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Extending Hamiltonian operators
to get bi-Hamiltonian coupled KdV systems

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Abstract

An analysis of the extension of Hamiltonian operators from lower order to higher order of the matrix paves the way for constructing Hamiltonian pairs which may result in hereditary operators. Based on a specific choice of Hamiltonian operators of lower order, new local bi-Hamiltonian coupled KdV systems are proposed. As a consequence of the bi-Hamiltonian structure, they all possess infinitely many symmetries and infinitely many conserved densities. © 1998 Elsevier Science B.V.

1. Introduction

The bi-Hamiltonian structure has been identified as one of the basic mechanisms supplying conservation laws and commuting flows and even constructing multi-gap solutions for a given nonlinear system. From a differential geometric point of view, the bi-Hamiltonian structure can lead to an interesting application which holds the key to describing integrable surfaces embedded in pseudo-Euclidean spaces. For example, in many cases a local bi-Hamiltonian structure provides additional information about some nonlocal Hamiltonian structures closely connected with metrics of constant curvatures. There exist other applications in handling quantization of Poisson brackets, and studying the problem of instability, for instance, in fluid mechanics. What we would like to develop in this paper is a more local bi-Hamiltonian structure which can engender hierarchies of coupled KdV systems.

Let u be a dependent variable $u = (u^1, \ldots, u^q)^T$, where u^i, $1 \leqslant i \leqslant q$, depend on the spatial variable $x = (x_1, \ldots, x_p)^T$ and on the temporal variable t. We use $\mathcal{A}^r$ ($r \geqslant 1$) to denote the space of r-dimensional column function vectors depending on u itself and its derivatives with respect to the spatial variable x. Sometimes we write $\mathcal{A}^r(u)$ in order to specify the dependent variable u.

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A natural inner product over $\mathcal{A}^r(u)$ is given by

$$\langle \alpha, \beta \rangle = \int_{\mathbb{R}^p} (\alpha)^T \beta \, dx, \qquad \alpha, \beta \in \mathcal{A}^r. \tag{1.1}$$

Moreover, on $\mathcal{A}^q(u)$ (note that the dimensions of both a function vector in $\mathcal{A}^q(u)$ and the dependent variable u are the same), a Lie product can be defined as follows,

$$[K, S] = \frac{\partial}{\partial \varepsilon} (K(u + \varepsilon S) - S(u + \varepsilon K)) \Big|_{\varepsilon=0}, \qquad K, S \in \mathcal{A}^q. \tag{1.2}$$

Two function vectors $K, S \in \mathcal{A}^q(u)$ are called commutative if $[K, S] = 0$. A fundamental conception that we need is Hamiltonian operators, which is shown in the following definition.

Definition 1.1. A linear skew-symmetric operator $J(u) : \mathcal{A}^q(u) \to \mathcal{A}^q(u)$ is called a Hamiltonian operator or a Hamiltonian, if the Jacobi identity

$$\langle \alpha, J'(u)[J(u)\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) = 0 \tag{1.3}$$

holds for all $\alpha, \beta, \gamma \in \mathcal{A}^q(u)$. A pair of operators $J(u), M(u) : \mathcal{A}^q(u) \to \mathcal{A}^q(u)$ is called a Hamiltonian pair if $J(u) + cM(u)$ is always Hamiltonian for any constant c.

Associated with a given Hamiltonian operator $J(u)$, the Poisson bracket is defined to be

$$\{\tilde{H}_1, \tilde{H}_2\}_J = \int \left(\frac{\delta \tilde{H}_1}{\delta u}\right)^T J \frac{\delta \tilde{H}_2}{\delta u} \, dx, \qquad \tilde{H}_1, \tilde{H}_2 \in \tilde{\mathcal{A}}, \tag{1.4}$$

where $\tilde{\mathcal{A}}$ consists of functionals $\tilde{H} = \int H \, dx$, $H \in \mathcal{A}$. Two functionals $\tilde{H}_1, \tilde{H}_2$ are called commutative under the Poisson bracket associated with J if $\{\tilde{H}_1, \tilde{H}_2\}_J = 0$. A Hamiltonian operator $J : \mathcal{A}^q \to \mathcal{A}^q$ has the nice property

$$\left[J \frac{\delta \tilde{H}_1}{\delta u}, J \frac{\delta \tilde{H}_2}{\delta u}\right] = J \frac{\delta \{\tilde{H}_1, \tilde{H}_2\}_J}{\delta u}, \qquad \tilde{H}_1, \tilde{H}_2 \in \tilde{\mathcal{A}}, \tag{1.5}$$

which gives rise to an important relation between symmetries and conserved densities for a Hamiltonian system with the Hamiltonian operator J.

