Available online at www.sciencedirect.com

ScienceDirect

Mathematics and Computers in Simulation 127 (2016) 166–177

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original Articles

Nonlinear bi-integrable couplings with Hamiltonian structures

Wen-Xiu Ma*, Jinghan Meng, Mengshu Zhang

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

Received 21 September 2011; received in revised form 24 November 2012; accepted 24 November 2013
Available online 12 April 2014

Abstract

Bi-integrable couplings of soliton equations are presented through introducing non-semisimple matrix Lie algebras on which there exist non-degenerate, symmetric and ad-invariant bilinear forms. The corresponding variational identity yields Hamiltonian structures of the resulting bi-integrable couplings. An application to the AKNS spectral problem gives bi-integrable couplings with Hamiltonian structures for the AKNS equations.

PACS: 02.30.Ik; 11.30.−j

Keywords: Bi-integrable coupling; Zero curvature equation; Hamiltonian structure

1. Introduction

Zero curvature equations associated with semisimple Lie algebras generate typical soliton equations such as the Korteweg-de Vries equation, the nonlinear Schrödinger equation and the Kadomtsev–Petviashvili equation [2]. In the case of non-semisimple Lie algebras, zero curvature equations result in integrable couplings of soliton equations [23,24], and the perturbation equations generalizing the symmetry equations are examples of integrable couplings [20,9]. There are very rich mathematical structures behind integrable couplings [20,9,36,31,34] and the study of integrable couplings provide clues towards complete classification of multicomponent integrable equations [20,9,10].

The variational identity on general loop algebras presents Hamiltonian structures for the associated integrable couplings [18,13,16]. Based on special semi-direct sums of Lie algebras, Lax pairs of block form and with several spectral parameters bring diverse interesting integrable couplings with Hamiltonian structures [9,28,29]. A key to construct Hamiltonian structures by the variational identity is the existence of non-degenerate, symmetric and ad-invariant bilinear forms on the underlying Lie algebras.

Let us consider an integrable evolution equation

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots), \tag{1.1}$$

* Corresponding author. Tel.: +1 813 9749563; fax: +1 813 9742700.
E-mail addresses: wma3@usf.edu, mawx@cas.usf.edu (W. X. Ma), jmeng@mail.usf.edu (J. H. Meng), mzhang@mail.usf.edu (M. S. Zhang).

http://dx.doi.org/10.1016/j.matcom.2013.11.007
0378-4754/

where u is a column vector of dependent variables. Assume that it has a zero curvature representation [3]:

$$U_t - V_x + [U, V] = 0, \tag{1.2}$$

where the Lax pair [6], U and V, belongs to a matrix loop algebra, let us say, g, i.e.,

$$U = U(u, \lambda),\ V = V(u, \lambda) \in g,\ \lambda \ - \ \text{spectral} \quad \text{parameter}. \tag{1.3}$$

An integrable coupling of Eq. (1.1) (see [20,9] for definition):

$$\overline{u}_t = \overline{K}_1(\overline{u}) = \begin{bmatrix} K(u) \\ S(u, v) \end{bmatrix}, \overline{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{1.4}$$

is called nonlinear, if $S(u, v)$ is nonlinear with respect to the sub-vector v of dependent variables [26,17]. An integrable system of the form

$$\overline{u}_t = \overline{K}(\overline{u}) = \begin{bmatrix} K(u) \\ S_1(u, v) \\ S_2(u, v, w) \end{bmatrix}, \overline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{1.5}$$

is called a bi-integrable coupling of Eq. (1.1). Note that in (1.5), S_1 does not depend on w, and the whole system is of triangular form. In this paper, we would like to explore some mathematical structures of Lie algebras and zero curvature equations, to construct bi-integrable couplings and their Hamiltonian structures by using the variational identity associated with the enlarged Lax pairs.

This paper is organized as follows. In Section 2, a kind of matrix Lie algebras will be introduced. Zero curvature equations on the resulting Lie algebras present bi-integrable couplings of soliton equations. In Section 3, an application to the AKNS soliton hierarchy will be made to generate nonlinear bi-integrable couplings and the corresponding variational identity yields Hamiltonian structures for the obtained integrable couplings. An important step in constructing Hamiltonian structures is to find non-degenerate, symmetric and ad-invariant bilinear forms on the underlying Lie algebras. In the last section, a few of concluding remarks will be given, along with discussion on a particular bi-integrable coupling.

2. Matrix Lie algebras and bi-integrable couplings

Let α be an arbitrary fixed constant, which could be zero. To generate bi-integrable couplings, we introduce a kind of block matrices

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & A_2 \\ 0 & 0 & A_1 \end{bmatrix}, \tag{2.1}$$

where $A_i = A_i(\lambda)$, $1 \le i \le 3$, are square matrices of the same order, depending on the free parameter λ. Obviously, we have the matrix commutator relation

$$[M(A_1, A_2, A_3), M(B_1, B_2, B_3)] = M(C_1, C_2, C_3), \tag{2.2}$$

with

$$\begin{cases} C_1 = [A_1, B_1], \\ C_2 = [A_1, B_2] + [A_2, B_1] + \alpha[A_2, B_2], \\ C_3 = [A_1, B_3] + [A_3, B_1] + [A_2, B_2]. \end{cases} \tag{2.3}$$

This closure property implies that all block matrices defined by (2.1) form a matrix Lie algebra. Such matrix Lie algebras create a basis for us to generate nonlinear Hamiltonian bi-integrable couplings. The block A_1 corresponds to the original integrable equation, and the other two blocks A_2 and A_3 are used to generate the supplementary vector fields S_1 and S_2. The commutator $[A_2, B_2]$ yields nonlinear terms in the resulting bi-integrable couplings.

