A Counterpart of the Wadati–Konno–Ichikawa Soliton Hierarchy Associated with so(3,R)

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A counterpart of the Wadati–Konno–Ichikawa (WKI) soliton hierarchy, associated with $\mathrm{so}(3,\mathbb{R})$, is presented through the zero curvature formulation. Its spectral matrix is defined by the same linear combination of basis vectors as the WKI one, and its Hamiltonian structures yielding Liouville integrability are furnished by the trace identity.

Key words: Spectral Problem; Hereditary Recursion Operator; Bi-Hamiltonian Structure.

1. Introduction

Soliton hierarchies consist of commuting nonlinear partial differential equations with Hamiltonian structures, and they are usually generated from given spectral problems associated with matrix Lie algebras (see, e. g., [1 – 3]). Typical examples include the Korteweg-de Vries hierarchy [4], the Ablowitz–Kaup–Newell–Segur hierarchy [5], the Kaup–Newell hierarchy [6], and the Wadati–Konno–Ichikawa (WKI) hierarchy [7].

When associated matrix Lie algebras are semisimple, the trace identity can be used to construct Hamiltonian structures of soliton hierarchies [8]. When associated matrix Lie algebras are non-semisimple, we obtain integrable couplings [10, 11], and the variational identity provides a basic technique to generate their Hamiltonian structures [12, 13]. Usually, the existence of bi-Hamiltonian structures [14] implies Liouville integrability, often generating hereditary recursion operators (see, e. g., [15 – 17]). The most widely used three-dimensional simple Lie algebra in soliton theory is the special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$. We would like to use the other three-dimensional simple Lie algebra, the special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$. Those two Lie algebras are only the two real three-dimensional Lie algebras, whose derived algebras are three-dimensional, too.

Let us briefly outline the steps of our procedure to construct soliton hierarchies by the zero curvature formulation (see, e. g., [8] for details).

Step 1 – Introducing a Spatial Spectral Problem

Take a matrix loop algebra $\tilde{\mathfrak{g}}$, associated with a given matrix Lie algebra $\mathfrak{g}$, often being semisimple. Then, introduce a spatial spectral problem

$$\phi_x = U\phi\,,\quad U = U(u,\lambda) \in \tilde{\mathfrak{g}}\,, \tag{1}$$

where u denotes a column dependent variable, and λ is the spectral parameter.

Step 2 – Computing Zero Curvature Equations

We search for a solution of the form

$$W = W(u,\lambda) = \sum_{i\geq 0} W_{0,i}\lambda^{-i}\,,\quad W_{0,i} \in \mathfrak{g}\,,\quad i \geq 0\,, \tag{2}$$

to the stationary zero curvature equation

$$W_x = [U,W]\,. \tag{3}$$

Then, use this solution W to introduce the Lax matrices

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}}\,,\quad m \geq 0\,, \tag{4}$$

where P_+ denotes the polynomial part of P in λ, and formulate the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi\,, \;\; m \geq 0\,. \tag{5}$$

The crucial point is to input the modification terms, $\Delta_m \in \tilde{\mathfrak{g}}$, $m \geq 0$, which aims to guarantee that the compatibility conditions of (1) and (5), i. e., the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0\,, \;\; m \geq 0\,, \tag{6}$$

will generate soliton equations. We write the resulting hierarchy of soliton equations of evolution type as follows:

$$u_{t_m} = K_m(u)\,, \;\; m \geq 0\,. \tag{7}$$

Step 3 – Constructing Hamiltonian Structures

Compute Hamiltonian functionals $\mathcal{H}_m$ by applying the trace identity [8]:

$$\begin{aligned} &\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right), \\ &\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left|\mathrm{tr}(W^2)\right|, \end{aligned} \tag{8}$$

or more generally, the variational identity [12, 18]:

$$\begin{aligned} &\frac{\delta}{\delta u}\int \left\langle\frac{\partial U}{\partial \lambda}, W\right\rangle \mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle\frac{\partial U}{\partial u}, W\right\rangle, \\ &\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\langle W, W\rangle|, \end{aligned} \tag{9}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric, and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$. Then, construct Hamiltonian structures for the whole hierarchy (7):

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}\,, \;\; m \geq 0\,. \tag{10}$$

The generating functional $\int \mathrm{tr}(\frac{\partial U}{\partial \lambda}W)\,\mathrm{d}x$ or $\int\langle\frac{\partial U}{\partial \lambda}, W\rangle\,\mathrm{d}x$ will be used to generate the Hamiltonian functionals $\{\mathcal{H}_m\}_0^\infty$ in the above Hamiltonian structures. Usually, the recursion structure of a soliton hierarchy leads to its bi-Hamiltonian structures and Liouville integrability.

