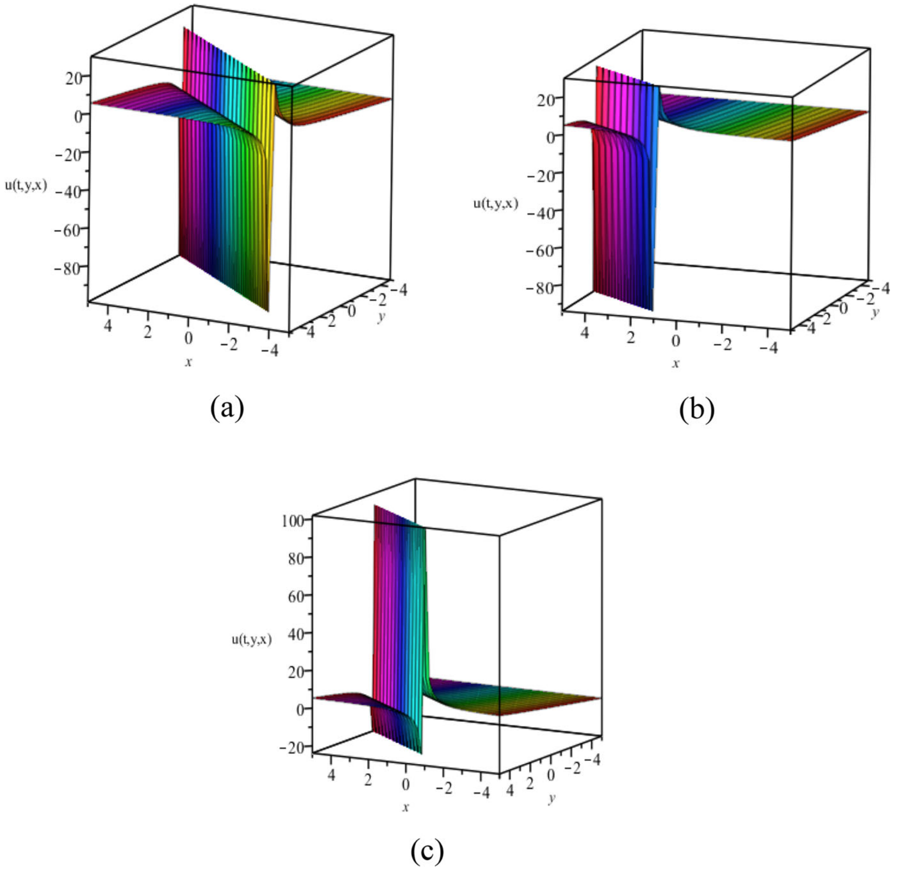

Fig. 1 Three dimensional plots for $u(t,\ y,\ x)$ at $\boldsymbol{c}_1 = \boldsymbol{c}_2 = 1$ and **a** $t = 0$, **b** $t = 3$ and**c**$t = 5$

where $\varphi(\xi)$ satisfies the Riccati equation and $a,\ b,\ c$ are constants which will determinate later;

Substituting from (19) in to (18) and set the coefficients of$\varphi(\xi)$to be zero, we obtain an algebraic system in $a,\ b,\ c$. By Solving the determinate system, we obtain sets of solutions;

$$\begin{cases} bc\left(1 + b^2\beta + c(\alpha + 8a\beta) + \lambda\right) = 0, \\ \left(b^4\beta + 2ac(1 + c(\alpha + 8a\beta) + \lambda) + b^2(1 + 2c(\alpha + 11a\beta) + \lambda)\right) = 0, \\ b\left(b^2\alpha + 60a^2c\beta + 3a\left(1 + 2c\alpha + 5b^2\beta + \lambda\right)\right) = 0, \\ 2a(2b^2\alpha + 20a^2c\beta + a(1 + 2c\alpha + 25b^2\beta + \lambda)) = 0, \\ 5a^2b(\alpha + 12a\beta) = 0, \\ 2a^3(\alpha + 12a\beta = 0. \end{cases} \tag{20}$$

This system has some solutions as;

Case 1

$$\beta \neq 0,\ a = -\frac{\alpha}{12\beta},\ \alpha \neq 0,\ c = -\frac{3(1 + b^2\beta + \lambda)}{\alpha} \tag{21}$$

