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An integrable generalization of the Kaup–Newell soliton hierarchy

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Abstract
A generalization of the Kaup–Newell spectral problem associated with $\mathrm{sl}(2, \mathbb{R})$ is introduced and the corresponding generalized Kaup–Newell hierarchy of soliton equations is generated. Bi-Hamiltonian structures of the resulting soliton hierarchy, leading to a common recursion operator, are furnished by using the trace identity, and thus, the Liouville integrability is shown for all systems in the new generalized soliton hierarchy. The involved bi-Hamiltonian property is explored by using the computer algebra system Maple.

Keywords: soliton equations, Hamiltonian structure, symmetries and conservation laws

1. Introduction

Soliton hierarchies are generated from matrix spectral problems or Lax pairs associated with matrix loop algebras (see, e.g., [1–5]). Among celebrated examples, with dependent variables less than three, are the Korteweg–de Vries hierarchy [6], the Ablowitz–Kaup–Newell–Segur hierarchy [7], the Dirac hierarchy [8], the Kaup–Newell hierarchy [9], the Wadati–Konno–Ichikawa hierarchy [10] and the Boiti–Pempinelli–Tu hierarchy [11]. They often possess bi-Hamiltonian structures and are Liouville integrable. The associated Hamiltonian structures can be furnished by the variational identity [12, 13], particularly by the trace identity when the underlying matrix loop algebra is semisimple [14].

Let us give an outline of the procedure for constructing soliton hierarchies by zero curvature equations (see, e.g., [14, 15]). The beginning is to take a matrix loop algebra $\tilde{\mathfrak{g}}$, associated with a matrix Lie algebra $\mathfrak{g}$ with the commutator

$$[A, B] = AB - BA, \quad A, B \in \mathfrak{g}, \tag{1.1}$$

and introduce spatial matrix spectral problem

$$\phi_x = U\phi, \quad U = U(u, \lambda) \in \tilde{\mathfrak{g}}, \tag{1.2}$$

where u is the potential vector and λ is the spectral parameter (see, e.g., [16] for examples of block matrix loop algebras). We then find a solution

$$W = W(u, \lambda) = \sum_{i\geqslant 0} W_{0,i}\lambda^{-i}, \quad W_{0,i} \in \mathfrak{g}, \quad i \geqslant 0, \tag{1.3}$$

to the stationary zero curvature equation

$$W_x = [U, W]. \tag{1.4}$$

Further, we try to formulate the temporal matrix spectral problems

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u, \lambda)\phi, \quad m \geqslant 0, \tag{1.5}$$

by introducing the Lax matrices

$$V^{[m]} = V^{[m]}(u, \lambda) = \left(\lambda^m W\right)_+ + \Delta_m, \quad \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geqslant 0,$$

P_+ denoting the polynomial part of P in λ, such that the compatibility conditions of equations (1.2) and (1.5), i.e., the zero curvature equations

$$U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0, \; m \geqslant 0, \tag{1.6}$$

will engender a hierarchy of soliton equations:

$$u_{t_m} = K_m(u), \;\; m \geqslant 0. \tag{1.7}$$

Such a soliton hierarchy usually possesses a recursion operator and Hamiltonian structures

$$u_{t_m} = K_m(u) = \Phi^m K_0 = J\frac{\delta \mathcal{H}_m}{\delta u}, \;\; m \geqslant 0. \tag{1.8}$$

The recursion operator Φ (see [17] for definition) is generated from the stationary zero curvature equation (1.4) and the Hamiltonian functionals $\mathcal{H}_m$'s can often be computed through the trace identity [14, 15]:

$$\begin{aligned} &\frac{\delta}{\delta u}\int \operatorname{tr}\left(\frac{\partial U}{\partial \lambda}W\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(\frac{\partial U}{\partial u}W\right),\\ &\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left|\operatorname{tr}\left(W^2\right)\right|, \end{aligned}\tag{1.9}$$

or generally, the variational identity [12, 13]:

$$\begin{aligned} &\frac{\delta}{\delta u}\int \left\langle \frac{\partial U}{\partial \lambda}, W\right\rangle \mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle \frac{\partial U}{\partial u}, W\right\rangle,\\ &\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle W, W\rangle|, \end{aligned}\tag{1.10}$$

where $\langle \cdot, \cdot\rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$ [12, 13].