Let us assume that an integrable equation

$$u_t = K(u) \tag{2.4}$$

possesses a zero curvature representation

$$U_t - V_x + [U, V] = 0, \tag{2.5}$$

where two square Lax matrices U and V usually belong to semisimple matrix Lie algebras (see, e.g., [4]). Now we introduce an enlarged spectral matrix

$$\overline{U} = \overline{U}(\overline{u}, \lambda) = \begin{bmatrix} U(u,\lambda) & U_1(u_1,\lambda) & U_2(u_2,\lambda) \\ 0 & U(u,\lambda) + \alpha U_1(u_1,\lambda) & U_1(u_1,\lambda) \\ 0 & 0 & U(u,\lambda) \end{bmatrix}, \tag{2.6}$$

where $\overline{u}$ consists of u, u_1 and u_2 (possibly, three vectors of dependent variables). Then an enlarged zero curvature equation

$$\overline{U}_t - \overline{V}_x + [\overline{U}, \overline{V}] = 0 \tag{2.7}$$

with

$$\overline{V} = \overline{V}(\overline{u}, \lambda) = \begin{bmatrix} V(u,\lambda) & V_1(u,u_1,\lambda) & V_2(u,u_1,u_2,\lambda) \\ 0 & V(u,\lambda) + \alpha V_1(u,u_1,\lambda) & V_1(u,u_1,\lambda) \\ 0 & 0 & V(u,\lambda) \end{bmatrix} \tag{2.8}$$

gives rise to

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{1,t} - V_{1,x} + [U, V_1] + [U_1, V] + \alpha[U_1, V_1] = 0, \\ U_{2,t} - V_{2,x} + [U, V_2] + [U_2, V] + [U_1, V_1] = 0. \end{cases} \tag{2.9}$$

This is a bi-integrable coupling of the evolution Eq. (2.4), noting the zero curvature representation (2.5) of (2.4). Normally, it is nonlinear with respect to u_1 and u_2, thereby providing a nonlinear bi-integrable coupling.

As usual, we take a solution

$$\overline{W} = \overline{W}(\overline{u}, \lambda) = \begin{bmatrix} W(u) & W_1(u,u_1) & W_2(u,u_1,u_2) \\ 0 & W(u) + \alpha W_1(u,u_1) & W_1(u,u_1) \\ 0 & 0 & W(u) \end{bmatrix} \tag{2.10}$$

to the enlarged stationary zero curvature equation

$$\overline{W}_x = [\overline{U}, \overline{W}]. \tag{2.11}$$

This equation is equivalent to

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W] + \alpha[U_1, W_1], \\ W_{2,x} = [U, W_2] + [U_2, W] + [U_1, W_1]. \end{cases} \tag{2.12}$$

We can often (see, e.g., [30,8]) have solutions of the form

$$W = \sum_{i=0}^{\infty} W_i \lambda^{-i}, \ W_1 = \sum_{i=0}^{\infty} W_{1,i} \lambda^{-i}, \ W_2 = \sum_{i=0}^{\infty} W_{2,i} \lambda^{-i}. \tag{2.13}$$

Then introduce a set of enlarged matrix modifications Δ_m, $m \geq 0$, and define the Lax matrices

$$\overline{V}^{[m]} = (\lambda^m \overline{W})_+ + \Delta_m, \tag{2.14}$$

where the subscript + denotes the polynomial part, such that the enlarged zero curvature equations

$$\overline{U}_{t_m} - \overline{V}^{[m]}_x + [\overline{U}, \overline{V}^{[m]}] = 0,\ m \geq 0, \tag{2.15}$$

generate a soliton hierarchy which provides nonlinear bi-integrable couplings of Eq. (2.4).

We further apply the associated variational identity [18,25]:

$$\frac{\delta}{\delta \overline{u}} \int \langle \overline{W}, \overline{U}_\lambda \rangle \, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle \overline{W}, \overline{U}_{\overline{u}} \rangle, \tag{2.16}$$

where γ is some constant, to furnish Hamiltonian structures for the bi-integrable couplings described above. In the variational identity (2.16), $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form over the underlying Lie algebra, which consists of square matrices of the form (2.6) (see [18,15] for details).

In the next section, we will apply the above computational paradigm to the AKNS hierarchy, thus generating a hierarchy of nonlinear Hamiltonian bi-integrable couplings for the AKNS equations. We remark that our general idea works for both positive and negative soliton hierarchies.

