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Tri-integrable Couplings by Matrix Loop Algebras

Abstract: We create a class of non-semisimple matrix loop algebras, and use the associated zero curvature equations to construct tri-integrable couplings. An application is made for the AKNS equations as an illustrative example. Hamiltonian structures of the resulting tri-integrable couplings are furnished by the variational identities over the presented matrix loop algebras, which implies the commutativity of the sequences of symmetries and conserved functionals.

Keywords: integrable coupling, non-semisimple Lie algebra, Hamiltonian structure

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1 Introduction

Zero curvature equations over semisimple matrix loop algebras produce integrable coupled systems (see, e.g., [1–4]). There are also vector or matrix generalizations of integrable equations formulated in terms of non-associative algebraic structures and their deformations (see, e.g., [5, 6]). Integrable couplings provide valuable new insights into the classification of integrable systems with multi-components [7, 8]. Zero curvature equations over non-semisimple matrix loop algebras are the basis for generating integrable couplings [9, 10], and the associated variational identities offer tools to furnish their Hamiltonian structures [11, 12]. A key step to success is to create non-semisimple matrix loop algebras and find bilinear forms required in the variational identities over the resulting matrix loop algebras.

Let a given integrable system

$$u_t = K(u) \tag{1.1}$$

be defined through a zero curvature equation

$$U_t - V_x + [U, V] = 0. \tag{1.2}$$

The Lax pair $U = U(u, \lambda)$ and $V = V(u, \lambda)$, with λ being the spectral parameter, are square matrices, often belonging to a semisimple matrix loop algebra. Integrable couplings are certain enlarged non-trivial integrable systems including the original system (1.1) as a subsystem [7, 8]. Since the integrability of a system has nothing to do with any arrangement of equations in the system, we will focus on triangular integrable systems, within which an initially given system is listed as the first subsystem.

By a tri-integrable coupling of a given integrable system (1.1), we mean an enlarged triangular integrable system of the following form:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2), \\ u_{3,t} = S_3(u, u_1, u_2, u_3). \end{cases} \tag{1.3}$$

We call this system a nonlinear integrable coupling, if at least one of $S_1(u, u_1)$, $S_2(u, u_1, u_2)$ and $S_3(u, u_1, u_2, u_3)$ is nonlinear with respect to any of the subvectors u_1, u_2, u_3 of new dependent variables.

Inspired by a study on bi-integrable couplings [13], we would like to explore non-semisimple loop algebras consisting of block matrices, associated with which enlarged zero curvature equations generate tri-integrable couplings. Non-semisimple Lie algebras are semi-direct sums of semi-simple Lie subalgebras and solvable Lie subalgebras [14]. The notion of semi-direct sums

$$\bar{g} = g \Subset g_c \tag{1.4}$$

means that the two Lie subalgebras g and g_c satisfy

$$[g, g_c] \subseteq g_c, \tag{1.5}$$

where $[g, g_c] = \{[A, B] \mid A \in g, B \in g_c\}$, with $[\cdot, \cdot]$ denoting the Lie bracket of $\bar{g}$. Obviously, g_c is an ideal Lie subalgebra of $\bar{g}$. The subscript c indicates a contribution to the construction of coupling systems. We also require the closure property between g and g_c under the matrix multiplication

$$gg_c, g_cg \subseteq g_c, \tag{1.6}$$

where $g_1 g_2 = \{AB \,|\, A \in g_1, B \in g_2\}$, while we use the discrete zero curvature equation over semi-direct sums of matrix loop algebras to generate discrete coupling systems [10]. Hamiltonian structures of the resulting coupling systems can usually be furnished through the variational identities on general loop algebras [11].

In this paper, we will create a class of non-semisimple matrix loop algebras consisting of 4×4 block matrices, and apply them to the construction of tri-integrable couplings, based on the associated zero curvature equations. An application will be made for the AKNS soliton hierarchy as an illustrative example. Hamiltonian structures of the resulting tri-integrable couplings of the AKNS equations will be established through the associated classical variational identities. The presented matrix loop algebras will be shown to be a starting point to construct integrable Hamiltonian couplings of given integrable systems.