We shall make use of the three-dimensional special linear Lie algebra $\mathrm{sl}(2, \mathbb{R})$, consisting of trace-free 2×2 matrices. This Lie algebra is simple and has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{1.11}$$

whose commutator relations are

$$[e_1, e_2] = 2e_2, \; [e_1, e_3] = -2e_3, \; [e_2, e_3] = e_1.$$

Its derived algebra is itself, and so, it is three-dimensional, too. The matrix loop algebra we shall adopt in what follows is

$$\begin{aligned} &\widetilde{\mathrm{sl}}(2, \mathbb{R})\\ &= \left\{\sum_{i\geqslant 0} M_i\lambda^{n-i} \,|\, M_i \in \mathrm{sl}(2, \mathbb{R}), \; i \geqslant 0, \; n \in \mathbb{Z}\right\}. \end{aligned}\tag{1.12}$$

It is the space of all Laurent series in λ with coefficients in $\mathrm{sl}(2, \mathbb{R})$ and a finite regular part. Particular examples of this matrix loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$ contain matrices of the form

$$\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$$

with arbitrary integers m, n, l. The loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$ lays a foundation for our study of soliton equations, from which many well-known soliton hierarchies are generated (see, e.g., [6–10]).

In this paper, we would like to consider a generalization of the Kaup–Newell spectral problem, which is associated with the matrix loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$, and compute an associated hierarchy of bi-Hamiltonian soliton equations by zero curvature equations. The corresponding bi-Hamiltonian structures will be furnished by applying the trace identity and a common recursion operator will be explicitly presented. All soliton equations in the resulting soliton hierarchy will be shown to be Liouville integrable. A few concluding remarks will round off the paper.

The resulting hierarchy provides another concrete example of soliton hierarchies associated with the matrix loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$. The generalized Kaup–Newell spectral problem is formulated by adding a multiple of the product of the two potentials to the e_1-part of the Kaup–Newell spectral problem, and interestingly, the perturbed spectral problem still generates an integrable hierarchy represented by local vector fields.

2. A generalization of the Kaup–Newell hierarchy

To present a generalized Kaup–Newell soliton hierarchy, let us introduce a new matrix spectral problem:

$$\phi_x = U\phi = U(u, \lambda)\phi, \;\; u = \begin{bmatrix} p \\ q \end{bmatrix}, \; \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.1}$$

with the spectral matrix U being chosen as

$$\begin{aligned} U &= \left(\lambda^2 + \alpha pq\right)e_1 + \lambda p e_2 + \lambda q e_3\\ &= \begin{bmatrix} \lambda^2 + \alpha pq & \lambda p \\ \lambda q & -\lambda^2 - \alpha pq \end{bmatrix} \in \widetilde{\mathrm{sl}}(2, \mathbb{R}), \end{aligned}\tag{2.2}$$

where α is an arbitrarily given constant. The case of $\alpha = 0$, where the perturbation αpq becomes zero, reduces to the Kaup–Newell spectral problem [9].

First, we solve the stationary zero curvature equation (1.4). The equation (1.4) reads as follows:

$$\begin{cases} a_x = \lambda(pc - qb),\\ b_x = 2\lambda^2 b - 2\lambda pa + 2\alpha pqb,\\ c_x = -2\lambda^2 c + 2\lambda qa - 2\alpha pqc, \end{cases}\tag{2.3}$$

if W is assumed to be

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2, \mathbb{R}). \tag{2.4}$$

Further let a, b and c possess the Laurent expansions:

$$\begin{aligned} a &= \sum_{i\geqslant 0} a_i\lambda^{-2i},\\ b &= \sum_{i\geqslant 0} b_i\lambda^{-2i-1},\\ c &= \sum_{i\geqslant 0} c_i\lambda^{-2i-1}, \end{aligned}\tag{2.5}$$

and take the initial data

$$a_0 = 1, \; b_0 = p, \; c_0 = q, \tag{2.6}$$

which are required by the equations on the highest powers of

λ in equation (2.3):