3. Application to the AKNS hierarchy

3.1. AKNS hierarchy

The spectral matrix

$$U = U(u, \lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \lambda \ - \ \text{spectral parameter}, \tag{3.1}$$

generates the AKNS hierarchy of soliton equations [1] (see also [35]). There are other integrable equations associated with $gl(2)$ (see, e.g., [37]). Once setting

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i \geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{3.2}$$

the stationary zero curvature equation $W_x = [U, W]$ gives

$$b_{i+1} = -\frac{1}{2} b_{i,x} - p a_i, \quad c_{i+1} = \frac{1}{2} c_{i,x} - q a_i, \quad a_{i+1,x} = p c_{i+1} - q b_{i+1}, \quad i \geq 0. \tag{3.3}$$

Taking the initial data as

$$a_0 = -1,\ b_0 = c_0 = 0, \tag{3.4}$$

and assuming $a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0$, $i \geq 1$ (equivalently selecting constants of integration to be zero), the recursion relation (3.3) uniquely defines all differential polynomial functions a_i, b_i and c_i, $i \geq 1$. The first few sets are listed as follows:

$$b_1 = p,\ c_1 = q,\ a_1 = 0;$$

$$b_2 = -\frac{1}{2} p_x,\ c_2 = \frac{1}{2} q_x,\ a_2 = \frac{1}{2} pq;$$

$$b_3 = \frac{1}{4} p_{xx} - \frac{1}{2} p^2 q,\ c_3 = \frac{1}{4} q_{xx} - \frac{1}{2} pq^2,\ a_3 = \frac{1}{4}(p q_x - p_x q);$$

$$b_4 = -\frac{1}{8} p_{xxx} + \frac{3}{4} p_x pq,\ c_4 = \frac{1}{8} q_{xxx} - \frac{3}{4} p q_x q,$$

$$a_4 = \frac{1}{8} p_{xx} q - \frac{1}{8} p_x q_x + \frac{1}{8} p q_{xx} - \frac{3}{8} p^2 q^2.$$

The zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad V^{[m]} = (\lambda^m W)_+,\ m\geq 0, \tag{3.5}$$

generate the AKNS hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ 2q \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u},\ m\geq 0, \tag{3.6}$$

with the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functions:

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},\ \Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \mathcal{H}_m = \int \frac{2a_{m+2}}{m+1}\,dx, \tag{3.7}$$

where $m \geq 0$ and $\partial = \frac{\partial}{\partial x}$.

3.2. Integrable couplings

3.2.1. An integrable coupling hierarchy

We begin with an enlarged spectral matrix

$$\overline{U} = \overline{U}(\overline{u}, \lambda) = \begin{bmatrix} U & U_1 & U_2 \\ 0 & U + \alpha U_1 & U_1 \\ 0 & 0 & U \end{bmatrix}, \quad \overline{u} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}, \tag{3.8}$$

where U is defined as in (3.1) and the supplementary spectral matrices U_1 and U_2 read

$$\begin{cases} U_1 = U_1(u_1) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, & u_1 = \begin{bmatrix} r \\ s \end{bmatrix}, \\ U_2 = U_2(u_2) = \begin{bmatrix} 0 & v \\ w & 0 \end{bmatrix}, & u_2 = \begin{bmatrix} v \\ w \end{bmatrix}. \end{cases} \tag{3.9}$$

To solve the enlarged stationary zero curvature Eq. (2.11), we take a solution of the following form

$$\overline{W} = \begin{bmatrix} W & W_1 & W_2 \\ 0 & W + \alpha W_1 & W_1 \\ 0 & 0 & W \end{bmatrix}, \tag{3.10}$$

where W, defined by (3.2), solves $W_x = [U, W]$, and

$$\begin{cases} W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \\ W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix}. \end{cases} \tag{3.11}$$

Then, the second and third equations in (2.12) equivalently generate

$$\begin{cases} e_x = pg - qf + \alpha rg - \alpha sf + rc - sb, \\ f_x = -2\lambda f - 2pe - 2\alpha re - 2ra, \\ g_x = 2qe + 2\lambda g + 2\alpha se + 2sa, \end{cases}$$

and

$$\begin{cases} e'_x = pg' - qf' + rg - sf + vc - wb, \\ f'_x = -2\lambda f' - 2pe' - 2re - 2va, \\ g'_x = 2qe' + 2\lambda g' + 2se + 2wa, \end{cases}$$

respectively. Trying a formal series solution $\overline{W}$ by assuming

$$\begin{cases} e = \sum_{i=0}^{\infty} e_i \lambda^{-i}, \quad f = \sum_{i=0}^{\infty} f_i \lambda^{-i}, \quad g = \sum_{i=0}^{\infty} g_i \lambda^{-i}, \\ e' = \sum_{i=0}^{\infty} e'_i \lambda^{-i}, \quad f' = \sum_{i=0}^{\infty} f'_i \lambda^{-i}, \quad g' = \sum_{i=0}^{\infty} g'_i \lambda^{-i}, \end{cases} \tag{3.12}$$

we obtain

$$\begin{cases} f_{i+1} = -\frac{1}{2} f_{i,x} - pe_i - \alpha re_i - ra_i, \\ g_{i+1} = \frac{1}{2} g_{i,x} - qe_i - \alpha se_i - wa_i, \\ e_{i+1,x} = pg_{i+1} - qf_{i+1} + \alpha rg_{i+1} - \alpha sf_{i+1} + rc_{i+1} - sb_{i+1}, \\ f'_{i+1} = -\frac{1}{2} f'_{i,x} - pe'_i - re_i - va_i, \\ g'_{i+1} = \frac{1}{2} g'_{i,x} - qe'_i - se_i - wa_i, \\ e'_{i+1,x} = pg'_i - qf'_i + rg_i - sf_i + vc_i - wb_i, \end{cases} \tag{3.13}$$

where $i \geq 0$. We select the initial data to be

$$e_0 = -1, \quad f_0 = g_0 = 0; \quad e'_0 = -1, \quad f'_0 = g'_0 = 0; \tag{3.14}$$

and assume that

$$e_i|_{\overline{u}=0} = f_i|_{\overline{u}=0} = g_i|_{\overline{u}=0} = 0, \quad e'_i|_{\overline{u}=0} = f'_i|_{\overline{u}=0} = g'_i|_{\overline{u}=0} = 0, \quad i \geq 1. \tag{3.15}$$