Substitute in Eq. (19) and solve the equation;

$$\varphi(\xi) = 6\left(\frac{b\beta}{\alpha} + \frac{\sqrt{\beta}\sqrt{1+\lambda}\tan\left[\frac{1}{2}\left(-\frac{\xi\sqrt{1+\lambda}}{\sqrt{\beta}} + 12\alpha\sqrt{\beta}\sqrt{1+\lambda}C\right)\right]}{\alpha}\right) \tag{22}$$

Back substitution stage;

$$u(t,\ y,\ x) = 6\left(\frac{b\beta}{\alpha} + \frac{\sqrt{\beta}\sqrt{1+\lambda}\tan\left[\frac{1}{2}\left(-\frac{(x+y+t)\sqrt{1+\lambda}}{\sqrt{\beta}} + 12\alpha\sqrt{\beta}\sqrt{1+\lambda}C\right)\right]}{\alpha}\right) \tag{23}$$

Case 2

$$\beta \neq 0,\ a = -\frac{\alpha}{12\beta},\ \alpha \neq 0,\ b = \mp\frac{2\sqrt{3}\sqrt{a + a\lambda}}{\sqrt{\alpha}},\ c = 0 \tag{24}$$

Substitute in Eq. (19) and solve the equation;

$$\varphi(\xi) = \frac{24\sqrt{3}e^{\frac{-\sqrt{3}\frac{\alpha}{6\beta}\xi(1+\gamma)}{\sqrt{\alpha}\sqrt{a(1+\gamma)}}}\beta\sqrt{-\frac{\alpha}{12\beta}(1+\gamma)}}{-e^{\frac{-2\sqrt{3}\alpha(1+\gamma)C}{\sqrt{-\frac{\alpha}{12\beta}(1+\gamma)}}} + e^{\frac{-\sqrt{3}\frac{\alpha}{6\beta}\xi(1+\gamma)}{\alpha\sqrt{-\frac{1}{12\beta}(1+\gamma)}}}\alpha^{3/2}} \tag{25}$$

Back substitution stage;

$$u(t,\ y,\ x) = \frac{24\sqrt{3}e^{\frac{-\sqrt{3}\frac{\alpha}{6\beta}(x+y+t)(1+\gamma)}{\sqrt{\alpha}\sqrt{-\frac{\alpha}{12\beta}(1+\gamma)}}}\beta\sqrt{-\frac{\alpha}{12\beta}(1+\gamma)}}{-e^{\frac{-2\sqrt{3}\alpha(1+\gamma)C}{\sqrt{-\frac{\alpha}{12\beta}(1+\gamma)}}} + e^{\frac{-\sqrt{3}\frac{\alpha}{6\beta}(x+y+t)(1+\gamma)}{\alpha\sqrt{-\frac{1}{12\beta}(1+\gamma)}}}\alpha^{3/2}} \tag{26}$$

The result in (26) plotted in Fig. 2, as follow;

The results depicted in Fig. 2 shows a train of peaks decreasing with time without and distorting in its properties.

