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An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems

Wen-xiu Ma

CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China, Institute of Mathematics, Fudan University, Shanghai 200433, China [1]

Walter Strampp

Fachbereich 17, Mathematik, GH-Universität Kassel, W-3500 Kassel, Germany

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Abstract

An explicit symmetry constraint is proposed for the Lax pairs and the adjoint Lax pairs of AKNS systems. The corresponding Lax pairs and adjoint Lax pairs are nonlinearized into a hierarchy of commutative, finite-dimensional integrable Hamiltonian systems in the Liouville sense and thus an involutive representation of solutions of AKNS systems is obtained. The purpose of this Letter is to elucidate that the nonlinearization method (i.e. a kind of symmetry constraint method) of integrable systems can be applied to the Lax pairs and the adjoint Lax pairs associated with integrable systems.

1. Introduction

The symmetry constraints give the interrelations between higher-dimensional integrable systems and lower-dimensional integrable systems and provide a direct method to construct solutions of higher-dimensional integrable systems by solving lower-dimensional integrable systems. The interesting symmetry constraints are those on the eigenfunctions of Lax pairs and the potentials of integrable systems [1,2]. The theory includes two lines: constraining $(1+1)$-dimensional integrable systems to finite-dimensional integrable systems (see, for example, Refs. [1,3–6]) and constraining $(1+2)$-dimensional integrable systems to $(1+1)$-dimensional integrable systems (see, for example, Refs. [2,7,8]). Its key is to search for proper symmetry constraints. Two such simplest examples in the cases of the KdV and KP equations are as follows,

$$u_x(x,t)=\partial\sum_{j=1}^{N}E_j\phi_j^2,\qquad E_j=\text{const},\qquad \partial=\frac{\mathrm{d}}{\mathrm{d}x},\tag{1.1}$$

where ϕ_j are the eigenfunctions with eigenvalues λ_j governed by the spectral problem (i.e. the spatial part of the Lax pair) $\phi_{xx}=\lambda\phi+u\phi$, and

[1] Mailing address.

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$$u_x(x,y,t)=\partial \sum_{j=1}^{N} E_j\phi_j\psi_j, \qquad E_j=\mathrm{const}, \qquad \partial=\frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.2}$$

where ϕ_j and ψ_j are the eigenfunctions governed respectively by the spectral problem $\phi_y=\phi_{xx}+u\phi$ and the adjoint spectral problem $\psi_y=-\psi_{xx}-u\psi$.

It is known [9] that for a hierarchy of (1+1)-dimensional integrable systems

$$u_{t_n}=K_n=J\frac{\delta H_n}{\delta u}=JL^n\frac{\delta H_0}{\delta u}, \qquad n\geqslant 0, \tag{1.3}$$

possessing the Lax pairs $\phi_x=U(u,\lambda)\phi$, $\phi_{t_n}=V^{(n)}(u,\lambda)\phi$, the set of solutions of

$$\frac{\delta I_0}{\delta u}=\sum_{j=1}^{N} E_j\frac{\delta I_j}{\delta u}, \qquad E_j=\mathrm{const},$$

where I_j, $0\leqslant j\leqslant N$, and N conserved densities of (1.3), determines an invariant submanifold of (1.3) and thus leads to important symmetry constraints,

$$K_n=\sum_{j=1}^{N} E_jJ\frac{\delta\lambda_j}{\delta u} \qquad \text{or} \qquad \frac{\delta H_n}{\delta u}=\sum_{j=1}^{N} E_j\frac{\delta \lambda_j}{\delta u}, \qquad n\geqslant 0, \tag{1.4}$$

where λ_j, $1\leqslant N\leqslant N$, are all the eigenvalues of the spectral problem $\phi_x=U(u,\lambda)\phi$. Under the constraint of (1.4), the Lax pairs of (1.3) are nonlinearized into a finite-dimensional integrable Hamiltonian system and a hierarchy of finite-dimensional integrable Hamiltonian systems, together with the symmetry constraint (1.4). Moreover the flows of these systems are certain to commute mutually.

In this Letter we shall show by an example of the AKNS case that the symmetry constraint (1.4) can be applied to discuss the nonlinearization problem of the Lax pairs and the adjoint Lax pairs associated with (1+1)-dimensional integrable systems. By taking the Bargmann constraint associated with AKNS systems [10], we successfully generate a new involutive system with $4N$ dependent variables and thus establish an involutive representation of solutions of AKNS systems. In Section 2 we first propose some required fundamental properties related to the adjoint representation equation $V_x=[U,V]$. Then, in Section 3, we consider the Bargmann constraint problem of the Lax pairs and the adjoint Lax pairs of the AKNS systems in detail.