Then the recursion relation (3.13) determines the sequence of e_i, f_i, g_i and e_i', f_i', g_i', $i \geq 1$, uniquely. The first few sets of functions are computed as follows:

$$f_1 = p + r + \alpha r, \quad g_1 = q + s + \alpha s, \quad e_1 = 0;$$
$$f_2 = -\frac{1}{2}p_x - \frac{1}{2}\alpha r_x - \frac{1}{2}r_x, \quad g_2 = \frac{1}{2}q_x + \frac{1}{2}\alpha s_x + \frac{1}{2}s_x,$$
$$e_2 = \frac{1}{2}pq + \frac{1}{2}ps + \frac{1}{2}qr + \frac{\alpha}{2}ps + \frac{\alpha}{2}qr + \frac{\alpha+\alpha^2}{2}rs;$$
$$f_3 = \frac{1}{4}p_{xx} + \frac{\alpha+1}{4}r_{xx} - \frac{1}{2}p^2q - \frac{\alpha+1}{2}p^2s - (\alpha+1)p(q+\alpha s)r - \frac{\alpha(\alpha+1)}{2}qr^2 - \frac{\alpha^2(\alpha+1)}{2}r^2s,$$
$$g_3 = \frac{1}{4}q_{xx} + \frac{\alpha+1}{4}s_{xx} - \frac{1}{2}q^2p - \frac{\alpha+1}{2}q^2r - (\alpha+1)(p+\alpha r)qs - \frac{\alpha(\alpha+1)}{2}ps^2 - \frac{\alpha^2(\alpha+1)}{2}rs^2,$$
$$e_3 = -\frac{1}{4}p_xq + \frac{1}{4}pq_x - \frac{\alpha+1}{4}p_xs + \frac{\alpha+1}{4}ps_x + \frac{\alpha+1}{4}q_xr - \frac{\alpha+1}{4}qr_x - \frac{\alpha(\alpha+1)}{4}r_xs + \frac{\alpha(\alpha+1)}{4}rs_x;$$
$$f_1' = p + r + v, \quad g_1' = q + s + w, \quad e_1' = 0;$$
$$f_2' = -\frac{1}{2}p_x - \frac{1}{2}r_x - \frac{1}{2}v_x, \quad g_2' = \frac{1}{2}q_x + \frac{1}{2}s_x + \frac{1}{2}w_x,$$
$$e_2' = \frac{1}{2}pq + \frac{1}{2}ps + \frac{1}{2}pw + \frac{1}{2}qr + \frac{1}{2}qv + \frac{\alpha+1}{2}rs;$$
$$f_3' = \frac{1}{4}p_{xx} + \frac{1}{4}r_{xx} + \frac{1}{4}v_{xx} - \frac{1}{2}p^2q - \frac{1}{2}p^2s - \frac{1}{2}p^2w - pqr - pqv - (\alpha+1)prs - \frac{\alpha+1}{2}qr^2 - \frac{\alpha(\alpha+1)}{2}r^2s,$$
$$g_3' = \frac{1}{4}q_{xx} + \frac{1}{4}s_{xx} + \frac{1}{4}w_{xx} - \frac{1}{2}pq^2 - \frac{1}{2}q^2r - \frac{1}{2}q^2v - pqs - pqw - (\alpha+1)qrs - \frac{\alpha+1}{2}ps^2 - \frac{\alpha(\alpha+1)}{2}rs^2,$$
$$e_3' = -\frac{1}{4}p_xq + \frac{1}{4}pq_x - \frac{1}{4}p_xs + \frac{1}{4}ps_x - \frac{1}{4}p_xw + \frac{1}{4}pw_x + \frac{1}{4}q_xr - \frac{1}{4}qr_x + \frac{1}{4}q_xv - \frac{1}{4}qv_x - \frac{\alpha+1}{4}r_xs + \frac{\alpha+1}{4}rs_x.$$

Note that they are all differential polynomials.