In this paper, starting from the three-dimensional special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$, we would like to present a counterpart of the WKI soliton hierarchy. The counterpart soliton hierarchy consists of commuting bi-Hamiltonian evolution equations, which are of differential function type but not of differential polynomial type, and its corresponding Hamiltonian structures will be furnished by the trace identity. Therefore, all equations in the counterpart soliton hierarchy provide a new example of soliton hierarchies associated with $\mathrm{so}(3,\mathbb{R})$ (see [19, 20] for two examples of Ablowitz–Kaup–Newell–Segur and Kaup–Newell types). A few concluding remarks will be given in the final section.

2. A Counterpart of the WKI Soliton Hierarchy

2.1. The WKI Hierarchy

Let us recall the WKI soliton hierarchy [7, 21] for comparison's sake. Its corresponding special matrix reads

$$U = U(u,\lambda) = \lambda e_1 + \lambda p e_2 + \lambda q e_3\,, \tag{11}$$

where e_1, e_2, and e_3, forming a basis of the special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$, are defined as follows:

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \;\; e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \;\; e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{12}$$

whose commutator relations are

$$[e_1, e_2] = 2e_2\,, \;\; [e_1, e_3] = -2e_3\,, \;\; [e_2, e_3] = e_1\,.$$

A solution of the form

$$\begin{aligned} W &= aU + b_x e_2 + c_x e_3 \\ &= \lambda a e_1 + (\lambda p a + b_x)e_2 + (\lambda q a + c_x)e_3 \end{aligned} \tag{13}$$

to the stationary zero curvature equation (3) is determined by

$$\begin{aligned} &a_x = p c_x - q b_x\,, \;\; \lambda(pa)_x + b_{xx} = 2\lambda b_x\,, \\ &\lambda(qa)_x + c_{xx} = -2\lambda c_x\,. \end{aligned} \tag{14}$$

Upon setting

$$\begin{aligned} &a = \sum_{i\geq 0} a_i \lambda^{-i}\,, \;\; b = \sum_{i\geq 0} b_i \lambda^{-i}\,, \\ &c = \sum_{i\geq 0} c_i \lambda^{-i}\,, \;\; i \geq 0\,, \end{aligned} \tag{15}$$

and choosing the initial values

$$a_0=\frac{1}{\sqrt{pq+1}}\,,\ b_0=\frac{p}{2\sqrt{pq+1}}\,,\quad c_0=-\frac{q}{2\sqrt{pq+1}}\,, \tag{16}$$

the system (14) recursively defines the infinite sequence of $\{a_i,b_i,c_i|i\geq 1\}$ as follows:

$$\begin{bmatrix}-c_{i+1}\\ b_{i+1}\end{bmatrix}=\Psi\begin{bmatrix}-c_i\\ b_i\end{bmatrix}, \tag{17}$$
$$\Psi=\begin{bmatrix}-\frac12\partial+\frac14\bar q\partial^{-1}\bar p\partial^2 & -\frac14\bar q\partial^{-1}\bar q\partial^2\\ \frac14\bar p\partial^{-1}\bar p\partial^2 & \frac12\partial-\frac14\bar p\partial^{-1}\bar q\partial^2\end{bmatrix},\ i\geq 0,$$

and

$$a_{i+1,x}=pc_{i+1,x}-qb_{i+1,x}\,,\ i\geq 0\,, \tag{18}$$

with $\bar p$ and $\bar q$ being given by

$$\bar p=\frac{p}{\sqrt{pq+1}}\,,\ \bar q=\frac{q}{\sqrt{pq+1}}\,. \tag{19}$$

We impose the conditions on constants of integration,

$$a_i|_{u=0}=b_i|_{u=0}=c_i|_{u=0}=0\,,\ i\geq 1\,, \tag{20}$$

which guarantee the uniqueness of the infinite sequence of $\{a_i,b_i,c_i|i\geq 1\}$. So, the first two sets can be computed as follows:

$$a_1=\frac{pq_x-qp_x}{4(pq+1)^{\frac32}}\,,\ b_1=\frac{p_x}{4(pq+1)^{\frac32}}\,,$$
$$c_1=\frac{q_x}{4(pq+1)^{\frac32}}\,,$$
$$a_2=\frac{1}{32(pq+1)^{\frac72}}\Big[5q^2p_x^2+(14pq+4)p_xq_x+5p^2q_x^2 -4q(pq+1)p_{xx}-4p(pq+1)q_{xx}\Big],$$
$$b_2=-\frac{1}{64(pq+1)^{\frac72}}\Big[q(7pq+12)p_x^2-2p(pq-4)p_xq_x -5p^3q_x^2-4(pq+1)(pq+2)p_{xx}+4p^2(pq+1)q_{xx}\Big],$$
$$c_2=-\frac{1}{64(pq+1)^{\frac72}}\Big[5q^3p_x^2+2q(pq-4)p_xq_x-p(7pq +12)q_x^2-4q^2(pq+1)p_{xx}+4(pq+1)(pq+2)q_{xx}\Big].$$