If we have a Hamiltonian pair J and M, one of which is invertible, for example J, then $\Phi := MJ^{-1}$ is a hereditary operator [1,2]. Furthermore, if the adjoint operator $\Psi := \Phi^\dagger = J^{-1}M$ of this hereditary operator $\Phi = MJ^{-1}$ maps a gradient vector $f_0 = \delta \tilde{H}_0 / \delta u$ to another gradient vector $f_1 = \Psi f_0 = \delta \tilde{H}_1 / \delta u$, then all vectors $\Psi^n f_0$, $n \geqslant 0$, are gradient, i.e. there exist functionals $\tilde{H}_i$, $i \geqslant 2$, such that $\Psi^i f_0 = \delta \tilde{H}_i / \delta u$, $i \geqslant 2$ (see Refs. [3–5] for more information). Then it follows from the bi-Hamiltonian structure that all systems of evolution equations in the hierarchy $u_t = \Phi^n f_0$, $n \geqslant 0$, commute with each other, i.e. $[\Phi^m f_0, \Phi^n f_0] = 0$, $m, n \geqslant 0$, and they have infinitely many common conserved densities being commutative under two Poisson brackets. Therefore, Hamiltonian pairs can pave the way for constructing integrable systems. However, the problem still exists and we will turn to finding Hamiltonian pairs.

In this paper we want to develop a way to generate Hamiltonian pairs. We are successful in constructing bi-Hamiltonian coupled KdV systems in such a way. The paper is organized as follows. We first concentrate on the techniques for extending Hamiltonian operators from lower order to higher order of the matrix, motivated by the idea of Refs. [6–8]. Then we will proceed by analyzing a class of Hamiltonian pairs which can yield hereditary operators and eventually new bi-Hamiltonian coupled KdV systems. A few concluding remarks are given in the final section.

2. Extending Hamiltonian operators

Let us specify our dependent variables

$$u_k = (u_k^1, \ldots, u_k^q)^T, \qquad 1 \leqslant k \leqslant N,$$

$$u = (u_1^T, \ldots, u_N^T)^T = (u_1^1, \ldots, u_1^q, \ldots, u_N^1, \ldots, u_N^q)^T,$$

and introduce a condition

$$J_k'(u_k) = J_l'(u_l), \qquad 1 \leqslant k, l \leqslant N \tag{2.1}$$

on a set of given Hamiltonian operators $J_k(u_k) : \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k)$, $1 \leqslant k \leqslant N$. This condition requires a kind of linearity property of the operators involved with regard to their dependent variables. Such sets of Hamiltonian operators $J_k(u_k)$ do exist. For example, we can choose

$$J_k(u_k) = -\tfrac{1}{4}\partial^3 + 2\partial u_k \partial^{-1} + 2u_k, \qquad \partial = \frac{\partial}{\partial x}, \qquad 1 \leqslant k \leqslant N.$$

The problem that we want to handle here is how to generate Hamiltonian operators starting from a given set of $J_k(u_k)$, $1 \leqslant k \leqslant N$. The simplest solution is to make a big operator $J(u) = \mathrm{diag}(J_1(u_1), \ldots, J_N(u_N))$. What we want to develop below is a more general structure of Hamiltonian operators. To the end, we introduce the following structure of candidates for Hamiltonian operators,

$$J(u) = \left(\sum_{k=1}^N c_{ij}^k J_k(u_k) \right)_{N \times N}, \tag{2.2}$$

where $\{c_{ij}^k | i, j, k = 1, 2, \ldots N\}$ is a set of given constants. Obviously this big operator $J(u)$ may be viewed as a linear operator

$$J(u) : \mathcal{A}^{Nq}(u) = \underbrace{\mathcal{A}^q(u) \times \ldots \times \mathcal{A}^q(u)}_{N} \to \mathcal{A}^{Nq}(u) = \underbrace{\mathcal{A}^q(u) \times \ldots \times \mathcal{A}^q(u)}_{N},$$

where a vector function of $\mathcal{A}^q(u)$ depends on all the dependent variables $u_1, \ldots, u_N$, not just on a certain dependent variable u_k, by defining

$$J\alpha = \left(((J\alpha)_1)^T, \ldots, ((J\alpha)_N)^T \right)^T, \qquad (J\alpha)_i = \sum_{j,k=1}^N c_{ij}^k J_k(u_k)\alpha_j, \qquad 1 \leqslant i \leqslant N, \tag{2.3}$$

where $\alpha = (\alpha_1^T, \ldots, \alpha_N^T)^T$, $\alpha_i \in \mathcal{A}^q(u)$, $1 \leqslant i \leqslant N$. To guarantee the skew-symmetric property of the big operator J, a symmetric condition on $\{c_{ij}^k\}$,

$$c_{ij}^k = c_{ji}^k, \qquad 1 \leqslant i, j, k \leqslant N, \tag{2.4}$$

suffices. The following theorem provides us with a sufficient condition for keeping the Jacobi identity (1.3).

Theorem 2.1. If all $J_k(u_k) : \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k)$, $1 \leqslant k \leqslant N$, are Hamiltonian operators having the linearity condition (2.1) and the constants c_{ij}^k, $1 \leqslant i, j, k \leqslant N$, satisfy the symmetric condition (2.4) and the following coupled condition,

$$\sum_{k=1}^N c_{ij}^k c_{kl}^n = \sum_{k=1}^N c_{li}^k c_{kj}^n, \qquad 1 \leqslant i, j, l, n \leqslant N, \tag{2.5}$$

then the operator $J(u) : \mathcal{A}^{Nq}(u) \to \mathcal{A}^{Nq}(u)$ defined by (2.2) and (2.3) is a Hamiltonian operator.