$$a_{0,x} = pc_0 - qb_0,\ b_0 = pa_0,\ c_0 = qa_0.$$

Then, the system in equation (2.3) becomes

$$\begin{cases} a_{i+1,x} = -\frac{1}{2}\left(qb_{i,x} + pc_{i,x}\right) + \alpha pq\left(qb_i - pc_i\right), \\ b_{i+1} = \frac{1}{2}b_{i,x} + pa_{i+1} - \alpha pqb_i, \\ c_{i+1} = -\frac{1}{2}c_{i,x} + qa_{i+1} - \alpha pqc_i, \end{cases} \quad i \geqslant 0, \quad (2.7)$$

the first of which is because from equation (2.3), we have

$$\begin{aligned} qb_x + pc_x &= q\left(2\lambda^2 b - 2\lambda pa + 2\alpha pqb\right) + p\left(-2\lambda^2 c + 2\lambda qa - 2\alpha pqc\right) \\ &= 2\lambda^2(qb - pc) + 2\alpha pq(qb - pc) \\ &= -2\lambda a_x + 2\alpha pq(qb - pc). \end{aligned}$$

The recursion relations in equation (2.7) also tell that

$$\begin{bmatrix} c_{i+1} \\ b_{i+1} \end{bmatrix} = L\begin{bmatrix} c_i \\ b_i \end{bmatrix},\ L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},\ i \geqslant 0, \quad (2.8)$$

where

$$\begin{cases} L_{11} = -\frac{1}{2}\partial - \alpha pq - \frac{1}{2}q\partial^{-1}p\partial - \alpha q\partial^{-1}p^2 q, \\ L_{12} = -\frac{1}{2}q\partial^{-1}q\partial + \alpha q\partial^{-1}pq^2, \\ L_{21} = -\frac{1}{2}p\partial^{-1}p\partial - \alpha p\partial^{-1}p^2 q, \\ L_{22} = \frac{1}{2}\partial - \alpha pq - \frac{1}{2}p\partial^{-1}q\partial + \alpha p\partial^{-1}pq^2. \end{cases} \quad (2.9)$$

While using the above recursion relations in equation (2.7), we impose the following conditions on constants of integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geqslant 1, \quad (2.10)$$

to determine the sequence of $\{a_i, b_i, c_i | i \geqslant 1\}$ uniquely. This way, the first two sets can be computed as follows:

$$\begin{aligned} a_1 &= -\frac{1}{2}pq,\ b_1 = \frac{1}{2}p_x - \frac{1}{2}p^2 q - \alpha p^2 q, \\ c_1 &= -\frac{1}{2}q_x - \frac{1}{2}pq^2 - \alpha pq^2; \\ a_2 &= \frac{1}{4}pq_x - \frac{1}{4}p_x q + \frac{3}{8}p^2 q^2 + \alpha p^2 q^2, \\ b_2 &= \frac{1}{4}p_{xx} - \frac{3}{4}pp_x q + \frac{3}{8}p^3 q^2 - \frac{3}{2}\alpha pp_x q \\ &\quad - \frac{1}{2}\alpha p^2 q_x + \alpha^2 p^3 q^2 + \frac{3}{2}\alpha p^3 q^2, \\ c_2 &= \frac{1}{4}q_{xx} + \frac{3}{4}pqq_x + \frac{3}{8}p^2 q^3 + \frac{1}{2}\alpha p_x q^2 \\ &\quad + \frac{3}{2}\alpha pqq_x + \alpha^2 p^2 q^3 + \frac{3}{2}\alpha p^2 q^3. \end{aligned}$$

We saw above the localness of the first two sets of $\{a_i, b_i, c_i | i \geqslant 1\}$. This is not an accident. Actually, we have the following result on the localness of the whole sequence of $\{a_i, b_i, c_i | i \geqslant 1\}$.