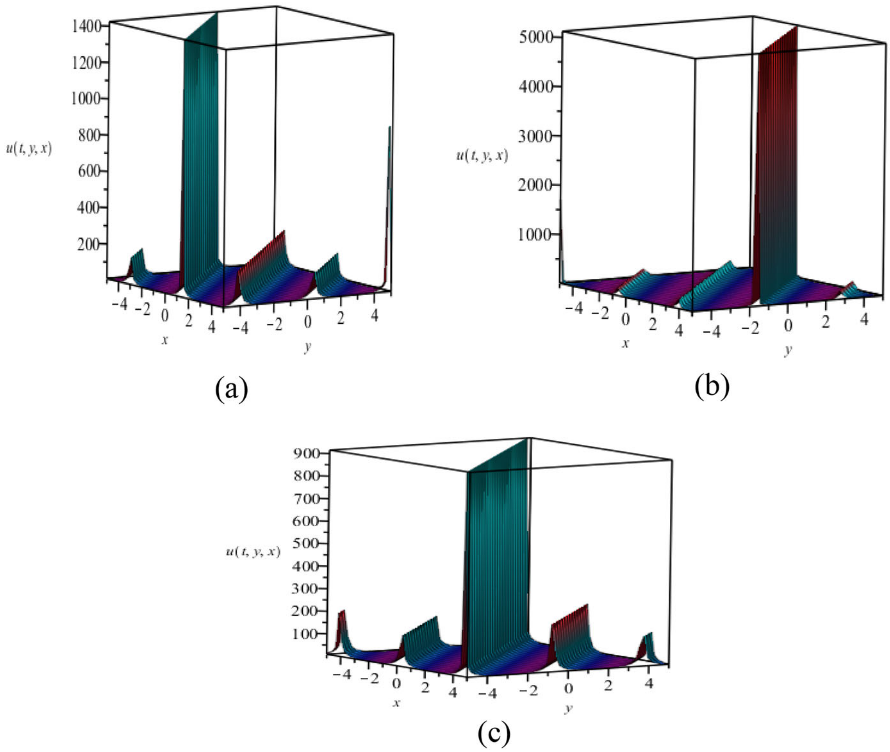

Fig. 2 Three dimensional plots for$u(t,\ y,\ x)$ at $\beta=1,\ \alpha=1,\ \gamma=1,\ c=1,$ **a** $t=0$, **b** $t=2$ and **c** $t=10$

4 Nonlocal Potential Transformation Method

Bluman and Kumei [27] explored a method for finding a nonlocally potential system when at least one equation of a given system of PDEs is written in a conserved form. This conserved form naturally leads to find the auxiliary potential variables and to construct an auxiliary system of PDEs which is the potential system. Nonlocally related systems are important in the construction of solutions for a given system of PDEs that arise from symmetry reductions due to the nonlocal symmetries but do not arise as invariant solutions from symmetry reductions due to point symmetries. In the following section, we will get the conservation laws for Eq. (1);

4.1 Conservative Forms for the (2 + 1)-dimensional pKP Equation

We start with detecting a set of local multipliers $\wedge_j(t,\ x,\ y,\ U)$ by applying a direct method based on Euler operator as follows;

$$E_u\big(\wedge_j(t,\ x,\ y,\ U)\big(U_{xt}+\alpha U_x U_{xx}+\beta U_{xxxx}+\gamma U_{yy}\big)\big)=0. \qquad (27)$$

Table 3 Conservation laws corresponding to the multipliers

Multiplier	Conservation law
$\wedge_1 = 1$	$(u_x)_t + \left(\frac{\alpha}{2}u_x{}^2 + \beta u_{xxx}\right)_x + \left(\gamma u_y\right)_y = 0.$
$\wedge_2 = t$	$(tu_x)_t + \left(\frac{\alpha}{2}tu_x{}^2 + \beta tu_{xxx} - u\right)_x + \left(\gamma tu_y\right)_y = 0.$
$\wedge_3 = y$	$(yu_x)_t + \left(\frac{\alpha}{2}yu_x{}^2 + \beta yu_{xxx}\right)_x + \left(\gamma yu_y - \gamma u\right)_y = 0.$
$\wedge_4 = ty$	$(tyu_x)_t + \left(\frac{\alpha}{2}tyu_x{}^2 + \beta tyu_{xxx} - yu\right)_x + \left(\gamma tyu_y - \gamma tu\right)_y = 0.$

Solving (26), with the aid of the symbolic computation software Maple, reveals a set of determining equations, whose solution leads to a family of conservation multipliers;

$$\wedge_j(t, x, y, U) = F1(t)y + F2(t) \tag{28}$$

where $F1$, and $F2$ are arbitrary functions. Consequently, we have the following local multipliers:

$$\wedge_1 = 1, \wedge_2 = t, \wedge_3 = y, \wedge_4 = ty. \tag{29}$$

Theorem 2 Conservation laws corresponding to multipliers $\wedge_j$, $j = 1, 0.4$ are presented in Table 3.