Proof. We only need to prove that $J(u)$ satisfies the Jacobi identity (1.3), because the linearity property and the skew-symmetric property of J have already been shown. Noting that $\mathcal{A}^q(u)$ is composed of column function vectors, we suppose for $\alpha, \beta, \gamma \in \mathcal{A}^{Nq}(u)$ that

$$\alpha = (\alpha_1^T, \ldots, \alpha_N^T)^T, \qquad \beta = (\beta_1^T, \ldots, \beta_N^T)^T, \qquad \gamma = (\gamma_1^T, \ldots, \gamma_N^T)^T,$$

$$\alpha_i, \beta_i, \gamma_i \in \mathcal{A}^q(u), \qquad 1 \leqslant i \leqslant N.$$

Moreover we will utilize a convenient notation $(X)_i = X_i$, $X_i \in \mathcal{A}^q(u)$, $1 \leqslant i \leqslant N$, when a function vector $X \in \mathcal{A}^{Nq}(u)$ itself is complicated.

First from the definition (2.2), we have

$$J'(u)[J\beta] = \left(\sum_{k=1}^{N} c_{ij}^k J_k'(u_k)[(J\beta)_k] \right)_{N \times N},$$

$$(J'(u)[J\beta]\gamma)_i = \sum_{j,k=1}^{N} c_{ij}^k J_k'(u_k)[(J\beta)_k]\gamma_j, \qquad 1 \leqslant i \leqslant N.$$

Then taking advantage of the concrete definition (2.3), the linearity condition (2.1) and the coupled condition (2.5), we can make the following computation,

$$\langle \alpha, J'(u)[J\beta]\gamma \rangle + \text{cycle}(\alpha, \beta, \gamma) = \sum_{i,j,k=1}^{N} c_{ij}^k \langle \alpha_i, J_k'(u_k)[(J\beta)_k]\gamma_j \rangle + \text{cycle}(\alpha, \beta, \gamma)$$

$$= \sum_{i,j,k=1}^{N} c_{ij}^k \left\langle \alpha_i, J_k'(u_k) \left[\sum_{l,n=1}^{N} c_{kl}^n J_n(u_n)\beta_l \right] \gamma_j \right\rangle + \text{cycle}(\alpha, \beta, \gamma)$$

$$= \sum_{i,j,k,l,n=1}^{N} c_{ij}^k c_{kl}^n \langle \alpha_i, J_k'(u_k)[J_n(u_n)\beta_l]\gamma_j \rangle + \text{cycle}(\alpha, \beta, \gamma)$$

$$= \sum_{i,j,l,n=1}^{N} \sum_{k=1}^{N} (c_{ij}^k c_{kl}^n) \langle \alpha_i, J_k'(u_k)[J_n(u_n)\beta_l]\gamma_j \rangle + \text{cycle}(\alpha, \beta, \gamma)$$

$$= \sum_{i,j,l,n=1}^{N} \sum_{k=1}^{N} (c_{ij}^k c_{kl}^n) \langle \alpha_i, J_n'(u_n)[J_n(u_n)\beta_l]\gamma_j \rangle + \text{cycle}(\alpha, \beta, \gamma)$$

$$= \sum_{i,j,l,n=1}^{N} \sum_{k=1}^{N} (c_{ij}^k c_{kl}^n) \langle \alpha_i, J_n'(u_n)[J_n(u_n)\beta_l]\gamma_j \rangle + \sum_{i,j,l,n=1}^{N} \sum_{k=1}^{N} (c_{ij}^k c_{kl}^n) \langle \beta_i, J_n'(u_n)[J_n(u_n)\gamma_l]\alpha_j \rangle$$

$$+ \sum_{i,j,l,n=1}^{N} \sum_{k=1}^{N} (c_{ij}^k c_{kl}^n) \langle \gamma_i, J_n'(u_n)[J_n(u_n)\alpha_l]\beta_j \rangle$$

$$= \sum_{i,j,l,n=1}^{N} \sum_{k=1}^{N} (c_{ij}^k c_{kl}^n) \left(\langle \alpha_i, J_n'(u_n)[J_n(u_n)\beta_l]\gamma_j \rangle + \text{cycle}(\alpha_i, \beta_l, \gamma_j) \right).$$

Recall that each $J_i(u_i)$ is a Hamiltonian and it follows from the above equality that the big operator J defined by (2.2) and (2.3) satisfies the Jacobi identity (1.3). This completes the proof. $\qquad\square$

We remark that if we consider the coefficients $\{c_{ij}^k\}$ as the structural constants of a finite-dimensional algebra with a basis e_1, e_2, $\ldots$, e_N as follows,

$$e_i * e_j = \sum_{k=1}^{N} c_{ij}^k e_k, \qquad 1 \leqslant i,j \leqslant N, \tag{2.6}$$

then the symmetric condition (2.4) and the coupled condition (2.5) can become

$$e_i * e_j = e_j * e_i, \qquad 1 \leqslant i,j \leqslant N, \tag{2.7}$$

$$(e_i * e_j) * e_k = (e_k * e_i) * e_j, \qquad 1 \leqslant i,j,k \leqslant N, \tag{2.8}$$

respectively. They reflect two specific properties of the related algebra.