2 Matrix loop algebras and tri-integrable couplings

2.1 Soliton hierarchy

A soliton hierarchy is usually associated with a spectral problem

$$\phi_x = U\phi, \quad U = U(u,\lambda) \in g, \tag{2.1}$$

where g is often a semi-simple matrix loop algebra. Suppose that the stationary zero curvature equation

$$W_x = [U, W] \tag{2.2}$$

has a solution of the form

$$W = W(u,\lambda) = \sum_{i \ge 0} W_{0,i}\lambda^{-i}, \tag{2.3}$$

where $W_{0,i} \in g$, $i \ge 0$. Introduce the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi, \quad m \ge 0, \tag{2.4}$$

with the Lax matrices being defined by

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \Delta_m \in g, \quad m \ge 0, \tag{2.5}$$

where P_+ denotes the polynomial part of P in λ. The introduction of the modification terms Δ_m aims to guarantee that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \ge 0, \tag{2.6}$$

engender a soliton hierarchy with a Hamiltonian structure:

$$u_{t_m} = K_m(u) = J\frac{\delta \mathscr{H}_m}{\delta u}, \quad m \ge 0, \tag{2.7}$$

of which the first system $u_{t_1} = K_1$ is assumed to be the given system (1.1) with $V = V^{[1]}$. The above Hamiltonian functionals $\mathscr{H}_m$ are generally presented by using the variational identity [11, 12]:

$$\frac{\delta}{\delta u}\int \left\langle \frac{\partial U}{\partial \lambda}, W \right\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle \frac{\partial U}{\partial u}, W \right\rangle, \tag{2.8}$$

where γ is a constant, W is a solution of (2.2), and $\langle \cdot, \cdot \rangle$ is a bilinear form on the loop algebra g which is non-degenerate, symmetric and ad-invariant [12]. If g is non-semisimple, $\langle \cdot, \cdot \rangle$ must not be the Killing form. If $\langle A, A \rangle$ is positive for every non-zero matrix $A \in g$, then the Lie algebra $(g, \langle \cdot, \cdot \rangle)$ becomes quadratic.

2.2 Matrix loop algebras

Let α, β, μ and ν be four arbitrarily given constants, which could be zero. We create a class of triangular block matrices

$$M(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 + \alpha A_2 & \alpha A_3 & \beta A_2 + \alpha A_4 \\ 0 & 0 & A_1 + \alpha A_2 + \mu A_3 & \nu A_3 \\ 0 & 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{2.9}$$

where A_1, A_2, A_3, A_4 are square submatrices of the same order. Under the matrix commutator

$$[M_1, M_2] = M_1 M_2 - M_2 M_1, \tag{2.10}$$

we have the closure property

$$[M(A_1, A_2, A_3, A_4), M(B_1, B_2, B_3, B_4)] = M(C_1, C_2, C_3, C_4), \tag{2.11}$$

where

$$\begin{cases} C_1 = [A_1, B_1], \\ C_2 = [A_1, B_2] + [A_2, B_1] + \alpha[A_2, B_2], \\ C_3 = [A_1, B_3] + \alpha[A_2, B_3] + [A_3, B_1] \\ \qquad + \alpha[A_3, B_2] + \mu[A_3, B_3], \\ C_4 = [A_1, B_4] + \beta[A_2, B_2] + \alpha[A_2, B_4] \\ \qquad + \nu[A_3, B_3] + [A_4, B_1] + \alpha[A_4, B_2]. \end{cases} \tag{2.12}$$

This property guarantees that all block matrices defined by (2.9) form a matrix Lie algebra for fixed constants α, β, μ and ν. We point out that any larger number of blocks in creating matrix Lie algebras gives us more difficulties in theoretical verification. Now, we introduce a class of matrix loop algebras possessing a semi-direct sum decomposition

$$\bar{g} = g \Subset g_c, \tag{2.13}$$

with

$$\begin{cases} g = \{M(A_1(\lambda), 0, 0, 0) \,|\, \text{entries of} \\ \qquad A_1 - \text{Laurent series in } \lambda\}, \\ g_c = \{M(0, A_2(\lambda), A_3(\lambda), A_4(\lambda)) \,|\, \text{entries of} \\ \qquad A_i - \text{Laurent series in } \lambda, 2 \le i \le 4\}, \end{cases} \tag{2.14}$$

and thus, they must be non-semisimple. Obviously, one of non-trivial ideals of every $\bar{g}$ is g_c.