2. Some basic properties

Let $\mathscr{B}$ denote the differential algebra of differential functions $u=u(x,t)$ and write

$$\mathscr{V}_0^r=\sum_{k=0}^{\infty}\mathscr{V}_{(k)}^r, \qquad \mathscr{V}_{(k)}^r=\{(P^{ij}\partial^k)_{r\times r}\mid P^{ij}\in\mathscr{B}\}, \qquad \partial=\frac{\mathrm{d}}{\mathrm{d}x},$$

$$\tilde{\mathscr{V}}_0^r=\mathscr{V}_0^r\otimes\mathrm{C}[\lambda,\lambda^{-1}]=\sum_{k=0}^{\infty}\tilde{\mathscr{V}}_{(k)}^r, \qquad \tilde{\mathscr{V}}_{(k)}^r=\mathscr{V}_{(k)}^r\otimes\mathrm{C}[\lambda,\lambda^{-1}].$$

In this section we want to discuss some basic properties connected with a general spectral problem,

$$\phi_x=U\phi=U(u,\lambda)\phi, \qquad U\in\tilde{\mathscr{V}}_{(0)}^r, \tag{2.1}$$

and its adjoint representation equation

$$V_x=[U,V], \qquad V=V(u,\lambda)\in\tilde{\mathscr{V}}_{(0)}^r. \tag{2.2}$$

Proposition 2.1. Let $U, V, W\in\tilde{\mathscr{V}}_{(0)}^r$. If the adjoint representation equation (2.2) and the zero curvature equation

$$U_t - W_x + [U, W] = 0 \tag{2.3}$$

hold, then the matrix $\Lambda = V_t - [W, V]$ also satisfies the adjoint representation equation (2.2).

Proof. By using (2.2) and (2.3), we have

$$\begin{aligned}\Lambda_x &= V_{tx} - [W_x, V] - [W, V_x] = [U_t, V] + [U, V_t] - [W_x, V] - [W, [U, V]] \\ &= [W_x - [U, W], V] + [U, V_t] - [W_x, V] - [W, [U, V]] = -[[V, W], U] + [U, V_t] = [U, \Lambda] .\end{aligned}$$

The proof is completed.

We note that the spectral problem (2.1) generally possesses the uniqueness property [11]: if $V_x = [U, V]$, $V \in \tilde{\mathscr{V}}^r_{(0)}$ and $V|_{u=0} = 0$, then $V = 0$. This property corresponds to the uniqueness of the associated integrable hierarchy of (2.1) and thus we call it the uniqueness property. The following result is a direct corollary of proposition 2.1 and the uniqueness property.

Corollary 2.1. For the spectral problem (2.1) possessing the above uniqueness property, we have $\Lambda = V_t - [W, V] = 0$ if $\Lambda|_{u=0} = (V_t - [W, V])|_{u=0} = 0$.

The above result plays a dominant role in the deduction of commutability of the associated flows and the constrained flows.

Proposition 2.2. If $U, V \in \tilde{\mathscr{V}}^r_{(0)}$ satisfy the adjoint representation equation (2.2), then we have $(V^n)_x = [U, V^n]$, $n \geqslant 1$.

Proof. It follows from $V_x = [U, V]$ that

$$(V^n)_x = \sum_{i=1}^{n} \underbrace{V \ldots \overset{i}{V_x} \ldots V}_{n} = \sum_{i=1}^{n} \underbrace{V \ldots \overset{i}{V(UV - VU)} V \ldots V}_{n} = UV^n - V^n U = [U, V^n] , \qquad n \geqslant 1 ,$$

which completes the proof.

For the spectral problem (2.1) we introduce its adjoint spectral problem

$$\psi_x = U^* \psi = -U^{\mathrm{T}} \psi , \tag{2.4}$$

where T means transposition of the matrix. Obviously

$$U_t - W_x + [U, W] = 0 \quad \Leftrightarrow \quad U^*_t - W^*_x + [U^*, W^*] = 0, \qquad W^* = -W^{\mathrm{T}} . \tag{2.5}$$

Moreover, we can easily reach the following conclusion.

Proposition 2.3. Let $\phi = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ and $\psi = (\psi_1, \ldots, \psi_r)^{\mathrm{T}}$ satisfy (2.1) and (2.4), respectively, and set $\bar{V} = \phi \psi^{\mathrm{T}} = (\phi_i \psi_j)_{r \times r}$. Then we have $\bar{V}_x = [U, \bar{V}]$.

Proposition 2.4. Let $U, V, W \in \tilde{\mathscr{V}}^r_{(0)}$. Then the compatibility condition of the system

$$V_x = [U, V], \qquad V_t = [W, V] \tag{2.6}$$

is the equation

$$[U_t - W_x + [U, W], V] = 0 . \tag{2.7}$$

Proof. By (2.6), we have

$$V_{xt}=[U_t,V]+[U,V_t]=[U_t,V]+[U,[W,V]],\qquad V_{tx}=[W_x,V]+[W,V_x]=[W_x,V]+[W,[U,V]].$$

Therefore $V_{tx}=V_{xt}$ is equivalent to Eq. (2.7). The proof is completed.