Apparently, basic scalar Hamiltonian operators satisfying the linearity condition (2.1) can be the following set,

$$J_i(u_i) = c_i \partial^3 + d_i \partial + 2u_{ix} + 4u_i \partial, \qquad 1 \leqslant i \leqslant N, \tag{2.9}$$

where $\partial = \partial/\partial x$ and c_i, d_i, $1 \leqslant i \leqslant N$, are arbitrary constants. Of course, matrix Hamiltonian operators having the linearity condition (2.1) may be chosen. Actually, such sets of Hamiltonian operators may be presented directly from the above operators by Theorem 2.1 or by perturbation around solutions as in Refs. [9,10].

In what follows, we give examples of applications of Theorem 2.1 to two specific choices of $\{c_{ij}^k\}$. The analysis below is quite similar to the one made for the extension of hereditary operators in Ref. [11].

Example 1. Let us first choose

$$c_{ij}^k = \delta_{i+j,k-p}, \tag{2.10}$$

where p is an integer and δ_{kl} denotes the Kronecker symbol. The corresponding big operator formed by (2.2) becomes

$$J(u) = \begin{bmatrix} J_{p+2}(u_{p+2}) & J_{p+3}(u_{p+3}) & \ldots & J_{p+N+1}(u_{p+N+1}) \\ J_{p+3}(u_{p+3}) & & \cdot^{\cdot^{\cdot}} & J_{p+N+2}(u_{p+N+2}) \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ J_{p+N+1}(u_{p+N+1}) & J_{p+N+2}(u_{p+N+2}) & \ldots & J_{p+2N}(u_{p+2N}) \end{bmatrix}, \tag{2.11}$$

where we accept that $J_i(u_i) = 0$ if $i \leqslant 0$ or $i \geqslant N+1$.

For two cases of $-2N+1 \leqslant p \leqslant -N$ and $-1 \leqslant p \leqslant N-2$, the coupled condition (2.5) can be satisfied, because we have

$$\sum_{k=1}^{N} c_{ij}^k c_{kl}^n = \sum_{k=1}^{N} c_{li}^k c_{kj}^n = 1, \qquad \text{when } n-i-j-l = 2p,$$

$$= 0, \qquad \text{otherwise}.$$

While proving the above equality, we should keep in mind that we have

$$1 \leqslant i+j+p = n-l-p \leqslant N, \qquad 1 \leqslant i+l+p = n-j-p \leqslant N,$$

when $n - i - j - l = 2p$. But for the case of $-N < p < -1$, upon choosing $i = -p + 1$, $j = -p - 1$, $n = l = 1$, we obtain

$$\sum_{k=1}^{N} c_{ij}^{k} c_{kl}^{n} = 1, \qquad \sum_{k=1}^{N} c_{li}^{k} c_{kj}^{n} = 0,$$

and thus the coupled condition (2.5) can not be satisfied.

Note that when $p \geqslant N - 1$ or $p \leqslant -2N$, the resulting operators are all zero operators. Therefore, among the operators defined by (2.11), we can obtain only two sets of candidates for Hamiltonian operators,

$$J(u) = \begin{bmatrix} 0 & & & J_{p+N+1}(u_{p+N+1}) \\ & & J_{p+N+1}(u_{p+N+1}) & J_{p+N+2}(u_{p+N+2}) \\ & \cdot\cdot\cdot & \cdot\cdot\cdot & \vdots \\ J_{p+N+1}(u_{p+N+1}) & J_{p+N+2}(u_{p+N+2}) & \cdots & J_{p+2N}(u_{p+2N}) \end{bmatrix},$$
$$-2N + 1 \leqslant p \leqslant -N, \tag{2.12}$$

$$J(u) = \begin{bmatrix} J_{p+2}(u_{p+2}) & J_{p+3}(u_{p+3}) & \cdots & J_{p+N+1}(u_{p+N+1}) \\ J_{p+3}(u_{p+3}) & & \cdot\cdot\cdot & \\ \vdots & \cdot\cdot\cdot & & \\ J_{p+N+1}(u_{p+N+1}) & & & 0 \end{bmatrix}, \qquad -1 \leqslant p \leqslant N - 2, \tag{2.13}$$

where we still accept that $\Phi_i(u_i) = 0$ if $i \leqslant 0$ or $i \geqslant N + 1$. These two sets of operators can be changed into each other by a simple transformation $(u_1, u_2, \ldots, u_N) \leftrightarrow (u_N, u_{N-1}, \ldots, u_1)$.

Example 2. Let us now choose

$$c_{ij}^{k} = \delta_{kl}, \qquad l = i + j - p \pmod{N}, \tag{2.14}$$

where $1 \leqslant p \leqslant N$ is fixed and δ_{kl} denotes the Kronecker symbol again. In this case, we have

$$\sum_{k=1}^{N} c_{ij}^{k} c_{kl}^{n} = \sum_{k=1}^{N} c_{li}^{k} c_{kj}^{n} = 1, \qquad \text{when } i + j + l - n = 2p \pmod{N},$$

$$= 0, \qquad \text{otherwise}, \tag{2.15}$$

which allows us to conclude that the coupled condition (2.5) holds indeed. Therefore we obtain a set of candidates for Hamiltonian operators