Such presented matrix loop algebras serve as a foundation upon which zero curvature equations generate nonlinear Hamiltonian tri-integrable couplings, while many other existing loop algebras lead to linear Hamiltonian integrable couplings (see, e.g., [15–20]). The block A_1 corresponds to the original integrable system, and the other three blocks A_2, A_3 and A_4 are used to generate the supplementary vector fields S_1, S_2 and S_3. We remark that the commutators $[A_2, B_2]$ and $[A_3, B_3]$ will yield nonlinear terms in the resulting tri-integrable couplings.

2.3 Tri-integrable couplings

In order to generate tri-integrable couplings, defined by (1.3), for a given integrable system (1.1) possessing a zero curvature representation (1.2) with a Lax pair $U = U(u, \lambda)$ and $V = V(u, \lambda)$, we use the corresponding enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{g}, \tag{2.15}$$

and the corresponding enlarged Lax matrix

$$\bar{V} = \bar{V}(\bar{u}, \lambda) = M(V, V_1, V_2, V_3) \in \bar{g}, \tag{2.16}$$

where $\bar{u} = (u^T, u_1^T, u_2^T, u_3^T)^T$, λ is the spectral parameter, and

$$U_i = U_i(u_i, \lambda), \quad V_i = V_i(u, u_1, \dots, u_i, \lambda), \quad 1 \le i \le 3, \tag{2.17}$$

are square matrices of the same order as U and V. Then, associated with this new enlarged Lax pair $\bar{U}$ and $\bar{V}$, the enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0 \tag{2.18}$$

presents the following triangle system

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{1,t} - V_{1,x} + [U, V_1] + [U_1, V] + \alpha[U_1, V_1] = 0, \\ U_{2,t} - V_{2,x} + [U, V_2] + \alpha[U_1, V_2] + [U_2, V] \\ \qquad + \alpha[U_2, V_1] + \mu[U_2, V_2] = 0, \\ U_{3,t} - V_{3,x} + [U, V_3] + \beta[U_1, V_1] + \alpha[U_1, V_3] \\ \qquad + \nu[U_2, V_2] + [U_3, V] + \alpha[U_3, V_1] = 0. \end{cases} \tag{2.19}$$

The first equation above precisely gives the originally given integrable system (1.1), and thus, the above whole system (2.19) provides a coupling system for the system (1.1). This shows a basic idea of enlarging given integrable systems by using the presented class of matrix loop algebras $\bar{g}$.

Following the general scheme for constructing soliton hierarchies [21, 22], we solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{2.20}$$

by setting the following form

$$\begin{cases} W_1 = W_1(u_1, \lambda) = \sum_{i \ge 0} W_{1,i} \lambda^{-i}, \\ W_2 = W_2(u, u_1, u_2, \lambda) = \sum_{i \ge 0} W_{2,i} \lambda^{-i}, \\ W_3 = W_3(u, u_1, u_2, u_3, \lambda) = \sum_{i \ge 0} W_{3,i} \lambda^{-i}, \end{cases} \tag{2.21}$$

for

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2, W_3) \in \bar{g}, \tag{2.22}$$

of which W is defined by (2.3). The enlarged stationary zero curvature equation (2.20) equivalently presents

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W] + \alpha[U_1, W_1], \\ W_{2,x} = [U, W_2] + \alpha[U_1, W_2] + [U_2, W] \\ \qquad + \alpha[U_2, W_1] + \mu[U_2, W_2], \\ W_{3,x} = [U, W_3] + \beta[U_1, W_1] + \alpha[U_1, W_3] \\ \qquad + \nu[U_2, W_2] + [U_3, W] + \alpha[U_3, W_1]. \end{cases} \tag{2.23}$$