Proposition 2.1: *Let a_0, b_0, c_0 be given by equation (2.6). Then all functions $a_i, b_i, c_i,\ i \geqslant 1$, determined by equation (2.7) under the conditions in equation (2.10), are differential polynomials in u with respect to x, and so, they are all local.*

Proof. First from the stationary zero curvature equation $W_x = [U, W]$, we can compute that

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{tr}\left(W^2\right) = 2\operatorname{tr}\left(WW_x\right) = 2\operatorname{tr}(W[U, W]) = 0,$$

and so, noting $\operatorname{tr}\left(W^2\right) = 2\left(a^2 + bc\right)$, we have

$$a^2 + bc = \left(a^2 + bc\right)\Big|_{u=0} = 1, \quad (2.11)$$

the last step of which follows from the initial data in equation (2.6). Now through the Laurent expansions in equation (2.5), this gives

$$a_i = -\frac{1}{2}\sum_{k+l=i,\ k,l\geqslant 1} a_k a_l - \frac{1}{2}\sum_{k+l=i-1,\ k,l\geqslant 0} b_k b_l,\ i \geqslant 1. \quad (2.12)$$

Finally, based on this recursion relation in equation (2.12) and the last two recursion relations in equation (2.7), an application of the mathematical induction tells that all functions $a_i, b_i, c_i,\ i \geqslant 1$, are differential polynomials in u with respect to x, and thus, they are all local. The proof is finished. □

Now, envisaged by the recursion relations in equation (2.7), we, as usual, introduce

$$\begin{aligned} W^{[m]} &= \lambda\left(\lambda^{2m+1}W\right)_+ = \left(\lambda^{2m+2}W\right)_+ \\ &\quad - a_{m+1}e_1 \in \widetilde{\mathrm{sl}}(2, \mathbb{R}),\ m \geqslant 0, \end{aligned}$$

and consequently, we have

$$W_x^{[m]} - \left[U, W^{[m]}\right] = \begin{bmatrix} 0 & \lambda\left(b_{m,x} - 2\alpha pqb_m\right) \\ \lambda\left(c_{m,x} + 2\alpha pqc_m\right) & 0 \end{bmatrix},\ m \geqslant 0.$$

This is not the same type matrix as the Gateaux derivative operator U', and so, noting

$$\left[U, e_1\right] = -2\lambda pe_2 + 2\lambda qe_3,$$

we take a sequence of Lax operators with modification terms:

$$\begin{aligned} V^{[m]} &= \left(\lambda^{2m+2}W\right)_+ + \Delta_m \\ &= W^{[m]} + \delta_m e_1 \in \widetilde{\mathrm{sl}}(2, \mathbb{R}),\ m \geqslant 0, \end{aligned} \quad (2.13)$$

where $\delta_m,\ m \geqslant 0$, are functions to be determined and

$\Delta_m = (\delta_m - a_{m+1}) e_1,\ \ m \geqslant 0$. Then, we find that

$$
\begin{aligned}
& V_x^{[m]} - \left[U, V^{[m]}\right] \\
&= \begin{bmatrix} \delta_{m,x} & \lambda\left(b_{m,x} - 2\alpha pqb_m + 2p\delta_m\right) \\ \lambda\left(c_{m,x} + 2\alpha pqc_m - 2q\delta_m\right) & -\delta_{m,x} \end{bmatrix}, \\
& m \geqslant 0,
\end{aligned}
$$

and therefore, the corresponding zero curvature

$$U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0,\ \ m \geqslant 0, \tag{2.14}$$

equivalently engender a hierarchy of equations:

$$\begin{cases} p_{t_m} = b_{m,x} - 2\alpha pqb_m + 2p\delta_m, \\ q_{t_m} = c_{m,x} + 2\alpha pqc_m - 2q\delta_m,\ \ m \geqslant 0. \\ \alpha(pq)_{t_m} = \delta_{m,x}, \end{cases} \tag{2.15}$$