$$J(u) = \begin{bmatrix} J_{2-p}(u_{2-p}) & J_{3-p}(u_{3-p}) & \cdots & J_{N-p+1}(u_{N-p+1}) \\ J_{3-p}(u_{3-p}) & & & J_{N-p+2}(u_{N-p+2}) \\ \vdots & \cdot\cdot\cdot & \cdot\cdot\cdot & \vdots \\ J_{N-p+1}(u_{N-p+1}) & J_{N-p+2}(u_{N-p+2}) & \cdots & J_{2N-p}(u_{2N-p}) \end{bmatrix}, \tag{2.16}$$

where we accept $J_i(u_i) = J_j(u_j)$ if $i = j \pmod{N}$, while determining the operators involved, for example, $J_{2-p}(u_{2-p}) = J_N(u_N)$ when $p = 2$. The special choice of $p = 1$ leads to a candidate of Hamiltonian operators

$$J(u) = \begin{bmatrix} J_1(u_1) & J_2(u_2) & \ldots & J_N(u_N) \\ J_2(u_2) & & \cdot^{\cdot^\cdot} & J_1(u_1) \\ \vdots & \cdot^{\cdot^\cdot} & \cdot^{\cdot^\cdot} & \vdots \\ J_N(u_N) & J_1(u_1) & \ldots & J_{N-1}(u_{N-1}) \end{bmatrix}. \tag{2.17}$$

This operator will be our starting point for constructing new integrable systems in the next section.

3. New bi-Hamiltonian coupled KdV systems

From now on we focus on the candidate of the Hamiltonian operators defined by (2.17). Let us pick out the Hamiltonian operators formed by (2.17) with the choice of (2.9). Then the following Hamiltonian pair can be engendered,

$$J(u) = A\partial = \begin{bmatrix} a_1 & a_2 & \ldots & a_N \\ a_2 & & \cdot^{\cdot^\cdot} & a_1 \\ \vdots & \cdot^{\cdot^\cdot} & \cdot^{\cdot^\cdot} & \vdots \\ a_N & a_1 & \ldots & a_{N-1} \end{bmatrix} \partial, \tag{3.1}$$

$$M(u) = \begin{bmatrix} M_1(u_1) & M_2(u_2) & \ldots & M_N(u_N) \\ M_2(u_2) & & \cdot^{\cdot^\cdot} & M_1(u_1) \\ \vdots & \cdot^{\cdot^\cdot} & \cdot^{\cdot^\cdot} & \vdots \\ M_N(u_N) & M_1(u_1) & \ldots & M_{N-1}(u_{N-1}) \end{bmatrix}, \tag{3.2}$$

where $\partial = \partial/\partial x$, $u = (u_1, u_2, \ldots, u_N)^T$, $a_i = \text{const}$, $1 \leqslant i \leqslant N$, and the operators $M_i(u_i)$, $1 \leqslant i \leqslant N$, are given by

$$M_i(u_i) = c_i \partial^3 + d_i \partial + 2u_{ix} + 4u_i \partial, \qquad c_i, d_i = \text{const}, \qquad 1 \leqslant i \leqslant N, \tag{3.3}$$

which are all Hamiltonian.

We assume that the constant matrix A is invertible to guarantee the invertibility of J, and its inverse matrix is given by

$$B = \begin{bmatrix} b_1 & b_2 & \ldots & b_N \\ b_2 & & \cdot^{\cdot^\cdot} & b_1 \\ \vdots & \cdot^{\cdot^\cdot} & \cdot^{\cdot^\cdot} & \vdots \\ b_N & b_1 & \ldots & b_{N-1} \end{bmatrix}, \tag{3.4}$$

where b_i, $1 \leqslant i \leqslant N$, can be determined by solving a specific linear system

$$a_1 b_1 + a_2 b_2 + \ldots + a_N b_N = 1, \qquad a_2 b_1 + a_3 b_2 + \ldots + a_N b_{N-1} + a_1 b_N = 0, \qquad \ldots,$$
$$a_N b_1 + a_1 b_2 + \ldots + a_{N-1} b_N = 0.$$

Now the resulting hereditary operator $\Phi = MJ^{-1}$ reads

$$
\Phi(u) = \begin{bmatrix} M_1(u_1) & M_2(u_2) & \dots & M_N(u_N) \\ M_2(u_2) & & \cdot^{\cdot} & M_1(u_1) \\ \vdots & \cdot^{\cdot} & \cdot^{\cdot} & \vdots \\ M_N(u_N) & M_1(u_1) & \dots & M_{N-1}(u_{N-1}) \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_N \\ b_2 & & \cdot^{\cdot} & b_1 \\ \vdots & \cdot^{\cdot} & \cdot^{\cdot} & \vdots \\ b_N & b_1 & \dots & b_{N-1} \end{bmatrix} \partial^{-1}
$$

$$
= \begin{bmatrix} \Phi_1(u_1) & \Phi_2(u_2) & \dots & \Phi_N(u_N) \\ \Phi_2(u_2) & & \cdot^{\cdot} & \Phi_1(u_1) \\ \vdots & \cdot^{\cdot} & \cdot^{\cdot} & \vdots \\ \Phi_N(u_N) & \Phi_1(u_1) & \dots & \Phi_{N-1}(u_{N-1}) \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_N \\ b_2 & & \cdot^{\cdot} & b_1 \\ \vdots & \cdot^{\cdot} & \cdot^{\cdot} & \vdots \\ b_N & b_1 & \dots & b_{N-1} \end{bmatrix} \tag{3.5}
$$

with

$$
\Phi_i(u_i) = M_i(u_i)\partial^{-1} = c_i\partial^2 + d_i + 2u_{ix}\partial^{-1} + 4u_i, \qquad 1 \leqslant i \leqslant N. \tag{3.6}
$$