This proposition shows that when $U_t-W_x+[U,W]=0$, (2.6) is integrable for V.

3. A symmetry constraints of the AKNS case

We consider the AKNS spectral problem

$$\phi_x=U\phi,\qquad U=U(u,\lambda)=\begin{pmatrix}-\lambda & v\\ w & \lambda\end{pmatrix},\qquad \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\qquad u=\begin{pmatrix}v\\ w\end{pmatrix},\tag{3.1}$$

and its adjoint spectral problem

$$\psi_x=U^*\psi,\qquad U^*=-U^{\mathrm{T}}=\begin{pmatrix}\lambda & -w\\ -v & -\lambda\end{pmatrix},\qquad \psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.\tag{3.2}$$

Let us first recall the construction of AKNS systems [10,12]. Setting

$$V=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}=\sum_{i=0}^{\infty}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{-i},\tag{3.3}$$

we see that the adjoint representation equation $V_x=[U,V]$ becomes

$$a_{ix}=vc_i-wb_i,\qquad i\geqslant 0,\tag{3.4a}$$

$$b_{ix}=-2b_{i+1}-2va_i,\qquad i\geqslant 0,\tag{3.4b}$$

$$c_{ix}=2c_{i+1}+2wa_i,\qquad i\geqslant 0.\tag{3.4c}$$

We choose $a_0=-1$, $b_0=c_0=0$ and assume $a_i|_{u=0}=b_i|_{u=0}=c_i|_{u=0}=0$, $i\geqslant 1$ (or equivalently select constants of integration to be zero). In this way, the recursion relation (3.4) uniquely gives a series of polynomial functions with respect to u, u_x, ... For example, we have

$$a_1=0,\qquad b_1=v,\qquad c_1=w,\qquad a_2=\tfrac{1}{2}(vw),\qquad b_2=-\tfrac{1}{2}v_x,\qquad c_2=\tfrac{1}{2}w_x.$$

Moreover we have $a^2+bc=1$ since $(a^2+bc)_x=\frac{1}{2}\operatorname{tr}(V^2)_x=\frac{1}{2}\operatorname{tr}[U,V^2]=0$ and $(a^2+bc)|_{u=0}=1$. At this point, the compatibility conditions of Lax pairs,

$$\phi_x=U\phi,\qquad \phi_{t_n}=V^{(n)}\phi,\qquad V^{(n)}=(\lambda^nV)_+,\qquad n\geqslant 0,\tag{3.5}$$

where the plus denotes the choice of the non-negative power of λ, determine a hierarchy of AKNS systems,

$$u_{t_n}=\begin{pmatrix}v\\ w\end{pmatrix}_{t_n}=K_n=\begin{pmatrix}-2b_{n+1}\\ 2c_{n+1}\end{pmatrix}=JL^n\begin{pmatrix}w\\ v\end{pmatrix}=J\frac{\delta H_n}{\delta u},\qquad n\geqslant 0,\tag{3.6}$$

where the Hamiltonian operator J, the recursion operator L and the Hamiltonian functions H_n read as

$$J=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix},\qquad L=\begin{pmatrix}\frac{1}{2}\partial-w\partial^{-1}v & w\partial^{-1}w\\ -v\partial^{-1}v & -\frac{1}{2}\partial+v\partial^{-1}w\end{pmatrix},\qquad H_n=\frac{2}{n+1}a_{n+2},\qquad n\geqslant 0.\tag{3.7}$$

When $u_{t_n}=K_n$, i.e. $U_{t_n}-V_x^{(n)}+[U, V^{(n)}]=0$, $n\geqslant 0$, by corollary 2.1 we can get [13]

$$V_{t_n}=[V^{(n)}, V], \qquad n\geqslant 0, \tag{3.8}$$

which underlines the commutability of the constrained systems. It follows from (3.8) and proposition 2.4 that $[V_{t_n}^{(m)}-V_{t_m}^{(n)}+[V^{(m)}, V^{(n)}], V]=0$. In fact, we have [11] $V_{t_n}^{(m)}-V_{t_m}^{(n)}+[V^{(m)}, V^{(n)}]=0$, which implies the commutability of the flows of (3.6).