For each integer $m \geq 0$, take

$$\overline{V}^{[m]} = (\lambda^m \overline{W})_+ = \begin{bmatrix} V^{[m]} & V_1^{[m]} & V_2^{[m]} \\ 0 & V^{[m]} + \alpha V_1^{[m]} & V_1^{[m]} \\ 0 & 0 & V^{[m]} \end{bmatrix}, \tag{3.16}$$

where $V_i^{[m]} = (\lambda^m W_i)_+, \quad i = 1, 2$, and then, the enlarged zero curvature equation

$$\overline{U}_{t_m} - (\overline{V}^{[m]})_x + [\overline{U}, \overline{V}^{[m]}] = 0$$

yields

$$U_{1,t_m} - V_{1,x}^{[m]} + [U, V_1^{[m]}] + [U_1, V^{[m]}] + \alpha[U_1, V_1^{[m]}] = 0,$$
$$U_{2,t_m} - V_{2,x}^{[m]} + [U, V_2^{[m]}] + [U_2, V^{[m]}] + [U_1, V_1^{[m]}] = 0,$$

together with the mth AKNS system in (3.6). This gives rise to

$$\overline{v}_{t_m} = S_m = S_m(\overline{u}) = \begin{bmatrix} S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix}, \quad \overline{v} = (r, s, v, w)^T, \quad m \geq 0, \tag{3.17}$$

where

$$S_{1,m}(u,u_1)=\begin{bmatrix}-2f_{m+1}\\ 2g_{m+1}\end{bmatrix},\ S_{2,m}(u,u_1,u_2)=\begin{bmatrix}-2f'_{m+1}\\ 2g'_{m+1}\end{bmatrix}.$$

Therefore, the hierarchy of enlarged zero curvature equations present a hierarchy of bi-integrable couplings:

$$\overline{u}_{t_m}=\begin{bmatrix}p\\ q\\ r\\ s\\ v\\ w\end{bmatrix}_{t_m}=\overline{K}_m(\overline{u})=\begin{bmatrix}K_m(u)\\ S_{1,m}(u,u_1)\\ S_{2,m}(u,u_1,u_2)\end{bmatrix}=\begin{bmatrix}-2b_{m+1}\\ 2c_{m+1}\\ -2f_{m+1}\\ 2g_{m+1}\\ -2f'_{m+1}\\ 2g'_{m+1}\end{bmatrix},\quad m\geq 0, \tag{3.18}$$

for the AKNS hierarchy (3.6). Except the first two, all bi-integrable couplings constructed above are nonlinear, since the supplementary systems (3.17) with $m\geq 2$ are nonlinear with respect to the four dependent variables r, s, v, w. This implies that (3.18) provides a hierarchy of nonlinear bi-integrable couplings for the AKNS hierarchy of soliton equations. The first nonlinear bi-integrable coupling system reads

$$\begin{cases} p_{t_2}=-\frac{1}{2}p_{xx}+p^2q,\quad q_{t_2}=\frac{1}{2}q_{xx}-pq^2,\\ r_{t_2}=-\frac{1}{2}p_{xx}-\frac{\alpha+1}{2}r_{xx}+p^2q+(\alpha+1)p^2s+2(\alpha+1)pqr+2\alpha(\alpha+1)prs+\alpha(\alpha+1)qr^2+\alpha^2(\alpha+1)r^2s,\\ s_{t_2}=\frac{1}{2}q_{xx}+\frac{\alpha+1}{2}s_{xx}-q^2p-2(\alpha+1)pqs-(\alpha+1)q^2r-2\alpha(\alpha+1)qrs-\alpha(\alpha+1)ps^2-\alpha^2(\alpha+1)rs^2,\\ v_{t_2}=-\frac{1}{2}p_{xx}-\frac{1}{2}r_{xx}-\frac{1}{2}v_{xx}+p^2q+p^2s+p^2w+2pqr+2pqv+2(\alpha+1)prs+(\alpha+1)qr^2+\alpha(\alpha+1)r^2s,\\ w_{t_2}=\frac{1}{2}q_{xx}+\frac{1}{2}s_{xx}+\frac{1}{2}w_{xx}-pq^2-q^2r-q^2v-2pqs-2pqw-2(\alpha+1)qrs-(\alpha+1)ps^2-\alpha(\alpha+1)rs^2. \end{cases} \tag{3.19}$$

3.2.2. Hamiltonian structures

In order to generate Hamiltonian structures of the obtained bi-integrable couplings, we have to compute non-degenerate, symmetric and ad-invariant bilinear forms on the adopted Lie algebra:

$$\overline{g}=\left\{\begin{bmatrix}A_1 & A_2 & A_3\\ 0 & A_1+\alpha A_2 & A_2\\ 0 & 0 & A_1\end{bmatrix}\middle|\ A_1,A_2,A_3\in\widetilde{\mathrm{sl}}(2)\right\}, \tag{3.20}$$

where the loop algebra $\widetilde{\mathrm{sl}}(2)$ is defined by

$$\widetilde{\mathrm{sl}}(2)=\{A(\lambda)\in\mathrm{sl}(2)|\,\mathrm{entries of}\,A(\lambda)\ -\mathrm{Laurent series in}\,\lambda\}.$$

As usual, we transform the Lie algebra $\overline{g}$ into a vector form through the mapping

$$\delta:\overline{g}\to\mathbb{R}^9,\ A\mapsto(a_1,a_2,\cdots,a_9)^T, \tag{3.21}$$

where

$$A=\begin{bmatrix}A_1 & A_2 & A_3\\ 0 & A_1+\alpha A_2 & A_2\\ 0 & 0 & A_1\end{bmatrix}\in\overline{g},\ A_i=\begin{bmatrix}a_{3i-2} & a_{3i-1}\\ a_{3i} & -a_{3i-2}\end{bmatrix},\ 1\leq i\leq 3. \tag{3.22}$$