To make the third equation to be compatible with the other two equations in the above system in equation (2.15), we observe that

$$
\begin{aligned}
\delta_{m,x} &= \alpha\left(p_{t_m} q + pq_{t_m}\right) \\
&= \alpha\Big[q\left(b_{m,x} - 2\alpha pqb_m + 2p\delta_m\right) \\
&\quad + p\left(c_{m,x} + 2\alpha pqc_m - 2q\delta_m\right)\Big] \\
&= \alpha\Big[q\left(b_{m,x} - 2\alpha pqb_m\right) + p\left(c_{m,x} + 2\alpha pqc_m\right)\Big] \\
&= -2\alpha a_{m+1,x},\ \ m \geqslant 0,
\end{aligned}
$$

the last equality of which is a consequence of equation (2.7). Hence, we can take

$$\delta_m = -2\alpha a_{m+1},\ \ m \geqslant 0, \tag{2.16}$$

and then from the zero curvature equations (2.14), we obtain a hierarchy of soliton equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} - 2\alpha pqb_m - 4\alpha pa_{m+1} \\ c_{m,x} + 2\alpha pqc_m + 4\alpha qa_{m+1} \end{bmatrix},\ \ m \geqslant 0. \tag{2.17}$$

All equations in this hierarchy are local since all functions $a_i, b_i, c_i,\ \ i \geqslant 1$, are differential polynomials in u. The first nonlinear system in the soliton hierarchy equation (2.17) is as follows:

$$
\begin{aligned}
u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\
&= \begin{bmatrix} -\frac{1}{2}p_{xx} - pp_x q - \frac{1}{2}p^2 q_x - 2\alpha pp_x q \\ -2\alpha p^2 q_x - 2\alpha^2 p^3 q^2 - \frac{1}{2}\alpha p^3 q^2 \\ \frac{1}{2}q_{xx} - pqq_x - \frac{1}{2}p_x q^2 - 2\alpha p_x q^2 \\ -2\alpha pqq_x + 2\alpha^2 p^2 q^3 + \frac{1}{2}\alpha p^2 q^3 \end{bmatrix}.
\end{aligned} \tag{2.18}
$$

3. Liouville integrability

We shall show that all systems in the genelalized Kaup–Newell soliton hierarchy in equation (2.17) are Liouville integrable (see [18] for definition).

To this end, let us first furnish Hamiltonian structures for the hierarchy in equation (2.17). We shall use the trace identity in equation (1.9) (or generally, the variational identity in equation (1.10)). It is direct to see that

$$
\begin{aligned}
\frac{\partial U}{\partial \lambda} &= \begin{bmatrix} 2\lambda & p \\ q & -2\lambda \end{bmatrix}, \\
\frac{\partial U}{\partial p} &= \begin{bmatrix} \alpha q & \lambda \\ 0 & -\alpha q \end{bmatrix}, \\
\frac{\partial U}{\partial q} &= \begin{bmatrix} \alpha p & 0 \\ \lambda & -\alpha p \end{bmatrix},
\end{aligned}
$$

and so, we have

$$
\begin{aligned}
\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) &= 4\lambda a + qb + pc, \\
\operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) &= 2\alpha qa + \lambda c, \\
\operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) &= 2\alpha pa + \lambda b.
\end{aligned}
$$

Now, the trace identity equation (1.9) in this situation gives

$$
\begin{aligned}
&\frac{\delta}{\delta u}\int (4\lambda a + qb + pc)\mathrm{d}x \\
&= \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} 2\alpha qa + \lambda c \\ 2\alpha pa + \lambda b \end{bmatrix}.
\end{aligned} \tag{3.1}
$$

Balancing coefficients of $\lambda^{-2m-1},\ \ m \geqslant 0$, in the equality leads to

$$
\begin{aligned}
&\frac{\delta}{\delta u}\int \left(4a_{m+1} + qb_m + pc_m\right)\mathrm{d}x \\
&= (\gamma - 2m)\begin{bmatrix} 2\alpha qa_m + c_m \\ 2\alpha pa_m + b_m \end{bmatrix},\ \ m \geqslant 0.
\end{aligned}
$$

The identity with $m = 1$ tells $\gamma = 0$, and thus, we obtain

$$\frac{\delta}{\delta u}\mathcal{H}_m = \begin{bmatrix} 2\alpha qa_m + c_m \\ 2\alpha pa_m + b_m \end{bmatrix},\ \ m \geqslant 0, \tag{3.2}$$

with the Hamiltonian functionals being defined by

$$
\begin{aligned}
\mathcal{H}_0 &= \int (1 + 2\alpha)pq\ \mathrm{d}x, \\
\mathcal{H}_m &= \int\left(-\frac{4a_{m+1} + qb_m + pc_m}{2m}\right)\mathrm{d}x,\ \ m \geqslant 1,
\end{aligned} \tag{3.3}
$$

the first functional $\mathcal{H}_0$ of which was determined directly from the vector $\left(2\alpha qa_0 + c_0, 2\alpha pa_0 + b_0\right)^T$.