4.2 Potential Systems

Consider a scalar PDE which can be expressed as a conservation law:

$$\sum_{i=1}^{n} D_i f^i\left(x, u_{(k-1)}\right) = 0. \tag{30}$$

with independent variable $x = (x_1, x_2, \ldots, x_n)$ and a single dependent variable u; This conserved form allows us to introduce $(n-1)$ new variable $v = \left(v^{(1)}, v^{(2)}, \ldots, v^{(n-1)}\right)$. These $(n-1)$ variables are used to build up a new system of equations, which are the potential systems. Explicitly, we find:

$f^1\left(x, u_{(k-1)}\right) = v^{(1)}_{x_2}$,

$$f^l\left(x, u_{(k-1)}\right) = (-1)^{l-1}\left\{v^{(l)}_{x_{l+1}} + v^{(l-1)}_{x_{l-1}}\right\} 1 < l < n, \tag{31}$$

$f^n\left(x, u_{(k-1)}\right) = (-1)^{n-1} v^{(n-1)}_{x_{n-1}}$.

- For the first integrating multiplier $\wedge_1(t, x, y, u) = 1$,

$$\psi^1\left(t, x, y; u, v^1, v^2\right) = \begin{cases} u_x = v_x^{(1)}, \\ \frac{\alpha}{2}{u_x}^2 + \beta u_{xxx} = -\left(v_y^{(2)} + v_t^{(1)}\right), \\ \gamma u_y = v_x^{(2)}. \end{cases} \tag{32}$$

$$\psi^2\left(t, x, y; u, v^3, v^4\right) = \begin{cases} tu_x = v_x^{(3)}, \\ \frac{\alpha}{2}t{u_x}^2 + \beta tu_{xxx} - u = -\left(v_y^{(4)} + v_t^{(3)}\right), \\ \gamma tu_y = v_x^{(4)}. \end{cases} \tag{33}$$

$$\psi^3\left(t, x, y; u, v^5, v^6\right) = \begin{cases} yu_x = v_x^{(5)}, \\ \frac{\alpha}{2}y{u_x}^2 + \beta yu_{xxx} = -\left(v_y^{(6)} + v_t^{(5)}\right), \\ \gamma yu_y - \gamma u = v_x^{(6)}. \end{cases} \tag{34}$$

$$\psi^4\left(t, x, y; u, v^7, v^8\right) = \begin{cases} tyu_x = v_x^{(7)}, \\ \frac{\alpha}{2}ty{u_x}^2 + +\beta tyu_{xxx} - yu = -\left(v_y^{(8)} + v_t^{(7)}\right), \\ \gamma tyu_y - \gamma tu = v_x^{(8)}. \end{cases} \tag{35}$$

where, $\left(v^{(1)}, v^{(2)}, \ldots, v^{(8)}\right)$ are the potential variables.

4.3 Point Symmetries and Invariant Solutions of the Auxiliary Potential Systems:

Point symmetries and their invariant solutions corresponding to the four potential systems will be discussed in this section.

Theorem 3 *Invariant solutions corresponding to potential system.*

The symmetries of the potential system (32) ψ^1, are:

$$\begin{aligned} \xi_x &= \frac{c_1 x}{3} - \frac{y {f_1}'}{2\gamma} + f_2(t), \ \xi_y = \frac{2c_1 y}{3} + f_1(t), \ \xi_t = c_1 t + c_2, \ \eta_u \\ &= \frac{y^3 {f_1}'''}{12\alpha\gamma^2} - \frac{xy{f_1}''}{2\alpha\gamma} - \frac{y^2 {f_2}''}{2\alpha\gamma} + \frac{x{f_2}'}{\alpha} + yf_3(t) + f_4(t) - \frac{c_1 u}{3}, \\ \eta_{v1} &= -\frac{v_1 c_1}{3} - \frac{xy{f_1}''}{2\alpha\gamma} + \frac{x{f_2}'}{\alpha} + \frac{d}{dy} f_5(y, t), \ \eta_{v2} \\ &= \frac{xy^2 {f_1}'''}{4\alpha\gamma} - \frac{x^2 {f_1}''}{4\alpha\gamma} - \frac{xy{f_2}''}{\alpha\gamma} - \frac{{f_5}'}{12\gamma} + \frac{{f_1}'}{12\gamma}\left(-\frac{u}{2} + v_1\right) + \frac{xf_3(t)}{12} - \frac{v_2 c_1}{18} + \frac{f_6(t)}{12} \end{aligned} \tag{36}$$