The mapping δ induces a Lie algebraic structure on $\mathbb{R}^9$, isomorphic to the above matrix Lie algebra $\overline{g}$. The corresponding commutator $[\cdot,\cdot]$ on the resulting Lie algebra $\mathbb{R}^9$ reads

$$[a,b]^T = a^T R(b),\ a = (a_1, a_2, \ldots, a_9)^T,\ b = (b_1, b_2, \ldots, b_9)^T \in \mathbb{R}^9, \tag{3.23}$$

where $R(b)$ is given by

$$R(b) = \begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & R_1 + \alpha R_2 & R_2 \\ 0 & 0 & R_1 \end{bmatrix},\ R_i = \begin{bmatrix} 0 & 2b_{3i-1} & -2b_{3i} \\ b_{3i} & -2b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2b_{3i-2} \end{bmatrix}, \quad 1 \le i \le 3.$$

Let us consider an arbitrary bilinear form on $\mathbb{R}^9$:

$$\langle a,b\rangle = a^T F b, \tag{3.24}$$

where F is a constant matrix. Two of the required properties, the symmetric property $\langle a,b\rangle = \langle b,a\rangle$ and the ad-invariance property

$$\langle a,[b,c]\rangle = \langle [a,b],c\rangle, \tag{3.25}$$

tell that $F^T = F$ and

$$(R(b)F)^T = -R(b)F \text{ forall } b \in \mathbb{R}^9.$$

This matrix equation on F yields a system of linear equations on the elements of F. Solving the resulting linear system gives

$$F = \begin{bmatrix} \frac{1}{2}\eta_1 & \frac{1}{2}\eta_2 & \eta_3 \\ \frac{1}{2}\eta_2 & \frac{\alpha}{2}\eta_2 + \eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.26}$$

where η_1, η_2 and η_3 are arbitrary constants and $\otimes$ denotes the Kronecker product of matrices.

This way, a required bilinear form on the underlying Lie algebra $\overline{g}$ is given by

$$\begin{aligned}\langle A,B\rangle_{\overline{g}} &= \langle \delta(A),\delta(B)\rangle_{\mathbb{R}^9} = (a_1,a_2,\cdots,a_9)F(b_1,b_2,\cdots,b_9)^T = \eta_1(a_1b_1 + \frac{1}{2}a_2b_3 + \frac{1}{2}a_3b_2) \\ &+ \eta_2[a_1b_4 + \frac{1}{2}a_2b_6 + \frac{1}{2}a_3b_5 + a_4(b_1+\alpha b_4) + \frac{1}{2}a_5(b_3+\alpha b_6) + \frac{1}{2}a_6(b_2+\alpha b_5)] + \eta_3[2a_1b_7 + a_2b_9 \\ &+ a_3b_8 + 2a_4b_4 + a_5b_6 + a_6b_5 + 2a_7b_1 + a_8b_3 + a_9b_2],\end{aligned} \tag{3.27}$$

where A and B are two block matrices of the form defined by (3.22). This bilinear form (3.27) is symmetric and ad-invariant:

$$\langle A,B\rangle = \langle B,A\rangle,\ \langle A,[B,C]\rangle = \langle [A,B],C\rangle,\ A,B,C \in \overline{g},$$

and it is non-degenerate if and only if

$$\det(F) = (\alpha\eta_2 + 2\eta_3)^3\eta_3^6 \ne 0. \tag{3.28}$$

To use the variational identity, let us further compute that

$$\langle \overline{W}, \overline{U}_\lambda\rangle = -\eta_1 a - \eta_2 e - 2\eta_3 e',$$

$$\langle \overline{W}, \overline{U}_{\overline{u}}\rangle = \left(\frac{1}{2}\eta_1 c + \frac{1}{2}\eta_2 g + \eta_3 g', \frac{1}{2}\eta_1 b + \frac{1}{2}\eta_2 f + \eta_3 f', \frac{1}{2}\eta_2(c+\alpha g) + \eta_3 g', \frac{1}{2}\eta_2(b+\alpha f) + \eta_3 f', \eta_3 c, \eta_3 b\right)^T,$$

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \overline{W}, \overline{W}\rangle| = 0,$$

where $\overline{W}$ is given by (3.10). Therefore, the corresponding variational identity (2.16) leads to

$$\frac{\delta}{\delta\bar{u}}\int\frac{\eta_1 a_{m+1}+\eta_2 e_{m+1}+2\eta_3 e'_{m+1}}{m}\,dx=\Big(\frac{1}{2}\eta_1 c_m+\frac{1}{2}\eta_2 g_m+\eta_3 g'_m,\ \frac{1}{2}\eta_1 b_m+\frac{1}{2}\eta_2 f_m$$
$$+\eta_3 f'_m,\ \frac{1}{2}\eta_2(c_m+\alpha g_m)+\eta_3 g'_m,\ \frac{1}{2}\eta_2(b_m+\alpha f_m)+\eta_3 f'_m,\ \eta_3 c_m,\ \eta_3 b_m\Big)^T, \tag{3.29}$$

where $m\geq 1$. It follows thus that the AKNS bi-integrable couplings in (3.18) possess the following Hamiltonian structures:

$$\bar{u}_{t_m}=\overline{K}_m(\bar{u})=\overline{J}\frac{\delta\overline{\mathcal{H}}_m}{\delta\bar{u}},\ m\geq 0, \tag{3.30}$$

where the Hamiltonian operator is

$$\overline{J}=\begin{bmatrix}\frac{1}{2}\eta_1 & \frac{1}{2}\eta_2 & \eta_3\\ \frac{1}{2}\eta_2 & \frac{\alpha}{2}\eta_2+\eta_3 & 0\\ \eta_3 & 0 & 0\end{bmatrix}^{-1}\otimes\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix} \tag{3.31}$$

and the Hamiltonian functionals read

$$\overline{\mathcal{H}}_m=\int\frac{\eta_1 a_{m+2}+\eta_2 e_{m+2}+2\eta_3 e'_{m+2}}{m+1}\,dx,\ m\geq 0. \tag{3.32}$$