Then, we define the enlarged Lax matrices $\bar{V}^{[m]}$ as

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{g}, \tag{2.24}$$

with the submatrices $V^{[m]}$ being defined by (2.5) and

$$V_i^{[m]} = (\lambda^m W_i)_+ + \Delta_{m,i}, \quad 1 \le i \le 3, \quad m \ge 0, \tag{2.25}$$

where P_+ again denotes the polynomial part of P in λ. An important step to construct a hierarchy of triangular integrable couplings is to choose the modification terms $\Delta_{m,i}$ such that the enlarged zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \quad m \ge 0, \tag{2.26}$$

yield a hierarchy of enlarged soliton equations

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}), \quad m \ge 0. \tag{2.27}$$

This hierarchy provides tri-integrable couplings for the given soliton hierarchy (2.7):

$$\bar{u}_{t_m} = \begin{bmatrix} u_{t_m} \\ u_{1,t_m} \\ u_{2,t_m} \\ u_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix}, \quad m \ge 0. \tag{2.28}$$

Hamiltonian structures of such tri-integrable couplings can be established through applying the associated variational identities [11, 12], which contains the trace identity [21] as a particular example.

3 Application to the AKNS equations

3.1 The AKNS hierarchy

Let us recall the AKNS soliton hierarchy [23]. The traditional spectral problem for the AKNS hierarchy is given by

$$\phi_x = U\phi, \quad U = U(u, \lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{3.1}$$

The stationary zero curvature equation

$$W_x = [U, W] \tag{3.2}$$

gives rise to

$$\begin{cases} a_x = pc - qb, \\ b_x = -2\lambda b - 2pa, \\ c_x = 2qa + 2\lambda c, \end{cases} \tag{3.3}$$

if we assume that W is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i \ge 0} W_{0,i}\lambda^{-i} = \sum_{i \ge 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}. \tag{3.4}$$

Upon taking the initial values

$$a_0 = -1, \quad b_0 = c_0 = 0, \tag{3.5}$$

the system (3.3) equivalently yields

$$\begin{cases} b_{i+1} = -\frac{1}{2} b_{i,x} - pa_i, \\ c_{i+1} = \frac{1}{2} c_{i,x} - qa_i, \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \end{cases} \quad i \ge 0. \tag{3.6}$$

We impose the integration conditions

$$a_i|_{u=0} = b_i|_{u=0} = b_i|_{u=0} = 0, \quad i \ge 1, \tag{3.7}$$

to determine the sequence of $\{a_i, b_i, c_i \,|\, i \ge 1\}$ uniquely. Then, the first few sets can be computed as follows:

$$\begin{aligned} &b_1 = p, \quad c_1 = q, \quad a_1 = 0; \\ &b_2 = -\tfrac{1}{2} p_x, \quad c_2 = \tfrac{1}{2} q_x, \quad a_2 = \tfrac{1}{2} pq; \\ &b_3 = \tfrac{1}{4} p_{xx} - \tfrac{1}{2} p^2 q, \quad c_3 = \tfrac{1}{4} q_{xx} - \tfrac{1}{2} pq^2, \\ &a_3 = \tfrac{1}{4}(pq_x - p_x q). \end{aligned}$$

Now, the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0 \quad \text{with} \quad V^{[m]} = (\lambda^m W)_+, \quad m \ge 0, \tag{3.8}$$

generate the AKNS hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ 2q \end{bmatrix} = J\frac{\delta \mathscr{H}_m}{\delta u}, \quad m \geq 0, \tag{3.9}$$

where the Hamiltonian operator, the hereditary recursion operator and the Hamiltonian functions are defined by

$$J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix},$$
$$\mathscr{H}_m = \int \frac{2a_{m+2}}{m+1}dx, \quad m \geq 0, \tag{3.10}$$

in which $\partial = \frac{\partial}{\partial x}$, respectively.

3.2 Tri-integrable couplings of the AKNS equations

We use a special non-semisimple matrix loop algebra $\bar{g} = g \Subset g_c$ with

$$\begin{cases} g = \{M(A_1, 0, 0, 0) \,|\, A_1 \in \widetilde{\mathrm{sl}}(2)\}, \\ g_c = \{M(0, A_2, A_3, A_4) \,|\, A_2, A_3, A_4 \in \widetilde{\mathrm{sl}}(2)\}, \end{cases} \tag{3.11}$$

where the loop algebra $\widetilde{\mathrm{sl}}(2)$ is defined by

$$\widetilde{\mathrm{sl}}(2) = \{A(\lambda) \in \mathrm{sl}(2) \,|\, \text{entries of } A(\lambda) - \text{Laurent series in } \lambda\}. \tag{3.12}$$