Starting with the vector $S_1 = \frac{\partial}{\partial t}$ to reduce the potential system ψ^1 to a system of nonlinear ODEs

$$\begin{cases} \frac{\alpha}{2}\left(H1_\eta\right)^2 + \beta H1_{\eta\eta\eta} = 0 \\ H1_\eta = H2_\eta \\ \frac{\alpha}{2}\left(H1_\eta\right)^2 + \beta H1_{\eta\eta\eta} = 0 \\ H3_\eta = 0 \end{cases} \tag{37}$$

where $\eta = x$, $H1(\eta) = u(t, y, x)$, $H2(\eta) = v1(t, y, x)$, $H3(\eta) = u(t, y, x)$.Solving for $H1(\eta)$ using the integrating factor method. With the aid of maple, we deduce the Integrating Factors of (37)

$$\mu = H1_{\eta\eta} \tag{38}$$

Multiplying by μ Eq. (36) can be rewritten as

$$\begin{cases} \frac{\alpha}{3\beta}\left(H1_\eta\right)^3 + \left(H1_{\eta\eta}\right)^2 + C_1 = 0, \\ H1(\eta) = \left(-3C_1\beta\alpha^2\right)^{\frac{1}{3}} * \frac{\eta}{\alpha} + C_2 \end{cases} \tag{39}$$

Then back substitution for $\eta = x$, $H1(\eta) = u(t, y, x)$ we could obtain an invariant solution of the pKP equation of the form;

$$u(t, x, y) = \left(-3C_1\beta\alpha^2\right)^{\frac{1}{3}} \frac{x}{\alpha} + C_2 \tag{40}$$

In the same manner, we obtained also the following invariant solutions for the pKP equation

$$\begin{aligned} u(t, x, y) &= C_1, \\ u(t, x, y) &= F(t) \\ u(t, x, y) &= \frac{-6x}{\sqrt{-6\alpha}} - C_1 \\ u(t, x, y) &= \frac{12\beta}{\alpha(x + C_1)} + C_2 \\ u(t, x, y) &= -3\frac{(t - x)}{\alpha} + C_4 \\ u(t, x, y) &= \frac{(t - 3x - 1)y}{\gamma} + C_2 \\ u(t, y, x) &= \frac{(t - 3x - 1)y}{\gamma} + C_2 + x \end{aligned} \tag{41}$$

$$u(t,\ y,\ x) = \frac{3x}{2\alpha} - \frac{9t}{8\alpha} + \frac{C_2}{t^{1/3}}$$

$$u(t,\ y,\ x) = \frac{\left(-3C_1\beta\alpha^2\right)^{1/3} y}{\alpha} + C_2 + \frac{t^2 y}{2}$$

It is noted that there exist some solutions of the potential system that do not solve the original pKP Eq. (1), as

$$u(t,\ x,\ y) = \frac{\left(12C_2\beta tanh\left(\frac{C_2(-y+t)\sqrt{\left(-16C_2{}^2\beta\gamma+1\right)}+(2\gamma x-t+y)C_2+2C_1\gamma}{2\gamma}\right)\right) + C_3\alpha}{\alpha} \tag{42}$$

For the rest of auxiliary systems, we will mention only the Invariant solutions $u(t,\ x,\ y)$that satisfy the original pKP equation. The result in Eq. (42) is plotted in Fig. 3 as follow;

This result imply two and three Lump soliton solutions moves together. When time t increases to big enough, the lump solitary wave solution disappears.

Theorem 4 *Invariant solutions corresponding to potential system*

$$\begin{gathered} u(t,\ x,\ y) = C_1 \\ u(t,\ x,\ y) = F(t) \\ u(t,\ x,\ y) = \frac{C_3}{t^{\frac{1}{3}}} \end{gathered} \tag{43}$$