Now, noting that

$$\begin{bmatrix} c_m \\ b_m \end{bmatrix} = R \begin{bmatrix} 2\alpha q a_m + c_m \\ 2\alpha p a_m + b_m \end{bmatrix},$$

$$R = \begin{bmatrix} 1 - 2\alpha q \partial^{-1} p & 2\alpha q \partial^{-1} q \\ -2\alpha p \partial^{-1} p & 1 + 2\alpha p \partial^{-1} q \end{bmatrix}, \quad m \geqslant 0, \tag{3.4}$$

$$\begin{cases} p_{t_m} = 2b_{m+1} - 2pa_{m+1} - 4\alpha p a_{m+1}, \\ q_{t_m} = -2c_{m+1} + 2qa_{m+1} + 4\alpha a_{m+1}, \end{cases} m \geqslant 0, \tag{3.5}$$

and

$$a_{m+1} = \partial^{-1}\left(pc_{m+1} - qb_{m+1}\right), \quad m \geqslant 0, \tag{3.6}$$

we find that

$$K_m = \begin{bmatrix} -2(1+2\alpha)p\partial^{-1}p & 2 + 2(1+2\alpha)p\partial^{-1}q \\ -2 + 2(1+2\alpha)q\partial^{-1}p & -2(1+2\alpha)q\partial^{-1}q \end{bmatrix} \begin{bmatrix} c_{m+1} \\ b_{m+1} \end{bmatrix}$$
$$= J \begin{bmatrix} 2\alpha q a_{m+1} + c_{m+1} \\ 2\alpha p a_{m+1} + b_{m+1} \end{bmatrix}, m \geqslant 0. \tag{3.7}$$

with

$$J = \begin{bmatrix} -2(1+4\alpha)p\partial^{-1}p & 2 + 2(1+4\alpha)p\partial^{-1}q \\ -2 + 2(1+4\alpha)q\partial^{-1}p & -2(1+4\alpha)q\partial^{-1}q \end{bmatrix}, \tag{3.8}$$

which is a Hamiltonian operator [19]. It follows then that the soliton hierarchy in equation (2.17) has the Hamiltonian structures:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u}, \quad m \geqslant 0, \tag{3.9}$$

where the Hamiltonian operator is defined by equation (3.8) and the Hamiltonian functionals $\mathcal{H}_m$'s are given by equation (3.3).

The resulting functionals can generate infinitely many conservation laws for each soliton system in the soliton hierarchy in equation (2.17). We remark that differential polynomial type conservation laws can also be computed either by computer algebra codes (see, e.g., [20]) or from some Riccati equation associated with the underlying matrix spectral problem (see, e.g., [21–25]).

Obviously, we have

$$\frac{\delta \mathcal{H}_{m+1}}{\delta u} = \Psi \frac{\delta \mathcal{H}_m}{\delta u}, \; \Psi = R^{-1}LR, \tag{3.10}$$

where the inverse operator of R (see [19, 26] for other applications of such an inverse operator) can be computed as follows:

$$R^{-1} = \begin{bmatrix} 1 + 2\alpha q\partial^{-1}p & -2\alpha q \partial^{-1} q \\ 2\alpha p \partial^{-1} p & 1 - 2\alpha p \partial^{-1} q \end{bmatrix}. \tag{3.11}$$