To construct tri-integrable couplings for the AKNS equations, we adopt the corresponding enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{g}, \tag{3.13}$$

with $U = U(u, \lambda)$ being defined as in (3.1) and

$$U_i = U_i(u_i) = \begin{bmatrix} 0 & r_i \\ s_i & 0 \end{bmatrix}, \quad u_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}, \quad 1 \leq i \leq 3, \tag{3.14}$$

where $\bar{u} = (u^T, u_1^T, u_2^T, u_3^T)^T$, and r_i and s_i, $1 \leq i \leq 3$, are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation (2.20), we search for solutions of the following form

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2, W_3) \in \bar{g}, \tag{3.15}$$

where W is given by (3.4). Then, the enlarged stationary zero curvature equation gives

$$\begin{cases} W_{1,x} = [U, W_1] + [U_1, W] + \alpha[U_1, W_1], \\ W_{2,x} = [U, W_2] + \alpha[U_1, W_2] + [U_2, W] \\ \qquad + \alpha[U_2, W_1] + \mu[U_2, W_2], \\ W_{3,x} = [U, W_3] + \beta[U_1, W_1] + \alpha[U_1, W_3] \\ \qquad + \nu[U_2, W_2] + [U_3, W] + \alpha[U_3, W_1]. \end{cases} \tag{3.16}$$

Assume that W_1, W_2, W_3 are of the form

$$\begin{cases} W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i \geq 0} W_{1,i}\lambda^{-i}, \\ W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix} \\ \quad = \sum_{i \geq 0} W_{2,i}\lambda^{-i}, \\ W_3 = W_3(u, u_1, u_2, u_3, \lambda) = \begin{bmatrix} e'' & f'' \\ g'' & -e'' \end{bmatrix} \\ \quad = \sum_{i \geq 0} W_{3,i}\lambda^{-i}. \end{cases} \tag{3.17}$$

The above system (3.16) equivalently leads to

$$\begin{cases} e_x = -s_1 b + r_1 c - (q + \alpha s_1)f + (p + \alpha r_1)g, \\ f_x = -2r_1 a - 2(p + \alpha r_1)e - 2\lambda f, \\ g_x = 2s_1 a + 2(q + \alpha s_1)e + 2\lambda g; \end{cases} \tag{3.18}$$

$$\begin{cases} e'_x = -s_2 b + r_2 c - (q + \alpha s_1 + \mu s_2)f' \\ \qquad + (p + \alpha r_1 + \mu r_2)g' - \alpha s_2 f + \alpha r_2 g, \\ f'_x = -2r_2 a - 2(p + \alpha r_1 + \mu r_2)e' - 2\lambda f' - 2\alpha r_2 e, \\ g'_x = 2s_2 a + 2(q + \alpha s_1 + \mu s_2)e' + 2\lambda g' + 2\alpha s_2 e; \end{cases} \tag{3.19}$$

and

$$\begin{cases} e''_x = -s_3 b + r_3 c - (\beta s_1 + \alpha s_3)f \\ \qquad + (\beta r_1 + \alpha r_3)g - \nu s_2 f' + \nu r_2 g' \\ \qquad - (q + \alpha s_1)f'' + (p + \alpha r_1)g'', \\ f''_x = -2\lambda f'' - 2(\alpha r_3 + \beta r_1)e - 2\nu r_2 e' \\ \qquad - 2(p + \alpha r_1)e'' - 2r_3 a, \\ g''_x = 2\lambda g'' + 2(\alpha s_3 + \beta s_1)e + 2\nu s_2 e' \\ \qquad + 2(q + \alpha s_1)e'' + 2s_3 a. \end{cases} \tag{3.20}$$