Theorem 5 *Invariant solutions corresponding to potential system*

$$\begin{gathered} u(t,\ x,\ y) = C_1 y \\ u(t,\ x,\ y) = \frac{C_1\gamma y + 1}{\gamma} \\ u(t,\ x,\ y) = \frac{C_1 y}{t} \\ u(t,\ x,\ y) = \frac{2C_2\alpha y + x^2}{2\alpha t} \\ u(t,\ x,\ y) = \left(-3C_1\beta\alpha^2\right)^{\frac{1}{3}} * \frac{x}{\alpha} + C_2 + \frac{t^2 y}{2} \end{gathered} \tag{44}$$

Theorem 5 *Invariant solutions corresponding to potential system*

$$u(t,\ x,\ y) = F(t) y$$

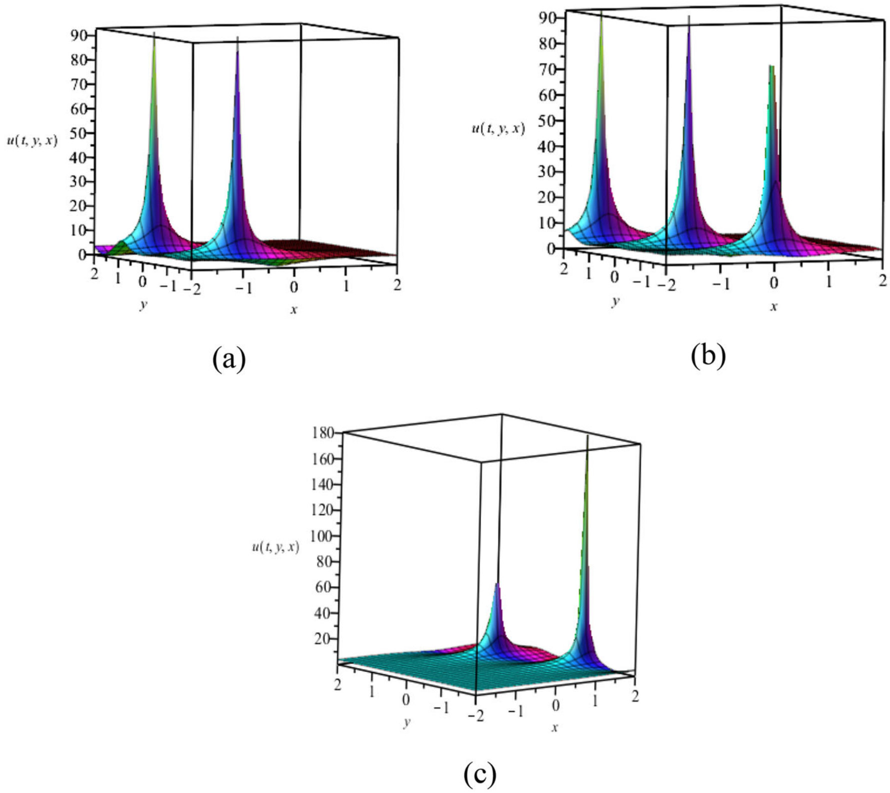

Fig. 3 Three dimensional plots for $u(t,\ y,\ x)$ at $\alpha = -3,\ \beta = 1,\ \gamma = 1,\ C_1 = 1,\ C_2 = 1,\ C_3 = 0$ and **a** $t = 0$, **b** $t = 1$, **c** $t = 5$

$$u(t,\ x,\ y) = \frac{t\left(C_2 \alpha y + x^2\right)\gamma - x y^2}{\alpha t^2 \gamma} \tag{45}$$

5 Conclusion

The (2 + 1) pKP equation) admits infinite number of infinitesimals through the commutative product between those vectors; we determined the optimal functions forms. Based on our method, we are examining new Lie vectors and via applying the integrating factor method single and double combinations of vectors, we are generating new solitary wave solutions.

We are exploiting a set of non-singular local multipliers; we present a set of local conservation laws for the studied equation. The nonlocally related (PDE) systems are found. Four nonlocally related systems are discussed reveal twenty-one interesting closed form solutions for this equation.

The summery of our results as follow;

- Figure (1.1,1.2, 1.3) shows one soliton solution that the amplitude decreasing with increasing the time and the wave peaks moves to lift direction.
- Figure (2.1, 2.2, 2.3) represent soliton wave peaks that decreasing with increasing time.
- Figure (3.1, 3.2) represent three Lump solitons solutions. They travel with decreasing in their amplitudes.

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