Finally, upon taking

$$V^{[m]}=\lambda\Big[(\lambda^m a)_+U+(\lambda^m b_x)_+e_2+(\lambda^m c_x)_+e_3\Big],\quad m\geq 0, \tag{21}$$

the corresponding zero curvature equations

$$U_{t_m}-V_x^{[m]}+\Big[U,V^{[m]}\Big]=0\,,\ m\geq 0\,, \tag{22}$$

present the WKI hierarchy of commuting Hamiltonian equations,

$$u_{t_m}=K_m=\begin{bmatrix}b_{m,xx}\\ c_{m,xx}\end{bmatrix}=J\begin{bmatrix}-c_m\\ b_m\end{bmatrix}=J\frac{\delta\mathcal{H}_m}{\delta u}\,,m\geq 0\,, \tag{23}$$

with the Hamiltonian operator J being defined by

$$J=\begin{bmatrix}0 & \partial^2\\ -\partial^2 & 0\end{bmatrix}, \tag{24}$$

and the Hamiltonian functionals $\mathcal{H}_m$ by

$$\mathcal{H}_0=\int 2\sqrt{pq+1}\,\mathrm{d}x\,,\quad \mathcal{H}_1=\int\frac{qp_x-pq_x}{4\sqrt{pq+1}\big(\sqrt{pq+1}+1\big)}\,\mathrm{d}x\,, \tag{25}$$

and

$$\mathcal{H}_{m+1}=\int\Big[-\frac{2(pq+1)a_{m+1}+pc_{m,x}+qb_{m,x}}{2m}\Big]\mathrm{d}x\,,\quad m\geq 1\,. \tag{26}$$

The above Hamiltonian functionals $\mathcal{H}_m$, $m\neq 1$, can be worked out by the trace identity (8) with

$$\mathrm{tr}\Big(W\frac{\partial U}{\partial\lambda}\Big)=2\lambda(pq+1)a+pc_x+qb_x\,,$$
$$\mathrm{tr}\Big(W\frac{\partial U}{\partial p}\Big)=\lambda(\lambda qa+c_x)=-2\lambda^2c\,,$$
$$\mathrm{tr}\Big(W\frac{\partial U}{\partial q}\Big)=\lambda(\lambda pa+b_x)=2\lambda^2b\,,$$

and $\mathcal{H}_1$ can be computed directly from $(-c_1,b_1)^{\mathrm{T}}$.

We point out that a generalized WKI soliton hierarchy was presented in [22] and its binary nonlinearization was carried out in [23]. A multi-component WKI hierarchy and a multi-component generalized WKI hierarchy and their integrable couplings were also analyzed in [24] and [25], respectively.

2.2. A Counterpart of the WKI Hierarchy

We will make use of the three-dimensional special orthogonal Lie algebra so$(3,\mathbb{R})$, consisting of 3×3 skew-symmetric real matrices. This Lie algebra is simple and has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

whose commutator relations are

$$[e_1, e_2] = e_3\,, \; [e_2, e_3] = e_1\,, \; [e_3, e_1] = e_2\,.$$

Its derived algebra is itself, and so, three-dimensional, too. The corresponding matrix loop algebra we will use is

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \left\{ \sum_{i\geq 0} M_i \lambda^{n-i} \middle| M_i \in \mathrm{so}(3,\mathbb{R}), \; i \geq 0, n \in \mathbb{Z} \right\}. \tag{28}$$

The loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ contains matrices of the form

$$\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$$

with arbitrary integers m, n, l, and it provides a good starting point to generate soliton equations.

Let us now introduce a spectral matrix

$$U = U(u,\lambda) = \lambda e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R})\,, \; u = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{29}$$

to formulate a matrix spatial spectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi\,, \; \phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}. \tag{30}$$

The spectral matrix above is defined by the same linear combination of basis vectors as the WKI one [7], but its underlying loop algebra is $\widetilde{\mathrm{so}}(3,\mathbb{R})$, not isomorphic to $\widetilde{\mathrm{sl}}(2,\mathbb{R})$. The other two examples associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$, as counterpart hierarchies of the Ablowitz–Kaup–Newell–Segur hierarchy and the Kaup–Newell hierarchy, were previously presented in [19] and [20], respectively.