Let us now turn to the nonlinearization problem of the Lax pairs and the adjoint Lax pairs of AKNS systems. From (3.1) and (3.2) we can directly obtain

$$\frac{\delta\lambda}{\delta u}=\frac{1}{E}\begin{pmatrix}\phi_2\psi_1\\ \phi_1\psi_2\end{pmatrix}, \qquad E=\int_{-\infty}^{\infty}(-\phi_1\psi_1+\phi_2\psi_2)\,\mathrm{d}x, \tag{3.9}$$

which is exactly what we need in the symmetry constraint (1.4). When $\lim_{|x|\to\infty}\phi=\lim_{|x|\to\infty}\psi=0$ we can show that

$$L\begin{pmatrix}\phi_2\psi_1\\ \phi_1\psi_2\end{pmatrix}=\lambda\begin{pmatrix}\phi_2\psi_1\\ \phi_1\psi_2\end{pmatrix}. \tag{3.10}$$

Therefore $(\phi_2\psi_1, \phi_1\psi_2)^{\mathrm{T}}$ belongs to the invariant space of the recursion operator L, which is a crucial point in the nonlinearization method. Introducing N distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$, we have

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}_x=U(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}, \qquad j=1, 2, ..., N, \tag{3.11a}$$

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_x=-U^{\mathrm{T}}(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}, \qquad j=1, 2, ..., N, \tag{3.11b}$$

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}_{t_n}=V^{(n)}(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}, \qquad j=1, 2, ..., N, \tag{3.12a}$$

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_{t_n}=-(V^{(n)})^{\mathrm{T}}(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}, \qquad j=1, 2, ..., N. \tag{3.12b}$$

The compatibility condition of (3.11) and (3.12) is the nth AKNS system $u_{t_n}=K_n$.

We impose the finite-dimensional invariant space for the AKNS flows

$$\frac{\delta H_0}{\delta u}=\sum_{j=1}^{N}E_j\frac{\delta\lambda_j}{\delta u}, \qquad E_j=\int_{-\infty}^{\infty}(-\phi_{1j}\psi_{1j}+\phi_{2j}\psi_{2j})\,\mathrm{d}x, \tag{3.13}$$

which engenders an explicit symmetry constraint

$$J\begin{pmatrix}w\\ v\end{pmatrix}=J\begin{pmatrix}\langle\Psi_1,\Phi_2\rangle\\ \langle\Psi_2,\Phi_1\rangle\end{pmatrix} \quad \text{or} \quad u=\begin{pmatrix}v\\ w\end{pmatrix}=\begin{pmatrix}\langle\Psi_2,\Phi_1\rangle\\ \langle\Psi_1,\Phi_2\rangle\end{pmatrix}, \tag{3.14}$$

where $\Psi_i=(\psi_{i1}, \psi_{i2}, ..., \psi_{iN})^{\mathrm{T}}$, $\Phi_i=(\phi_{i1}, \phi_{i2}, ..., \phi_{iN})^{\mathrm{T}}$, $i=1, 2$ and $\langle\ ,\ \rangle$ denotes the standard inner product of $\mathbb{R}^N$. By substituting (3.14) into Lax pairs (3.11) and (3.12) we obtain the constrained Lax pairs (or nonlinearized Lax pairs)

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}_x=U(u,\lambda_j)|_{\mathrm{B}}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\end{pmatrix}, \qquad j=1, 2, ..., N, \tag{3.15a}$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = -U^{\mathrm{T}}(u, \lambda_j)|_{\mathrm{B}} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j=1,2,\ldots,N, \tag{3.15b}$$

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda_j)|_{\mathrm{B}} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \quad j=1,2,\ldots,N, \tag{3.16a}$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -(V^{(n)})^{\mathrm{T}}(u, \lambda_j)|_{\mathrm{B}} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad j=1,2,\ldots,N, \tag{3.16b}$$

where the subscript B means substitution of (3.14) into the expression. System (3.15) is an ordinary differential equation and systems (3.16) for $n \geqslant 0$ are a hierarchy of nonlinear partial differential equations. Evidently, (3.15) may be expressed as the Hamiltonian form

$$\Psi_{ix} = -\frac{\partial H}{\partial \Psi_i}, \quad \Phi_{ix} = \frac{\partial H}{\partial \Psi_i}, \quad i=1,2, \tag{3.17}$$

with the Hamiltonian function $H = \langle A\Psi_2, \Phi_2 \rangle - \langle A\Psi_1, \Phi_1 \rangle + \langle \Psi_2, \Phi_1 \rangle \langle \Psi_1, \Phi_2 \rangle$, $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. In what follows we want to show that (3.17) is a finite-dimensional integrable system in the Liouville sense [14] and that under the control of (3.17), systems (3.16) for $n \geqslant 0$ are also a hierarchy of finite-dimensional integrable systems in the Liouville sense [14].