A direct computation shows a recursion relation

$$\overline{K}_{m+1}=\overline{\Phi}\overline{K}_m,\ m\geq 1, \tag{3.33}$$

where the recursion operator $\overline{\Phi}$ is given by

$$\overline{\Phi}=\begin{bmatrix}\Phi & 0 & 0\\ \Phi_1 & \Phi+\alpha\Phi_1 & 0\\ \Phi_2 & \Phi_1 & \Phi\end{bmatrix}, \tag{3.34}$$

with Φ being given by (3.7) and

$$\Phi_1=\begin{bmatrix}r\partial^{-1}q+(p+\alpha r)\partial^{-1}s & r\partial^{-1}p+(p+\alpha r)\partial^{-1}r\\ -s\partial^{-1}q-(q+\alpha s)\partial^{-1}s & -s\partial^{-1}p-(q+\alpha s)\partial^{-1}r\end{bmatrix}, \tag{3.35}$$

$$\Phi_2=\begin{bmatrix}v\partial^{-1}q+r\partial^{-1}s+p\partial^{-1}w & v\partial^{-1}p+r\partial^{-1}r+p\partial^{-1}v\\ -w\partial^{-1}q-s\partial^{-1}s-q\partial^{-1}w & -w\partial^{-1}p-s\partial^{-1}r-q\partial^{-1}v\end{bmatrix}. \tag{3.36}$$

Furthermore, we can show $\overline{J}\overline{\Phi}^\dagger=\overline{\Phi}\overline{J}$, where $\overline{\Phi}^\dagger$ denotes the adjoint operator of $\overline{\Phi}$. This tells that all bi-integrable couplings in (3.18) commute with each other and so do all conserved functionals in (3.32). We point out that the enlarged recursion operator (3.34) is closely related to the underlying Lie algebra (3.20).

It is also direct to verify that $\overline{J}$ and $\overline{\Phi}\overline{J}$ form a Hamiltonian pair [27,5], and so, $\overline{\Phi}$ is a common hereditary recursion operator for the hierarchy of Hamiltonian bi-integrable couplings (3.18). In particular, the bi-integrable coupling (3.19) has a bi-Hamiltonian structure.

4. Concluding remarks

We have introduced a class of Lie algebras consisting of specific block matrices, and presented a computational paradigm for construction of nonlinear bi-integrable couplings, starting from those suggested Lie algebras. The variational identity on the adopted Lie algebras was used to construct Hamiltonian structures of the resulting bi-integrable couplings. An application to the AKNS spectral problem resulted in a hierarchy of nonlinear bi-integrable Hamiltonian couplings for the AKNS equations. The obtained results complement well some of the ideas of generating linear and nonlinear integrable couplings [20,18,17,11], and show there can exist various kinds of integrable couplings for a given integrable equation.

We remark that high order block type matrix Lie algebras will allow us to generate multi-integrable couplings and more diverse integrable couplings, which can also supplement the spectral matrices of the other forms in the literature (see, e.g., [12,21]). Typical integrable properties such as Hirota bilinear forms can be discussed for the presented integrable couplings (see, e.g., [22]). Another interesting property is the linear superposition principle on subspaces of solutions, and all soliton solutions belong to the closure of such subspaces of exponential wave solutions [19].

We also point out that a particular bi-integrable coupling is

$$u_t = K(u), \quad v_t = K'(u)[v], \quad w_t = K'(u)[w],$$

where the Gateaux derivative is defined as follows

$$P'(u)[S] = \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} P(u + \varepsilon S, u_x + \varepsilon S_x, \cdots)$$

for an object $P = P(u, u_x, \ldots)$. It is open to us whether this system possesses any Hamiltonian structure. There are some enlarged zero curvature representations for this system [21], but all symmetric and ad-invariant bilinear forms are degenerate on the corresponding Lie algebras. It has been an important task for us to explore more about multi-integrable couplings including the above intriguing bi-integrable coupling to enrich multi-component integrable equations (see, e.g., [32,14,7,33,38]). It is expected that more particular new Lie algebras generating Hamiltonian integrable couplings can be presented, to understand and work towards complete classification of multi-component integrable equations.

Acknowledgments

The work was supported in part by the State Administration of Foreign Experts Affairs of China, the National Natural Science Foundation of China (Nos. 10971136, 10831003, 61072147 and 11071159), Chunhui Plan of the Ministry of Education of China, Zhejiang Innovation Project (Grant No. T200905), the Natural Science Foundation of Shanghai and the Shanghai Leading Academic Discipline Project (No. J50101).