Then, we solve the stationary zero curvature equation (3), and it becomes

$$a_x = pc_x - qb_x\,, \; \lambda(pa)_x + b_{xx} = -\lambda c_x\,, \quad \lambda(qa)_x + c_{xx} = \lambda b_x \tag{31}$$

if W is chosen as

$$W = aU + b_x e_2 + c_x e_3 = \lambda a e_1 + (\lambda pa + b_x) e_2 + (\lambda qa + c_x) e_3 = \begin{bmatrix} 0 & -(\lambda qa + c_x) & -\lambda a \\ \lambda qa + c_x & 0 & -(\lambda pa + b_x) \\ \lambda a & \lambda pa + b_x & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R})\,. \tag{32}$$

Further, we set

$$a = \sum_{i\geq 0} a_i \lambda^{-i}, \; b = \sum_{i\geq 0} b_i \lambda^{-i}, \quad c = \sum_{i\geq 0} c_i \lambda^{-i}, \; i \geq 0, \tag{33}$$

and take the initial values

$$a_0 = \frac{1}{\sqrt{p^2+q^2+1}}\,, \; b_0 = \frac{q}{\sqrt{p^2+q^2+1}}\,, \quad c_0 = -\frac{p}{\sqrt{p^2+q^2+1}}\,, \tag{34}$$

which are required by

$$a_{0,x} = pc_{0,x} - qb_{0,x}\,, \; pa_0 = -c_0\,, \; qa_0 = b_0\,.$$

The system (31) then leads to the following two recursion relations:

$$\begin{bmatrix} c_{i+1} \\ -b_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} c_i \\ -b_i \end{bmatrix}, \quad \Psi = \begin{bmatrix} \tilde{p}\partial^{-1}\tilde{q}\partial^2 & \partial - \tilde{p}\partial^{-1}\tilde{p}\partial^2 \\ -\partial + \tilde{q}\partial^{-1}\tilde{q}\partial^2 & -\tilde{q}\partial^{-1}\tilde{p}\partial^2 \end{bmatrix}, \; i \geq 0, \tag{35}$$

and

$$a_{i+1,x} = pc_{i+1,x} - qb_{i+1,x}\,, \; i \geq 0, \tag{36}$$

with $\tilde{p}$ and $\tilde{q}$ being defined by

$$\tilde{p} = \frac{p}{\sqrt{p^2+q^2+1}}\,, \; \tilde{q} = \frac{q}{\sqrt{p^2+q^2+1}}\,. \tag{37}$$

We will show in the next section that all vectors $(c_i, -b_i)^{\mathrm{T}}$, $i \geq 0$, are gradient and the adjoint operator of Ψ is hereditary. To determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely, we impose the following conditions on constants of integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0\,, \; i \geq 1\,. \tag{38}$$

This way, the first two sets can be computed as follows:

$$a_1 = \frac{qp_x - pq_x}{(p^2+q^2+1)^{\frac{3}{2}}},\ b_1 = -\frac{p_x}{(p^2+q^2+1)^{\frac{3}{2}}},$$

$$c_1 = -\frac{q_x}{(p^2+q^2+1)^{\frac{3}{2}}},$$

$$a_2 = -\frac{1}{(p^2+q^2+1)^{\frac{7}{2}}}\Big[\Big(3p^2+\frac{1}{2}q^2+\frac{1}{2}\Big)p_x^2+5pqp_xq_x + \Big(\frac{1}{2}p^2+3q^2+\frac{1}{2}\Big)q_x^2 - p(p^2+q^2+1)p_{xx} - q(p^2+q^2+1)q_{xx}\Big],$$

$$b_2 = \frac{1}{(p^2+q^2+1)^{\frac{7}{2}}}\Big[-\frac{1}{2}q(6p^2+q^2+1)p_x^2 + p(3p^2 - 2q^2+3)p_xq_x + \frac{5}{2}q(p^2+1)q_x^2 + pq(p^2+q^2+1) \cdot p_{xx} - (p^2+1)(p^2+q^2+1)q_{xx}\Big],$$

$$c_2 = \frac{1}{(p^2+q^2+1)^{\frac{7}{2}}}\Big[-\frac{5}{2}p(q^2+1)p_x^2 - q(3q^2-2p^2 +3)p_xq_x + \frac{1}{2}p(p^2+6q^2+1)q_x^2 + (q^2+1) \cdot (p^2+q^2+1)p_{xx} - pq(p^2+q^2+1)q_{xx}\Big].$$