By using (3.10) under the constraint (3.14) we have

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = L^n \begin{pmatrix} w \\ v \end{pmatrix} = L^n \begin{pmatrix} \langle \Psi_1, \Phi_2 \rangle \\ \langle \Psi_2, \Phi_1 \rangle \end{pmatrix} = L^{n-1} \begin{pmatrix} \langle A\Psi_1, \Phi_2 \rangle \\ \langle A\Psi_2, \Phi_1 \rangle \end{pmatrix} = \begin{pmatrix} \langle A^n\Psi_1, \Phi_2 \rangle \\ \langle A^n\Psi_2, \Phi_1 \rangle \end{pmatrix}, \quad n \geqslant 0, \tag{3.18a}$$

and thus

$$a_{n+1} = \partial^{-1}(vc_{n+1} - wb_{n+1}) = \tfrac{1}{2}(\langle A^n\Psi_1, \Phi_1 \rangle - \langle A^n\Psi_2, \Phi_2 \rangle), \quad n \geqslant 0. \tag{3.18b}$$

By proposition 2.2 we see that $(V^2)_x = [U, V^2]$. Therefore

$$F_x = (\tfrac{1}{2} \operatorname{tr} V^2)_x = \frac{\mathrm{d}}{\mathrm{d}x}(a^2 + bc) = 0,$$

i.e. F is a generating function of integrals of motion for (3.17). After setting $F = \sum_{n \geqslant 0} F_n \lambda^{-n}$, we obtain the following expressions,

$$F_n = \sum_{i=0}^{n} (a_i a_{n-i} + b_i c_{n-i}), \tag{3.19}$$

$$F_0 = -1, \quad F_1 = \langle \Psi_2, \Phi_2 \rangle - \langle \Psi_1, \Phi_1 \rangle, \tag{3.20a}$$

$$\begin{aligned} F_n &= \sum_{i=1}^{n-1} (a_i a_{n-i} + b_i c_{n-i}) - 2a_n \\ &= \sum_{i=1}^{n-1} [\tfrac{1}{4}(\langle A^{i-1}\Psi_1, \Phi_1 \rangle - \langle A^{i-1}\Psi_2, \Phi_2 \rangle)(\langle A^{n-i-1}\Psi_1, \Phi_1 \rangle - \langle A^{n-i-1}\Psi_2, \Phi_2 \rangle) \\ &\quad + \langle A^{i-1}\Psi_2, \Phi_1 \rangle \langle A^{n-i-1}\Psi_1, \Phi_2 \rangle] + \langle A^{n-1}\Psi_2, \Phi_2 \rangle - \langle A^{n-1}\Psi_1, \Phi_1 \rangle, \quad n \geqslant 2. \end{aligned} \tag{3.20b}$$

Here the polynomial functions F_n, $n \geqslant 0$, include $4N$ dependent variables. It is worth saying that we should not obtain new integrals of motion of (3.17) beginning with the generating functions $(1/n) \operatorname{tr} V^n$ since $V^3 = (a^2 + bc)V$.

Let us now consider the time part (3.16) of constrained Lax pairs. A direct calculation may give that systems (3.16) for $n \geqslant 0$ are cast into

$$\Psi_{it_n}=-\frac{\partial F_{n+1}}{\partial \Phi_i}, \qquad \Phi_{it_n}=\frac{\partial F_{n+1}}{\partial \Psi_i}, \qquad i=1,2, \qquad n\geqslant 0, \tag{3.21}$$

when $(\Psi_1, \Psi_2, \Phi_1, \Phi_2)$ satisfies the spatial part (3.15) of constrained Lax pairs and possesses the property $\lim_{|x|\to\infty}\Psi_i=\lim_{|x|\to\infty}\Phi_i=0$, $i=1,2$. For example, from (3.16) and (3.18) we have

$$\begin{aligned}\Psi_{1t_n}&=-\sum_{i=0}^{n}a_iA^{n-i}\Psi_1-\sum_{i=0}^{n}c_iA^{n-i}\Psi_2=A^n\Psi_1-\sum_{i=1}^{n}a_iA^{n-i}\Psi_1-\sum_{i=1}^{n}c_iA^{n-i}\Psi_2\\&=A^n\Psi_1-\sum_{i=1}^{n}\tfrac{1}{2}(\langle A^{i-1}\Psi_1,\Phi_1\rangle-\langle A^{i-1}\Psi_2,\Phi_2\rangle)A^{n-i}\Psi_1-\sum_{i=1}^{n}\langle A^{i-1}\Psi_1,\Phi_2\rangle A^{n-i}\Psi_2\\&=-\frac{\partial F_{n+1}}{\partial\Phi_1}, \qquad n\geqslant 1.\end{aligned}$$

Similarly, using (3.8), we know that $F=\frac{1}{2}\operatorname{tr}V^2$ is also a generating function of integrals of motion for (3.21). Therefore

$$\{F_{m+1},F_{n+1}\}=\frac{\partial}{\partial t_n}F_{m+1}=0, \qquad m,n\geqslant 0, \tag{3.22}$$

where the Poisson bracket is defined by

$$\{f,g\}=\sum_{i=1}^{2}\sum_{j=1}^{N}\left(\frac{\partial f}{\partial\psi_{ij}}\frac{\partial g}{\partial\phi_{ij}}-\frac{\partial f}{\partial\phi_{ij}}\frac{\partial g}{\partial\psi_{ij}}\right). \tag{3.23}$$