Then, from $K_{m+1} = \Phi K_m, \; m \geqslant 0$, and $J\Psi = \Phi J$, we obtain a common recursion operator for the soliton hierarchy in equation (2.17):

$$\Phi = \Psi^{\dagger} = R^{\dagger} L^{\dagger} \left(R^{-1}\right)^{\dagger},$$

where $Q^{\dagger}$ denotes the adjoint operator of Q. This recursion operator can be explicitly computed as follows:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \tag{3.12}$$

where

$$\begin{cases} \Phi_{11} = \frac{1}{2}\partial - \alpha pq + \alpha p \partial^{-1} q \partial - \frac{1}{2}(1+2\alpha)\partial p \partial^{-1} q \\ \quad + \alpha(1+2\alpha)p^2 q \partial^{-1} q \\ \quad - 2\alpha^2 p \partial^{-1} p q^2 - \alpha(1+2\alpha)p\partial^{-1} q \partial p \partial^{-1} q \\ \quad - \alpha(1+2\alpha)p\partial^{-1} p \partial q \partial^{-1} q, \\ \Phi_{12} = -\alpha p \partial^{-1} p \partial - \frac{1}{2}(1+2\alpha)\partial p \partial^{-1} p \\ \quad + \alpha(1+2\alpha)p^2 q \partial^{-1} p - 2\alpha^2 p \partial^{-1} p^2 q \\ \quad - \alpha(1+2\alpha)p\partial^{-1} q \partial p \partial^{-1} p \\ \quad - \alpha(1+2\alpha)p\partial^{-1} p \partial q \partial^{-1} p, \\ \Phi_{21} = -\alpha q \partial^{-1} q \partial - \frac{1}{2}(1+2\alpha)\partial q \partial^{-1} q \\ \quad - \alpha(1+2\alpha)pq^2 \partial^{-1} q + 2\alpha^2 q \partial^{-1} p q^2 \\ \quad + \alpha(1+2\alpha)q\partial^{-1} q \partial p \partial^{-1} q \\ \quad + \alpha(1+2\alpha)q\partial^{-1} p \partial q \partial^{-1} q, \\ \Phi_{22} = -\frac{1}{2}\partial - \alpha pq + \alpha q \partial^{-1} p \partial - \frac{1}{2}(1+2\alpha)\partial q \partial^{-1} p \\ \quad - \alpha(1+2\alpha)pq^2 \partial^{-1} p \\ \quad + 2\alpha^2 q \partial^{-1} p^2 q + \alpha(1+2\alpha)q\partial^{-1} q \partial p \partial^{-1} p \\ \quad + \alpha(1+2\alpha)q\partial^{-1} p \partial q \partial^{-1} p. \end{cases} \tag{3.13}$$

Various recursion operators can be found through Lax representations or by computer algebra codes for nonlinear partial differential (see, e.g., [27, 28]). There are also direct algorithms for computing symmetries of nonlinear systems of differential equations (see, e.g., [29]).

It is now a straightforward computation to show that all members in the soliton hierarchy in equation (2.17) are bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u} = M\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geqslant 0, \tag{3.14}$$

where the second Hamiltonian operator (see [30, 31] for a bi-Hamiltonian theory) reads

$$M = \Phi J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{3.15}$$

with

$$\begin{cases} M_{11} = 2\alpha p\partial^{-1}p\partial - 2\alpha\partial p\partial^{-1}p \\ \quad + 4\alpha^2p^2q\partial^{-1}p + 4\alpha^2p\partial^{-1}p^2q \\ \quad - 4\alpha^2p\partial^{-1}p\partial q\partial^{-1}p - 4\alpha^2p\partial^{-1}q\partial p\partial^{-1}p, \\ M_{12} = \partial - 2\alpha pq + 2\alpha\partial p\partial^{-1}q + 2\alpha p\partial^{-1}q\partial \\ \quad - 4\alpha^2p\partial^{-1}pq^2 \\ \quad - 4\alpha^2p^2q\partial^{-1}q + 4\alpha^2p\partial^{-1}p\partial q\partial^{-1}q \\ \quad + 4\alpha^2p\partial^{-1}q\partial p\partial^{-1}q, \\ M_{21} = \partial + 2\alpha pq - 2\alpha q\partial^{-1}p\partial - 2\alpha\partial q\partial^{-1}p \\ \quad - 4\alpha^2pq^2\partial^{-1}p \\ \quad - 4\alpha^2q\partial^{-1}p^2q + 4\alpha^2q\partial^{-1}p\partial q\partial^{-1}p \\ \quad + 4\alpha^2q\partial^{-1}q\partial p\partial^{-1}p, \\ M_{22} = 2\alpha\partial q\partial^{-1}q - 2\alpha q\partial^{-1}q\partial + 4\alpha^2pq^2\partial^{-1}q \\ \quad + 4\alpha^2q\partial^{-1}pq^2 \\ \quad - 4\alpha^2q\partial^{-1}p\partial q\partial^{-1}q - 4\alpha^2q\partial^{-1}q\partial p\partial^{-1}q. \end{cases} \tag{3.16}$$