Let us explain how to derive the recursion relations in (35). First from (31), we have

$$\begin{aligned} a_{i,x} &= pc_{i,x} - qb_{i,x} \\ &= p[-(pa_i)_x - b_{i-1,xx}] - q[(qa_i)_x + c_{i-1,xx}] \\ &= -(p^2+q^2)a_{i,x} - \frac{1}{2}(p^2+q^2)_x a_i - pb_{i-1,xx} \\ &\quad - qc_{i-1,xx},\ i \ge 1. \end{aligned}$$

This is equivalent to

$$\sqrt{p^2+q^2+1}\Big(\sqrt{p^2+q^2+1}a_i\Big)_x = - pb_{i-1,xx} - qc_{i-1,xx},\ i \ge 1,$$

which leads to

$$a_i = -\frac{1}{\sqrt{p^2+q^2+1}}\big(\partial^{-1}\tilde{p}\partial^2 b_{i-1} + \partial^{-1}\tilde{q}\partial^2 c_{i-1}\big),\quad i \ge 1. \tag{39}$$

Then again by (31) and using (38), we see that

$$c_{i+1} = -pa_{i+1} - b_{i,x},\ b_{i+1} = qa_{i+1} + c_{i,x},\ i \ge 0. \tag{40}$$

Now the recursion relations in (35) follows from the above recursion relation (39) for a_i.

Note that the first three sets of $\{a_i,b_i,c_i|i\ge 1\}$ are of differential function type. This is actually true for all sets. We prove here that the whole sequence of $\{a_i,b_i,c_i|i\ge 1\}$ is of differential function type. First from the stationary zero curvature equation (3), we can compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}(W^2) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U,W]) = 0.$$

Thus, by (38), we obtain an equality

$$(p^2+q^2+1)a^2\lambda^2 + 2a(pb_x+qc_x)\lambda + b_x^2 + c_x^2 = \lambda^2,$$

since for W defined by (32), we have

$$\frac{1}{2}\mathrm{tr}(W^2) = -(p^2+q^2+1)a^2\lambda^2 - 2a(pb_x+qc_x)\lambda - b_x^2 - c_x^2.$$

This equality gives a formula to define a_{i+1} by using the previous sets $\{a_j,b_j,c_j|j\le i\}$:

$$a_{i+1} = -\frac{1}{2a_0}\Big\{\sum_{k+l=i+1,k,l\ge 1} a_k a_l + \frac{1}{p^2+q^2+1} \cdot \Big[2\sum_{k+l=i,k,l\ge 0} a_k\big(pb_{l,x}+qc_{l,x}\big) + \sum_{k+l=i-1,k,l\ge 0}\big(b_{k,x}b_{l,x}+c_{k,x}c_{l,x}\big)\Big]\Big\},\ i\ge 1.$$

Combined with (40), a mathematical induction then shows that the whole sequence of $\{a_i,b_i,c_i|i\ge 1\}$ is of differential function type.

Now, based on both the recursion relations in (35) and (36) and the structure of the spectral matrix U in (29), we introduce

$$V^{[m]} = \lambda\Big[(\lambda^m a)_+ U + (\lambda^m b_x)_+ e_2 + (\lambda^m c_x)_+ e_3\Big],\quad m \ge 0, \tag{41}$$

and see that the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m\ge 0, \tag{42}$$

generate a hierarchy of soliton equations,

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,xx} \\ c_{m,xx} \end{bmatrix},\ m \ge 0, \tag{43}$$

where are all local. In the next section, we are going to show that all those soliton equations are Liouville integrable.

3. Bi-Hamiltonian Structures

3.1. Hamiltonian Structures

To construct Hamiltonian structures, we apply the trace identity (8) (or more generally the variational identity (9)). From the definition of U and W in (29) and (32), it is direct to see that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q & -1 \\ q & 0 & -p \\ 1 & p & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix},$$
$$\frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so, we have

$$\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -2\lambda(p^2+q^2+1)a - 2pb_x - 2qc_x,$$
$$\operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) = -2\lambda(\lambda pa + b_x) = 2\lambda^2 c,$$
$$\operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) = -2\lambda(\lambda qa + c_x) = -2\lambda^2 b.$$

Now, in this case, the trace identity (8), i. e.,

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(W\frac{\partial U}{\partial u}\right), \quad u = \begin{bmatrix} p \\ q \end{bmatrix},$$

presents

$$\frac{\delta}{\delta u}\int\left[-\lambda(p^2+q^2+1)a - pb_x - qc_x\right]\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda^2 c \\ -\lambda^2 b \end{bmatrix}.$$