This hereditary operator can be rewritten in the concise form

$$
\Phi(u) = MJ^{-1} = \left(\sum_{k=1}^{N} b_{k-i+j}\Phi_k(u_k) \right), \tag{3.7}
$$

where $b_i = b_j$ if $i \equiv j \pmod{N}$. It is also an example satisfying the extension scheme of hereditary operators in Ref. [11], because upon setting $c_{ij}^k = b_{k-i+j}$ we have three equal sums for all $1 \leqslant i,j,l,n \leqslant N$:

$$
\sum_{k=1}^{N} c_{ij}^k c_{kn}^l = \sum_{k=1}^{N} b_{k-i+j}b_{l-k+n} = \sum_{m=1}^{N} b_m b_{l+i-j+n-m},
$$

$$
\sum_{k=1}^{N} c_{ik}^l c_{kj}^n = \sum_{k=1}^{N} b_{l-i+k}b_{n-k+j} = \sum_{m=1}^{N} b_m b_{l+i-j+n-m},
$$

$$
\sum_{k=1}^{N} c_{in}^k c_{kj}^l = \sum_{k=1}^{N} b_{k-i+n}b_{l-k+j} = \sum_{m=1}^{N} b_m b_{l+i-j+n-m},
$$

which are sufficient for $\Phi(u)$ to be hereditary (see Ref. [11]).

We now turn our attention to investigating the nonlinear systems which can result from the above Hamiltonian pairs. Such a system can be the following,

$$
u_t = \Phi u_x = MJ^{-1}u_x, \tag{3.8}
$$

which can be represented as

$$
u_{it} = \sum_{k,j=1}^{N} b_{k-i+j}(c_k u_{jxxx} + d_k u_{jx} + 2u_{kx}u_j + 4u_k u_{jx}), \qquad 1 \leqslant i \leqslant N, \tag{3.9}
$$

where we again accept $b_i = b_j$ if $i \equiv j \pmod{N}$. It is easy to find that

$$
f_0 := J^{-1}u_x = Bu = \frac{\delta \tilde{H}_0}{\delta u}, \qquad \tilde{H}_0 = \int H_0\, dx, \qquad H_0 = \tfrac{1}{2}u^T Bu. \tag{3.10}
$$

We proceed by checking whether or not the following vector defined by

$$
f_1 := \Psi f_0 = \Phi^{\dagger} f_0 = B
\begin{bmatrix}
\Psi_1(u_1) & \Psi_2(u_2) & \cdots & \Psi_N(u_N) \\
\Psi_2(u_2) & & \cdots & \Psi_1(u_1) \\
\vdots & \cdots & \cdots & \vdots \\
\Psi_N(u_N) & \Psi_1(u_1) & \cdots & \Psi_{N-1}(u_{N-1})
\end{bmatrix}
Bu
\tag{3.11}
$$

is a gradient vector, where $\Psi_i(u_i) = \Phi_i^{\dagger}(u_i) = \partial^{-1} M_i(u_i)$, $1 \leqslant i \leqslant N$. That is true, indeed. Actually we have

$$
f_1 = \Psi f_0 = \frac{\delta \tilde{H}_1}{\delta u}, \qquad \tilde{H}_1 = \int H_1 \mathrm{d}x,
$$

$$
H_1 = (Bu)^T
\begin{bmatrix}
\Theta_1(u_1) & \Theta_2(u_2) & \cdots & \Theta_N(u_N) \\
\Theta_2(u_2) & & \cdots & \Theta_1(u_1) \\
\vdots & \cdots & \cdots & \vdots \\
\Theta_N(u_N) & \Theta_1(u_1) & \cdots & \Theta_{N-1}(u_{N-1})
\end{bmatrix}
Bu
$$

$$
= \sum_{j,k,l=1}^{N} \left(\sum_{i=1}^{N} b_{i+l-1} b_{k+j-i} \right) u_l \left(\tfrac{1}{2} c_k u_{jxx} + \tfrac{1}{2} d_k u_j + u_k u_j \right),
\tag{3.12}
$$

where the operators $\Theta_i(u_i)$, $1 \leqslant i \leqslant N$, are given by

$$
\Theta_i(u_i) = \tfrac{1}{2} c_i \partial^2 + \tfrac{1}{2} d_i + \tfrac{2}{3} u_i + \tfrac{2}{3} \partial^{-1} u_i \partial, \qquad 1 \leqslant i \leqslant N,
\tag{3.13}
$$

and $b_i = b_j$ if $i \equiv j \pmod{N}$. We can also choose the energy form for the functional $\tilde{H}_1$:

$$
\tilde{H}_1 = \int H_1 \mathrm{d}x, \qquad H_1 = \sum_{j,k,l=1}^{N} \left(\sum_{i=1}^{N} b_{i+l-1} b_{k+j-i} \right) \left(-\tfrac{1}{2} c_k u_{lx} u_{jx} + \tfrac{1}{2} d_k u_l u_j + u_l u_k u_j \right).
\tag{3.14}
$$