The involution (3.22) of the polynomial functions F_n, $n\geqslant 1$, may also be verified by direct computation, and thus does not depend on any boundary condition of Ψ_i, Φ_i, $i=1,2$. In addition, it is easy to prove that

$$\nabla F_n=((\partial F_n/\partial\Psi_1)^{\mathrm T},(\partial F_n/\partial\Psi_2)^{\mathrm T},(\partial F_n/\partial\Phi_1)^{\mathrm T},(\partial F_n/\partial\Phi_2)^{\mathrm T}), \qquad 1\leqslant n\leqslant 2N,$$

are linearly independent by observing that

$$\left.\frac{\partial F_n}{\partial\Phi_1}\right|_{\Phi_1=\Phi_2=0}=-A^{n-1}\Psi_1, \qquad \left.\frac{\partial F_n}{\partial\Phi_2}\right|_{\Phi_1=\Phi_2=0}=A^{n-1}\Psi_2, \qquad n\geqslant 1, \tag{3.24}$$

and that the Vandermode determinant $V(\lambda_1,\dots,\lambda_N)\neq 0$. Hence (3.15) and (3.16) are all finite-dimensional integrable systems in the Liouville sense [14]. Moreover when $\Psi_1(x,t_n)$, $\Psi_2(x,t_n)$, $\Phi_1(x,t_n)$ and $\Phi_2(x,t_n)$ with the property $\lim_{|x|\to\infty}\Psi_i=\lim_{|x|\to\infty}\Phi_i=0$, $i=1,2$, are an involutive solution [15] of the consistent systems (3.15) and (3.16), $u=(\langle\Psi_2,\Phi_1\rangle,\langle\Psi_1,\Phi_2\rangle)^{\mathrm T}$ solves the nth AKNS system $u_{t_n}=K_n$. This gives an involutive representation [15] of solutions of AKNS systems.

In what follows we consider the general case without the zero boundary conditions. In this case from (3.11) we have

$$L\begin{pmatrix}\phi_2\psi_1\\\phi_1\psi_2\end{pmatrix}=\lambda\begin{pmatrix}\phi_2\psi_1\\\phi_1\psi_2\end{pmatrix}+I\begin{pmatrix}w\\v\end{pmatrix}, \tag{3.25}$$

where E is an integral of motion of (3.11). Therefore, under the constraint (3.14), we obtain

$$\begin{pmatrix}c_{n+1}\\b_{n+1}\end{pmatrix}=L^n\begin{pmatrix}w\\v\end{pmatrix}=L^n\begin{pmatrix}\langle\Psi_1,\Phi_2\rangle\\\langle\Psi_2,\Phi_1\rangle\end{pmatrix}=\sum_{i=0}^{n}I_i\begin{pmatrix}\langle A^{n-i}\Psi_1,\Phi_2\rangle\\\langle A^{n-i}\Psi_2,\Phi_1\rangle\end{pmatrix}, \qquad n\geqslant 0. \tag{3.26}$$

Here $I_0=1$ and I_i, $1\leqslant i\leqslant n$, are integrals of motion of (3.15). From (3.4a), we can calculate that

$$a_{n+1}=\partial^{-1}(vc_{n+1}-wb_{n+1})=\tfrac{1}{2}\sum_{i=0}^{n}I_i(\langle A^{n-i}\Psi_1,\Phi_1\rangle-\langle A^{n-i}\Psi_2,\Phi_2\rangle)+Q_{n+1},\qquad n\geq 0\,,\tag{3.27}$$

where Q_{n+1} is also an integral of motion of (3.15). Equalities (3.4b) and (3.4c) require that $Q_n=-I_n$, $n\geq 1$, and thus the quantity $a_1=0$ leads to $I_1=-\frac{1}{2}F_1$. Furthermore, by $a^2+bc=1$ we obtain

$$2a_n=\sum_{i=1}^{n-1}(a_ia_{n-i}+b_ic_{n-i}),\qquad n\geq 2\,.$$

The substitution of (3.26) and (3.27) into the above equality gives rise to

$$\begin{aligned}&\sum_{i=0}^{n-1}I_i(\langle A^{n-1-i}\Psi_1,\Phi_1\rangle-\langle A^{n-1-i}\Psi_2,\Phi_2\rangle)-2I_n\\&=\sum_{i=1}^{n-1}\frac{1}{4}\left(\sum_{k=0}^{i-1}I_k(\langle A^{i-1-k}\Psi_1,\Phi_1\rangle-\langle A^{i-1-k}\Psi_2,\Phi_2\rangle)-2I_i\right)\\&\times\left(\sum_{l=0}^{n-i-1}I_l(\langle A^{n-i-1-l}\Psi_1,\Phi_1\rangle-\langle A^{n-i-1-l}\Psi_2,\Phi_2\rangle)-2I_{n-i}\right)\\&+\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}I_k\langle A^{i-1-k}\Psi_2,\Phi_1\rangle\sum_{l=0}^{n-i-1}I_l\langle A^{n-i-1-l}\Psi_1,\Phi_2\rangle,\qquad n\geq 2\,.\end{aligned}\tag{3.28}$$