Balancing coefficients of all powers of λ in the equality tells

$$\frac{\delta}{\delta u}\int\left[-(p^2+q^2+1)a_0\right]\mathrm{d}x = (\gamma+2)\begin{bmatrix} c_0 \\ -b_0 \end{bmatrix} \tag{44}$$

and

$$\frac{\delta}{\delta u}\int\left[-(p^2+q^2+1)a_m - pb_{m-1,x} - qc_{m-1,x}\right]\mathrm{d}x = (\gamma-m+2)\begin{bmatrix} c_m \\ -b_m \end{bmatrix}, \quad m \geq 1. \tag{45}$$

Checking a particular case in (44) yields $\gamma = -1$, and thus we obtain

$$\frac{\delta \mathcal{H}_m}{\delta u} = \begin{bmatrix} c_m \\ -b_m \end{bmatrix}, \quad m \geq 0, \tag{46}$$

where

$$\mathcal{H}_0 = \int\left(-\sqrt{p^2+q^2+1}\right)\mathrm{d}x,$$
$$\mathcal{H}_1 = \int \frac{qp_x - pq_x}{\sqrt{p^2+q^2+1}\left(\sqrt{p^2+q^2+1}+1\right)}\mathrm{d}x, \tag{47}$$

and

$$\mathcal{H}_{m+1} = \int \frac{(p^2+q^2+1)a_{m+1} + pb_{m,x} + qc_{m,x}}{m}\mathrm{d}x, \quad m \geq 1. \tag{48}$$

Here $\mathcal{H}_1$ was directly computed, since when $m = 1$, the coefficient on the right hand side of (45) is zero. It then follows that the soliton hierarchy (43) has the first Hamiltonian structures

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,xx} \\ c_{m,xx} \end{bmatrix} = J\begin{bmatrix} c_m \\ -b_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{49}$$

where the Hamiltonian operator is defined by

$$J = \begin{bmatrix} 0 & -\partial^2 \\ \partial^2 & 0 \end{bmatrix} \tag{50}$$

and the Hamiltonian functionals by (47) and (48).

The obtained functionals $\{\mathcal{H}_m\}_0^\infty$ generate an infinite sequence of conservation laws, not being of differential polynomial type, for each member in the counterpart hierarchy (43). We point out that conservation laws of differential polynomial type can be computed systematically through Bäcklund transformations (see, e. g., [26, 27]), from a Riccati equation generated from the underlying spectral problems (see, e. g., [17, 28]) or by using computer algebra systems (see, e. g., [29]).

3.2. Bi-Hamiltonian Structures

It is now a direct but lengthy computation to show by computer algebra systems that J defined by (50) and

$$M = J\Psi = \Psi^\dagger J = \begin{bmatrix} \partial^3 - \partial^2\tilde{q}\partial^{-1}\tilde{q}\partial^2 & \partial^2\tilde{q}\partial^{-1}\tilde{p}\partial^2 \\ \partial^2\tilde{p}\partial^{-1}\tilde{q}\partial^2 & \partial^3 - \partial^2\tilde{p}\partial^{-1}\tilde{p}\partial^2 \end{bmatrix} \tag{51}$$

constitute a Hamiltonian pair (see [14, 15] for examples), where Ψ is defined as in (35) and $\Psi^\dagger$ denotes the conjugate operator of Ψ. Consequently, any linear combination N of J and M satisfies

$$\int K^{\mathrm{T}} N'(u)[NS]T\,\mathrm{d}x + \mathrm{cycle}(K,S,T) = 0 \quad (52)$$

for all vector fields K, S, and T. This implies that the operator $\Phi = \Psi^{\dagger}$ is hereditary (see [30] for definition), i. e., it satisfies

$$\begin{aligned} &\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \\ &\Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K \end{aligned} \quad (53)$$

for all vector fields K and S. The condition (53) for the hereditariness is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K \Phi\,, \quad (54)$$

where K is an arbitrary vector field. The Lie derivative $L_K\Phi$ here is defined by

$$(L_K\Phi)S = \Phi[K,S] - [K,\Phi S]\,,$$

with $[\cdot,\cdot]$ being the Lie bracket of vector fields,

$$[K,S] = K'(u)[S] - S'(u)[K]\,,$$

where K' and S' denotes their Gateaux derivatives.