Therefore, according to the Magri scheme [3–5], there exist other functionals $\tilde{H}_n$, $n \geqslant 2$, such that $\Psi^n f_0 = \delta \tilde{H}_n / \delta u$, $n \geqslant 2$. All such functionals can be generated by computing the following integrals,

$$
\tilde{H}_n = \int \int_0^1 \langle (\Psi^n f_0)(\lambda u), u \rangle \, \mathrm{d}\lambda \, \mathrm{d}x, \qquad n \geqslant 0.
\tag{3.15}
$$

Further we can obtain a hierarchy of bi-Hamiltonian equations

$$
u_t = K_n := (\Phi(u))^{n+1} u_x = J \frac{\delta \tilde{H}_{n+1}}{\delta u} = M \frac{\delta \tilde{H}_n}{\delta u}, \qquad n \geqslant 0,
\tag{3.16}
$$

which includes the nonlinear system (3.9) as the first member. It follows that they have infinitely many commutative symmetries $\{K_m\}_0^{\infty}$ and infinitely many commutative conserved densities $\{H_m\}_0^{\infty}$. All systems of evolution equations can reduce to KdV equations once $u_j = c_i = d_j = 0$, $j \neq 1$, are selected. Therefore they are all N-component coupled KdV systems.

Let us now work out a concrete example for a choice of A with $a_1 = 1$, $a_i = 0$, $2 \leqslant i \leqslant N$. In this case, we have $b_1 = 1$, $b_i = 0$, $2 \leqslant i \leqslant N$. Then the first Hamiltonian structure is given by

$$
\partial_t u_1 = \partial_x \frac{\delta \tilde{H}_1}{\delta u_1}, \qquad \partial_t u_k = \partial_x \frac{\delta \tilde{H}_1}{\delta u_{N+2-k}}, \qquad \partial_x = \frac{\partial}{\partial x}, \qquad 2 \leqslant k \leqslant N.
\tag{3.17}
$$

This first Hamiltonian structure has the momentum

$$
\tilde{P} = \tilde{H}_0 = \tfrac{1}{2} \int \left(u_1^2 + \sum_{k=2}^{N} u_k u_{N+2-k} \right) dx,
\tag{3.18}
$$

which is of quadratic form with respect to its N Casimirs (annihilators of the first Poisson bracket) $\tilde{F}_k = \int u_k dx$, $k = 1, \ldots, N$. The conservation law of momentum is

$$
\partial_t H_1 = \partial_x \left(\sum_{k=1}^{N} u_k \frac{\delta \tilde{H}_1}{\delta u_k} - F \right),
\tag{3.19}
$$

where $\tilde{H}_1 = \int H_1 dx$, $\partial_x F = \sum_{k=1}^{N} (\delta \tilde{H}_1 / \delta u_k) u_{kx}$. For simplicity, in the selection (3.6) we can put $d_i = 0$, because we can eliminate those constants in the expressions of Φ_i by making shifts $u_i \rightarrow u_i - d_i/4$. Then our coupled KdV system reads

$$
\partial_t u_m = \partial_x \left[\partial_x^2 \left(\sum_{k=1}^{m} c_{m+1-k} u_k + \sum_{k=m+1}^{N} c_{N+m+1-k} u_k \right) + 3 \sum_{k=1}^{m} u_k u_{m+1-k} + 3 \sum_{k=m+1}^{N} u_k u_{N+m+1-k} \right],
\tag{3.20}
$$

which has the following Hamiltonian for the first Hamiltonian structure,

$$
\tilde{H}_1 = \int H_1 dx,
\tag{3.21}
$$

$$
H_1 = -\tfrac{1}{2} \left(c_1 w_1^2 + w_1 \sum_{k=2}^{N} c_k w_{N+2-k} + \sum_{m=1}^{N-1} c_m \sum_{k=2}^{N+1-m} w_k w_{N+3-m-k} + \sum_{m=3}^{N} c_m \sum_{k=N+3-m}^{N} w_k w_{2N-m+3-k} \right)
$$

$$
+ u_1^3 + 3 u_1 \sum_{k=2}^{N} u_k u_{N+2-k} + \sum_{k=2}^{N-1} u_k \sum_{m=2}^{N+1-k} u_m u_{N+3-k-m} + \sum_{m=0}^{N-3} u_{m+3} \sum_{k=0}^{m} u_{N-k} u_{N+k-m},
$$

where $w_k = \partial_x u_k$, $1 \leqslant k \leqslant N$, and N is assumed to be greater than two for producing nontrivial systems. The second Hamiltonian structure can be determined by its recursion operator

$$
\partial_t u_i = \sum_{k=1}^{i} \Phi_{i+1-k} \partial_x \frac{\delta \tilde{H}_0}{\delta u_k} + \sum_{k=i+1}^{N} \Phi_{N+1+i-k} \partial_x \frac{\delta \tilde{H}_0}{\delta u_k}, \qquad 1 \leqslant i \leqslant N,
\tag{3.22}
$$

where $\Phi_i = c_i \partial_x^2 + 2(2u_i + w_i \partial_x^{-1})$, $1 \leqslant i \leqslant N$. The second Hamiltonian structure has the momentum $F_1' = \int u_1 dx$ and the Hamiltonian $\tilde{H}_0$.