Thus, we see that the soliton hierarchy in equation (2.17) is Liouville integrable, upon noticing that the vector fields K_n, $n \geqslant 1$, possess distinct differential orders and that the common conserved functionals $\{\mathcal{H}_n\}_{n=0}^{\infty}$ and symmetries $\{K_n\}_{n=0}^{\infty}$ commute:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J \frac{\delta \mathcal{H}_l}{\delta u}\, \mathrm{d}x = 0, \quad k, l \geqslant 0, \tag{3.17}$$

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T M \frac{\delta \mathcal{H}_l}{\delta u}\, \mathrm{d}x = 0, \quad k, l \geqslant 0, \tag{3.18}$$

and

$$[K_k, K_l] = K_k{}'(u)[K_l] - K_l{}'(u)[K_k] = 0, \quad k, l \geqslant 0. \tag{3.19}$$

These commuting relations are also consequences of the Virasoro algebra of Lax operators [32–34], which is much easier to be established than a bi-Hamiltonian formulation since it is long to show that J and M constitute a Hamiltonian pair (see [30] for details on the bi-Hamiltonian theory).

4. Concluding remarks

Based on the matrix loop algebra $\tilde{\mathrm{sl}}(2, \mathbb{R})$, we introduced a generalization of the Kaup–Newell spectral problem, and generated a hierarchy of soliton differential equations from the associated zero curvature equations. The resulting soliton equations are bi-Hamiltonian and Liouville integrable, and their common Hamiltonian pair and recursion operator were explicitly given. The Kaup–Newell soliton hierarchy is the special case with $\alpha = 0$ of the resulting generalized soliton hierarchy.

A hierarchy of generalized AKNS equations was similarly made and its nonlinearization was carried out previously in [35]. Very recently, the special orthogonal Lie algebra so (3, $\mathbb{R}$) has also been used to generate new soliton hierarchies [36–38]. Among typical discussed spectral matrices are the following three:

$$U(u, \lambda) = \lambda e_1 + pe_2 + qe_3,$$
$$U(u, \lambda) = \lambda^2 e_1 + \lambda pe_2 + \lambda qe_3,$$
$$U(u, \lambda) = \lambda e_1 + \lambda pe_2 + \lambda qe_3,$$

where $u = (p, q)^T$ and e_1, e_2 and e_3 are the basis matrices of so (3, $\mathbb{R}$):

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are many other higher-order matrix spectral problems which lead to soliton hierarchies (see, e.g., [39–44]).

There has also been a growing interest in generating hierarchies of integrable couplings [45] from matrix spectral problems associated with non-semisimple matrix loop algebras [46]. Integrable couplings show very rich structures, bringing us inspiring thoughts and ideas to classify multi-component integrable systems [47]. Bi-integrable couplings and tri-integrable couplings do exhibit diverse nice structures on recursion operators in block matrix form [13, 47]. It should be significantly important to explore more algebraic and geometric mathematical structures on integrable couplings. There are many interesting questions on integrable couplings, which are open to us. For example, is there any Hamiltonian structure for the bi-integrable coupling

$$u_t = K(u), \quad v_t = K'(u)[v], \quad w_t = K'(u)[w],$$

where K' denotes the Gateaux derivative and $u_t = K$ is assumed to be Hamiltonian? How can one generally solve the perturbation system

$$u_t = K(u), \quad v_t = K'(u)[v],$$

even in the KdV case with $K(u) = 6uu_x + u_{xxx}$, which gives

$$u_t = 6uu_x + u_{xxx}, \quad v_t = 6(uv)_x + v_{xxx}?$$

There are, though, plenty of particular solutions to the perturbation system, and one immediate solution is to take v as a symmetry of $u_t = K(u)$.

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