Note that an autonomous operator $\Phi = \Phi(u,u_x,\cdots)$ is a recursion operator of a given evolution equation $u_t = K = K(u)$ if and only if Φ needs to satisfy

$$L_K\Phi = 0\,. \quad (55)$$

It is easy to see that the operator $\Phi = \Psi^{\dagger}$ satisfies

$$L_{K_0}\Phi = 0\,, \;\text{ where }\; K_0 = \begin{bmatrix} \left(\frac{q}{\sqrt{p^2+q^2+1}}\right)_{xx} \\ -\left(\frac{p}{\sqrt{p^2+q^2+1}}\right)_{xx} \end{bmatrix}, \quad (56)$$

and thus

$$\begin{aligned} L_{K_m}\Phi &= L_{\Phi K_{m-1}}\Phi = \Phi L_{K_{m-1}}\Phi \\ &= \cdots = \Phi^m L_{K_0}\Phi = 0\,,\; m \ge 1\,, \end{aligned}$$

where the K_m are defined by (43). This implies that the operator $\Phi = \Psi^{\dagger}$ is a common hereditary recursion operator for the counterpart soliton hierarchy (43). We point out that there are also a few direct symbolic algorithms for computing recursion operators of nonlinear partial differential equations by computer algebra systems (see, e. g., [31]).

It now follows that all members, except the first one, in the counterpart soliton hierarchy (43) are bi-Hamiltonian,

$$u_{t_m} = K_m = J\frac{\delta\mathcal{H}_m}{\delta u} = M\frac{\delta\mathcal{H}_{m-1}}{\delta u}\,,\; m \ge 1\,, \quad (57)$$

where J, M, and $\mathcal{H}_m$ are defined by (50), (51), (47), and (48). Therefore, the counterpart hierarchy (43) is Liouville integrable, i. e., it possesses infinitely many commuting symmetries and conservation laws. Particularly, we have the Abelian symmetry algebra,

$$[K_k,K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0\,,\; k,l \ge 0\,, \quad (58)$$

and the two Abelian algebras of conserved functionals,

$$\begin{aligned} \{\mathcal{H}_k,\mathcal{H}_l\}_J = \int \left(\frac{\delta\mathcal{H}_k}{\delta u}\right)^{\mathrm{T}} J\frac{\delta\mathcal{H}_l}{\delta u}\,\mathrm{d}x = 0\,, \\ k,l \ge 0\,, \end{aligned} \quad (59)$$

and

$$\begin{aligned} \{\mathcal{H}_k,\mathcal{H}_l\}_M = \int \left(\frac{\delta\mathcal{H}_k}{\delta u}\right)^{\mathrm{T}} M\frac{\delta\mathcal{H}_l}{\delta u}\,\mathrm{d}x = 0\,, \\ k,l \ge 0\,. \end{aligned} \quad (60)$$

The first nonlinear bi-Hamiltonian integrable system in the counterpart soliton hierarchy (43) is as follows:

$$\begin{aligned} u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} &= K_1 = -\begin{bmatrix} \left(\frac{p_x}{(p^2+q^2+1)^{\frac{3}{2}}}\right)_{xx} \\ \left(\frac{q_x}{(p^2+q^2+1)^{\frac{3}{2}}}\right)_{xx} \end{bmatrix} \\ &= J\frac{\delta\mathcal{H}_1}{\delta u} = M\frac{\delta\mathcal{H}_0}{\delta u}\,. \end{aligned} \quad (61)$$

This is a different system from the WKI system of nonlinear soliton equations presented in [7].

4. Concluding Remarks

Based on the real matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$, we formulated a spectral problem by the same linear combination of basis vectors as the WKI one and introduced a counterpart of the WKI soliton hierarchy by the zero curvature formulation, whose soliton equations are of differential function type but not of differential polynomial type. By the trace identity, the resulting counterpart soliton hierarchy has been shown to be bi-Hamiltonian and so Liouville integrable.

The real Lie algebra of the special orthogonal group, $\mathrm{so}(3,\mathbb{R})$, is not isomorphic to the real Lie algebra $\mathrm{sl}(2,\mathbb{R})$ over $\mathbb{R}$, and thus the newly presented soliton hierarchy (43) and the WKI soliton hierarchy [7] are not gauge equivalent over $\mathbb{R}$. The main difference between the WKI soliton hierarchy and the counterpart soliton hierarchy is that the second Hamiltonian operators are different, which are