The relationship between the gradients of $\tilde{H}_k$ and $\tilde{H}_{k+1}$ determined by the recursion operator provides the possibility for constructing an infinite set of conservation laws and commuting flows by iterations. At each step we need to compute integrals to construct $\tilde{H}_{k+1}$. They can be done by using formula (3.15) in variational analysis.

Moreover we have an alternative way to construct an infinite set of conservation laws and commuting flows. Let us introduce an eigenfunction problem for the recursion operator

$$
[\Phi(u) - \lambda] u_\tau = 0,
\tag{3.23}
$$

i.e.

$$
\sum_{k=1}^{i} \Phi_{i+1-k} \partial_\tau u_k + \sum_{k=i+1}^{N} \Phi_{N+1+i-k} \partial_\tau u_k = \lambda \partial_\tau u_i, \qquad 1 \leqslant i \leqslant N.
\tag{3.24}
$$

This eigenfunction problem can be equivalently rewritten as

$$\partial_\tau u_k = \partial_x v_k, \qquad 1 \leqslant k \leqslant N,$$

$$\sum_{k=1}^{i} \Phi_{i+1-k}\partial_x v_k + \sum_{k=i+1}^{N} \Phi_{N+1+i-k}\partial_x v_k = \lambda \partial_x v_i, \qquad 1 \leqslant i \leqslant N.$$

If we use formal series near $\lambda \to \infty$,

$$\partial_\tau = \partial_{t_0} + \frac{1}{\lambda}\partial_{t_1} + \frac{1}{\lambda^2}\partial_{t_2} + \dots, \tag{3.25}$$

$$v_i = v_i^{(0)} + \frac{1}{\lambda}v_i^{(1)} + \frac{1}{\lambda^2}v_i^{(2)} + \dots, \tag{3.26}$$

we can obtain directly from the above eigenfunction problem an infinite set of commuting flows. Furthermore we can use formal series near $\lambda \to 0$. Then the first resulting nontrivial commuting flow is a coupled long-wave equation

$$\sum_{k=1}^{i} \Phi_{i+1-k}\partial_\tau u_k + \sum_{k=i+1}^{N} \Phi_{N+1+i-k}\partial_\tau u_k = 0, \qquad 1 \leqslant i \leqslant N. \tag{3.27}$$

This is an N-component generalization of the long-wave equation (see Ref. [12] for more information about long-wave equations), which commutes with the KdV equation.

4. Concluding remarks

We have proposed new coupled KdV systems possessing bi-Hamiltonian structures by extending Hamiltonian operators from lower order to higher order of the matrix. Clearly we may make other choices of $J_i(u_i)$ to give more results based on Theorem 2.1.

Compared to the well-known coupled KdV systems (see, e.g., Ref. [13])

$$u_t = \Phi^n u_x, \qquad \Phi(u) = \begin{bmatrix} 0 & \dots & 0 & \Phi_1 \\ 1 & \dots & 0 & \Phi_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & \Phi_N \end{bmatrix}, \qquad n \geqslant 0, \tag{4.1}$$

and the quite new coupled KdV systems introduced in Ref. [14],

$$u_t = \Phi^n u_x, \qquad \Phi(u) = \begin{bmatrix} b_N\Phi_1 & & & 0 \\ b_{N-1}\Phi_1 + b_N\Phi_2 & b_N\Phi_1 & & \\ \vdots & & \ddots & \ddots \\ b_1\Phi_1 + \dots + b_N\Phi_N & \dots & b_{N-1}\Phi_1 + b_N\Phi_2 & b_N\Phi_1 \end{bmatrix}, \qquad n \geqslant 0, \tag{4.2}$$

where $u = (u_1, u_2, \dots, u_N)^T$, b_i, $1 \leqslant i \leqslant N$, are arbitrary constants except $b_N \neq 0$, $\Phi_i = \Phi_i(u_i)$, $1 \leqslant i \leqslant N$, are still defined by (3.6), our new coupled KdV systems (3.16) have similar nice integrable properties, for example, bi-Hamiltonian structures, dispersionless limits having bi-Hamiltonian structures (see Ref. [15] for the case of the well-known coupled KdV systems). But there are differences among the structures of the recursion operators corresponding to these three hierarchies of coupled KdV systems. The bi-Hamiltonian structure (see

Ref. [14]) of the hierarchy (4.2) can similarly be derived from the Hamiltonian operators formed by (2.12) in Example 1 of Section 2. This is why we did not give above a detailed analysis for constructing integrable systems starting from (2.12) or equivalently from (2.13). However, we do not know whether or not the two new coupled KdV hierarchies (4.2) and (3.16) have other integrable properties, for example, Lax pairs like (4.1).

In terms of the existence of recursion operators, other integrable couple KdV systems, say, Jordan KdV systems, have also been derived (see, for example, Refs. [16–19]). A natural question is whether there exist other hierarchies possessing bi-Hamiltonian structures among those coupled KdV systems. This will enrich the content of Hamiltonian theory for coupled KdV systems.

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