Interchanging the summations in this equality, i.e.

$$\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}=\sum_{k=0}^{n-2}\sum_{i=k+1}^{n-1},\qquad \sum_{i=1}^{n-1}\sum_{l=0}^{n-i-1}=\sum_{l=0}^{n-2}\sum_{i=1}^{n-1-l},\qquad \sum_{i=1}^{n-1}\sum_{k=0}^{i-1}\sum_{l=0}^{n-i-1}=\sum_{k=0}^{n-2}\sum_{l=0}^{(n-2)-k}\sum_{i=k+1}^{n-(l+1)},$$

we may acquire by a direct but lengthy computation that

$$I_n=-\tfrac{1}{2}\sum_{\substack{k+l\leq n-1\\ k,l\geq 0}}I_kI_lF_{n-(k+l)}-\tfrac{1}{2}\sum_{i=1}^{n-1}I_iI_{n-i},\qquad n\geq 2\,.\tag{3.29}$$

Based upon this equality and $I_1=-\frac{1}{2}F_1$ we can easily obtain

$$I_n=\sum_{m=1}^{n}d_m\sum_{\substack{i_1+\ldots+i_m=n\\ i_1,\ldots,i_m\geq 1}}F_{i_1}\ldots F_{i_m},\qquad n\geq 1\,,\tag{3.30}$$

where the constants d_m are defined by

$$d_1=-\tfrac{1}{2},\qquad d_2=\tfrac{3}{8},\qquad d_m=-d_{m-1}-\tfrac{1}{2}\sum_{s=1}^{m-2}d_sd_{m-s-1}-\tfrac{1}{2}\sum_{s=1}^{m-1}d_sd_{m-s},\qquad m\geq 3\,.\tag{3.31}$$

At this stage the time parts (3.16) of the constrained Lax pairs and adjoint Lax pairs may be cast into the form

$$\Psi_{it_n}=-\frac{\partial H_n}{\partial \Phi_i},\qquad \Phi_{it_n}=\frac{\partial H_n}{\partial \Psi_i},\qquad i=1,2\,,\tag{3.32a}$$

$$H_n=\sum_{m=0}^{n}\frac{d_m}{m+1}\sum_{\substack{i_1+\ldots+i_{m+1}=n+1\\ i_1,\ldots,i_{m+1}\geq 1}}F_{i_1}\ldots F_{i_{m+1}}(d_0=1),\qquad n\geq 0\,.\tag{3.32b}$$

For instance, for $n\geq 1$ we have

$$\Psi_{1t_n}=-\sum_{i=0}^{n}a_iA^{n-i}\Psi_1-\sum_{i=0}^{n}c_iA^{n-i}\Psi_2$$

$$=A^n\Psi_1-\sum_{i=1}^{n}\frac{1}{2}\left(\sum_{k=0}^{i-1}I_k(\langle A^{i-1-k}\Psi_1,\Phi_1\rangle-\langle A^{i-1-k}\Psi_2,\Phi_2\rangle)-2I_i\right)A^{n-i}\Psi_1$$

$$-\sum_{i=1}^{n}\left(\sum_{k=0}^{i-1}I_k\langle A^{i-1-k}\Psi_1,\Phi_2\rangle\right)A^{n-i}\Psi_2$$

$$=\sum_{k=0}^{n}I_kA^{n-k}\Psi_1-\sum_{k=0}^{n-1}I_k\sum_{i=k+1}^{n}\tfrac{1}{2}(\langle A^{i-1-k}\Psi_1,\Phi_1\rangle-\langle A^{i-1-k}\Psi_2,\Phi_2\rangle)A^{n-i}\Psi_1$$

$$-\sum_{k=0}^{n-1}I_k\sum_{i=k+1}^{n}\langle A^{i-1-k}\Psi_1,\Phi_2\rangle A^{n-i}\Psi_2$$

$$=-I_n\frac{\partial F_1}{\partial\Phi_1}-\sum_{k=0}^{n-1}I_k\frac{\partial F_{n-k+1}}{\partial\Phi_1}=-\sum_{k=0}^{n}I_k\frac{\partial F_{n-k+1}}{\partial\Phi_1}$$

$$=-I_0\frac{\partial F_{n+1}}{\partial\Phi_1}-\sum_{k=1}^{n}\sum_{m=1}^{k}d_m\sum_{\substack{i_1+\ldots+i_m=k\\ i_1,\ldots,i_m\geq 1}}F_{i_1}\ldots F_{i_m}\frac{\partial F_{n-k+1}}{\partial\Phi_1}$$

$$=-I_0\frac{\partial F_{n+1}}{\partial\Phi_1}-\sum_{m=1}^{n}d_m\sum_{k=m}^{n}\sum_{\substack{i_1+\ldots+i_m=k\\ i_1,\ldots,i_m\geq 1}}F_{i_1}\ldots F_{i_m}\frac{\partial F_{n-k+1}}{\partial\Phi_1}$$