$$M=\begin{bmatrix}\frac{1}{4}\partial^2\bar{p}\partial^{-1}\bar{p}\partial^2 & \frac{1}{2}\partial^3-\frac{1}{4}\partial^2\bar{p}\partial^{-1}\bar{q}\partial^2\\ \frac{1}{2}\partial^3-\frac{1}{4}\partial^2\bar{q}\partial^{-1}\bar{p}\partial^2 & \frac{1}{4}\partial^2\bar{q}\partial^{-1}\bar{q}\partial^2\end{bmatrix}$$

and

$$M=\begin{bmatrix}\partial^3-\partial^2\tilde{q}\partial^{-1}\tilde{q}\partial^2 & \partial^2\tilde{q}\partial^{-1}\tilde{p}\partial^2\\ \partial^2\tilde{p}\partial^{-1}\tilde{q}\partial^2 & \partial^3-\partial^2\tilde{p}\partial^{-1}\tilde{p}\partial^2\end{bmatrix},$$

where

$$\bar{p}=\frac{p}{\sqrt{pq+1}},\quad \bar{q}=\frac{q}{\sqrt{pq+1}},$$

and

$$\tilde{p}=\frac{p}{\sqrt{p^2+q^2+1}},\quad \tilde{q}=\frac{q}{\sqrt{p^2+q^2+1}}.$$

They constitute two Hamiltonian pairs with the first Hamiltonian operators

$$J=\begin{bmatrix}0 & \partial^2\\ -\partial^2 & 0\end{bmatrix},\quad J=\begin{bmatrix}0 & -\partial^2\\ \partial^2 & 0\end{bmatrix},$$

and generate two different hereditary recursion operators,

$$\Phi=\begin{bmatrix}\frac{1}{2}\partial-\frac{1}{4}\partial^2\frac{1}{\bar{p}}\partial^{-1}\frac{1}{\bar{q}} & -\frac{1}{4}\partial^2\frac{1}{\bar{p}}\partial^{-1}\frac{1}{\bar{p}}\\ \frac{1}{4}\partial^2\frac{1}{\bar{q}}\partial^{-1}\frac{1}{\bar{q}} & -\frac{1}{2}\partial+\frac{1}{4}\partial^2\frac{1}{\bar{q}}\partial^{-1}\frac{1}{\bar{p}}\end{bmatrix}$$

and

$$\Phi=\begin{bmatrix}-\partial^2\frac{1}{\tilde{q}}\partial^{-1}\frac{1}{\tilde{p}} & \partial-\partial^2\frac{1}{\tilde{q}}\partial^{-1}\frac{1}{\tilde{q}}\\ -\partial+\partial^2\frac{1}{\tilde{p}}\partial^{-1}\frac{1}{\tilde{p}} & \partial^2\frac{1}{\tilde{p}}\partial^{-1}\frac{1}{\tilde{q}}\end{bmatrix},$$

respectively.

We remark that for a bi-Hamiltonian soliton hierarchy, one can make a kind of non-holonomic constraint

$$M\begin{bmatrix}f\\ g\end{bmatrix}=K_0$$

by using the first vector field K_0 and the second Hamiltonian operator M. Starting with such functions f and g, normally being non-local, and applying the first Hamiltonian operator J, one can introduce a so-called negative system of soliton equations

$$u_{t_{-1}}=J\begin{bmatrix}f\\ g\end{bmatrix},$$

and step by step, the whole negative hierarchy, which still has zero curvature representations similar to the ones for a given soliton hierarchy. However, in our case associated with $\mathrm{so}(3,\mathbb{R})$, the non-holonomic constraint itself defines an integro-differential system for f and g, which goes beyond our focused scope.

We also point out that there has recently been a growing interest in soliton hierarchies generated from spectral problems associated with non-semisimple Lie algebras. Various examples of bi-integrable couplings and tri-integrable couplings offer inspiring insights into the role they play in classifying multi-component integrable systems [32]. Multi-integrable couplings do bring diverse structures on recursion operators in block matrix form [18, 32]. It is significantly important in helping to understand essential properties of integrable systems to explore more mathematical structures behind integrable couplings.

It is known that there exist Hamiltonian structures for the perturbation equations [33, 34], but it is not clear how one can generate Hamiltonian structures for general integrable couplings [35, 36]. There is no guarantee that there will exist non-degenerate bilinear forms required in the variational identity on the underlying non-semisimple matrix Lie algebras. It is particularly interesting to see when Hamiltonian structures can exist for bi- or tri-integrable couplings [37–39], based on algebraic structures of non-semisimple matrix loop algebras. A basic question in the Hamiltonian theory of integrable couplings is whether there is any Hamiltonian structure for the bi-integrable coupling

$$u_t=K(u),\quad v=K'(u)[v],\quad w_t=K'(u)[w],$$

where K' is the Gateaux derivative.

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