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315.
[2] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, PA, 1981.
[3] A. Das, Integrable Models, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.
[4] V.G. Drinfel'd, V.V. Sokolov, Equations of Korteweg-de Vries type and simple Lie algebras, Soviet Math. Dokl. 23 (1981) 457–462.
[5] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D 4 (1981–1982) 47–66.
[6] P. Lax, Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. 21 (1968) 467–490.
[7] Z. Li, H.H. Dong, Two integrable couplings of the Tu hierarchy and their Hamiltonian structures, Comput. Math. Appl. 55 (2008) 2643–2652.
[8] W.X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction, Chin. Ann. Math. Ser. A 13 (1992) 115–123.
[9] W.X. Ma, Integrable couplings of soliton equations by perturbations I – a general theory and application to the KdV hierarchy, Methods Appl. Anal. 7 (2000) 21–55.
[10] W.X. Ma, A bi-Hamiltonian formulation for triangular systems by perturbations, J. Math. Phys. 43 (2002) 1408–1421.
[11] W.X. Ma, Enlarging spectral problems to construct integrable couplings of soliton equations, Phys. Lett. A 316 (2003) 72–76.
[12] W.X. Ma, Integrable couplings of vector AKNS soliton equations, J. Math. Phys. 46 (2005) 033507, 19 pp.
[13] W.X. Ma, A discrete variational identity on semi-direct sums of Lie algebras, J. Phys. A: Math. Theoret. 40 (2007) 15055–15069.
[14] W.X. Ma, A Hamiltonian structure associated with a matrix spectral problem of arbitrary-order, Phys. Lett. A 367 (2007) 473–477.
[15] W.X. Ma, Variational identities and applications to Hamiltonian structures of soliton equations, Nonlinear Anal. TMA 71 (2009) e1716–e1726.
[16] W.X. Ma, Variational identities and Hamiltonian structures, in: W.X. Ma, X.B. Hu, Q.P. Liu (Eds.), Nonlinear and Modern Mathematical Physics, CP 1212, AIP, 2010, pp. 1–27.
[17] W.X. Ma, Nonlinear continuous integrable Hamiltonian couplings, Appl. Math. Comput. 217 (2011) 7238–7244.
[18] W.X. Ma, M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of lie algebras, J. Phys. A: Math. Gen. 39 (2006) 10787–10801.

[19] W.X. Ma, E.G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011) 950–959.
[20] W.X. Ma, B. Fuchssteiner, Integrable theory of the perturbation equations, Chaos Solitons Fractals 7 (1996) 1227–1250.
[21] W.X. Ma, L. Gao, Coupling integrable couplings, Modern Phys. Lett. B 23 (2009) 1847–1860.
[22] W.X. Ma, W. Strampp, Bilinear forms and Bäcklund transformations of the perturbation systems, Phys. Lett. A 341 (2005) 441–449.
[23] W.X. Ma, X.X. Xu, Y.F. Zhang, Semi-direct sums of Lie algebras and continuous integrable couplings, Phys. Lett. A 351 (2006) 125–130.
[24] W.X. Ma, X.X. Xu, Y.F. Zhang, Semidirect sums of Lie algebras and discrete integrable couplings, J. Math. Phys. 47 (2006) 16.
[25] W.X. Ma, Y. Zhang, Component-trace identities for Hamiltonian structures, Appl. Anal. 89 (2010) 457–472.
[26] W.X. Ma, Z.N. Zhu, Constructing nonlinear discrete integrable Hamiltonian couplings, Comput. Math. Appl. 60 (2010) 2601–2608.
[27] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162.
[28] S.Y. Sakovic, On integrability of a $(2+1)$-dimensional perturbed KdV equation, J. Nonlinear Math. Phys. 5 (1998) 230–233.
[29] S.Y. Sakovich, Coupled KdV equations of Hirota-Satsuma type, J. Nonlinear Math. Phys. 6 (1999) 255–262.
[30] G.Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy, J. Phys. A: Math. Gen. 22 (1989) 2375–2392.
[31] T.C. Xia, X.H. Chen, D.Y. Chen, A new Lax integrable hierarchy, *n* Hamiltonian structure and its integrable couplings system, Chaos Solitons Fractals 23 (2005) 451–458.
[32] T.C. Xia, F.J. Yu, Y. Zhang, The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions, Physica A 343 (2004) 238–246.
[33] X.X. Xu, Integrable couplings of relativistic Toda lattice systems in polynomial form and rational form, their hierarchies and bi-Hamiltonian structures, J. Phys. A: Math. Theoret. 42 (2009) 395201, 21 pp.
[34] Y.J. Yu, H.Q. Zhang, Hamiltonian structure of the integrable couplings for the multicomponent Dirac hierarchy, Appl. Math. Comput. 197 (2008) 828–835.
[35] V.E. Zakharov, A.B. Shabat, A scheme for generating the nonlinear equations of mathematical physics by the method of the inverse scattering problem I, Funct. Anal. Appl. 8 (1974) (1975) 226–235.
[36] Y.F. Zhang, A generalized multi-component Glachette-Johnson (GJ) hierarchy and its integrable coupling system, Chaos Solitons Fractals 21 (2004) 305–310.
[37] Y.F. Zhang, H.W. Tam, Applications of the Lie algebra gl(2), Modern Phys. Lett. B 23 (2009) 1763–1770.
[38] Y.F. Zhang, H.W. Tam, Coupling commutator pairs and integrable systems, Chaos Solitons Fractals 39 (2009) 1109–1120.