$$=-I_0\frac{\partial F_{n+1}}{\partial\Phi_1}-\sum_{m=1}^{n}\frac{d_m}{m+1}\frac{\partial}{\partial\Phi_1}\sum_{\substack{i_1+\ldots+i_{m+1}=n+1\\ i_1,\ldots,i_{m+1}\geq 1}}F_{i_1}\ldots F_{i_{m+1}}$$

$$=-\frac{\partial}{\partial\Phi_1}\sum_{m=0}^{n}\frac{d_m}{m+1}\sum_{\substack{i_1+\ldots+i_{m+1}=n+1\\ i_1,\ldots,i_{m+1}\geq 1}}F_{i_1}\ldots F_{i_{m+1}}=-\frac{\partial H_n}{\partial\Phi_1}\,.$$

The above manipulation also allows us to establish the more general involutive representations of solutions of AKNS systems. More precisely, we have the solution $v=\langle\Psi_2,\Phi_1\rangle$, $w=\langle\Psi_1,\Phi_2\rangle$ to the nth AKNS system $u_{t_n}=K_n$, once $\Psi_1(x,t_n)$, $\Psi_2(x,t_n)$, $\Phi_1(x,t_n)$, $\Phi_2(x,t_n)$ solve the constrained systems (3.15) and (3.16) simultaneously.

By now, we finish the nonlinearization procedure for the Lax pairs and the adjoint Lax pairs of AKNS systems. We would like to emphasize that the above nonlinearization trick for Lax pairs and adjoint Lax pairs may be applied to other 1+1 integrable systems (see Ref. [16]). Moreover, we wish that the deeper properties of integrable systems could be exploited by the constrained Lax systems. For instance, the resulting involutive representations of solutions possibly contain soliton solutions.

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References

[1] C.W. Cao, Chin. Q. J. Math. 3 (1988) 90; Sci. China A 33 (1990) 528.
[2] B. Konopelchenko, J. Sidorenko and W. Strampp, Phys. Lett. A 157 (1991) 17;
Y. Cheng and Y.S. Li, Phys. Lett. A 157 (1991) 22.
[3] C.W. Cao and X.G. Geng, J. Phys. A 23 (1990) 4117; J. Math. Phys. 32 (1991) 2323;
Y.B. Zeng and Y.S. Li, J. Math. Phys. 30 (1989) 1679;
Y.B. Zeng, Phys. Lett. A 160 (1991) 541.
[4] M. Antonowicz and S. Wojciechowski, Phys. Lett. A 147 (1990) 455; J. Math. Phys. 33 (1992) 211; Inverse Probl. 9 (1993) 201;
O. Ragnisco and S. Wojciechowski, Inverse Probl. 8 (1992) 245.
[5] W.X. Ma, in: Nonlinear physics, eds. C.H. Gu, Y.S. Li and G.Z. Tu (Springer, Berlin, 1990) p. 79; Acta Math. Appl. Sin. 9 (1993) 92.
[6] Z.Q. Gu, J. Math Phys. 31 (1990) 1374;
T.X. Xu and W.H. Mu, Phys. Lett. A 147 (1990) 125;
X.G. Geng, Physica A 180 (1992) 241;
Z.J. Qiao, Phys. Lett. A 172 (1993) 224.
[7] B. Konopelchenko and W. Strampp, Inverse Probl. 7 (1991) L17; J. Math. Phys. 33 (1992) 3676;
J. Sidorenko and W. Strampp, Inverse Probl. 7 (1991) L37.
[8] B. Xu and Y.S. Li, J. Phys. A 25 (1992) 2957;
W. Oevel and W. Strampp, Constrained KP hierarchy and bi-Hamiltonian structures, to appear in Commun. Math. Phys.
[9] P.D. Lax, Commun. Pure Appl. Math. 28 (1975) 141; SIAM Rev. 18 (1976) 351.
[10] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Math. 53 (1974) 249.
[11] W.X. Ma, J. Math. Phys. 33 (1992) 2464; J. Phys. A 25 (1992) 5329.
[12] G.Z. Tu, J. Math. Phys. 30 (1989) 330; J. Phys. A 22 (1989) 2375.
[13] A.C. Newell, Solitons in mathematics and physics (SIAM, Philadelphia, 1985).
[14] V.I. Arnold, Mathematical methods of classical mechanics (Springer, Berlin, 1978);
R. Abraham and J. Marsden, Foundations of mechanics, 2nd Ed. (Addison-Wesley, Reading, MA, 1978).
[15] C.W. Cao, Acta Math. Sin., New Series 7 (1991) 216.
[16] W.X. Ma, New finite dimensional integrable systems by the symmetry constraint of the KdV equations, preprint (1993); Binary nonlinearization for the Dirac systems, preprint (1993).