Trying a solution $\bar{W}$ with

$$\begin{cases} e = \sum_{i \geq 0} e_i\lambda^{-i}, \quad f = \sum_{i \geq 0} f_i\lambda^{-i}, \quad g = \sum_{i \geq 0} g_i\lambda^{-i}, \\ e' = \sum_{i \geq 0} e'_i\lambda^{-i}, \quad f' = \sum_{i \geq 0} f'_i\lambda^{-i}, \quad g' = \sum_{i \geq 0} g'_i\lambda^{-i}, \\ e'' = \sum_{i \geq 0} e''_i\lambda^{-i}, \quad f'' = \sum_{i \geq 0} f''_i\lambda^{-i}, \quad g'' = \sum_{i \geq 0} g''_i\lambda^{-i}, \end{cases} \tag{3.21}$$

and taking

$$\begin{aligned} &e_0 = e_0' = e_0'' = -1, \\ &f_0 = g_0 = f_0' = g_0' = f_0'' = g_0'' = 0, \end{aligned} \tag{3.22}$$

then we can have

$$\begin{cases} f_{i+1} = -\frac{1}{2} f_{i,x} - r_1 a_i - (p + \alpha r_1) e_i, \\ g_{i+1} = \frac{1}{2} g_{i,x} - s_1 a_i - (q + \alpha s_1) e_i, \\ e_{i+1,x} = -s_1 b_{i+1} + r_1 c_{i+1} - (q + \alpha s_1) f_{i+1} \\ \qquad + (p + \alpha r_1) g_{i+1}; \end{cases} \tag{3.23}$$

$$\begin{cases} f'_{i+1} = -\frac{1}{2} f'_{i,x} - r_2 a_i - (p + \alpha r_1 + \mu r_2) e'_i - \alpha r_2 e_i, \\ g'_{i+1} = \frac{1}{2} g'_{i,x} - s_2 a_i - (q + \alpha s_1 + \mu s_2) e'_i - \alpha s_2 e_i, \\ e'_{i+1,x} = -s_2 b_{i+1} + r_2 c_{i+1} - (q + \alpha s_1 + \mu s_2) f'_{i+1} \\ \qquad + (p + \alpha r_1 + \mu r_2) g'_{i+1} - \alpha s_2 f_{i+1} + \alpha r_2 g_{i+1}; \end{cases} \tag{3.24}$$

and

$$\begin{cases} f''_{i+1} = -\frac{1}{2} f''_{i,x} - (\alpha r_3 + \beta r_1) e_i - \nu r_2 e'_i \\ \qquad - (p + \alpha r_1) e''_i - r_3 a_i, \\ g''_{i+1} = \frac{1}{2} g''_{i,x} - (\alpha s_3 + \beta s_1) e_i - \nu s_2 e'_i \\ \qquad - (q + \alpha s_1) e''_i - s_3 a_i, \\ e''_{i+1,x} = -s_3 b_{i+1} + r_3 c_{i+1} - (\beta s_1 + \alpha s_3) f_{i+1} \\ \qquad + (\beta r_1 + \alpha r_3) g_{i+1} - \nu s_2 f'_{i+1} + \nu r_2 g'_{i+1} \\ \qquad - (q + \alpha s_1) f''_{i+1} + (p + \alpha r_1) g''_{i+1}, \end{cases} \tag{3.25}$$

where $i \ge 0$. Then under the integration conditions

$$\begin{cases} e_i|_{\bar u=0} = f_i|_{\bar u=0} = g_i|_{\bar u=0} = 0, \\ e'_i|_{\bar u=0} = f'_i|_{\bar u=0} = g'_i|_{\bar u=0} = 0, \quad i \ge 1, \\ e''_i|_{\bar u=0} = f''_i|_{\bar u=0} = g''_i|_{\bar u=0} = 0, \end{cases} \tag{3.26}$$

the recursion relations in (3.23), (3.24) and (3.25) uniquely generate three sequences of $\{f_i, g_i, e_i\}_{i\ge1}$, $\{f'_i, g'_i, e'_i\}_{i\ge1}$ and $\{f''_i, g''_i, e''_i\}_{i\ge1}$, respectively. In the first sequence, the first three sets of functions are as follows:

$$\begin{cases} f_1 = p + (1+\alpha) r_1, \\ g_1 = q + (1+\alpha) s_1, \\ e_1 = 0; \end{cases}$$

$$\begin{cases} f_2 = -\frac{1}{2} p_x - \frac{1}{2}(1+\alpha) r_{1,x}, \\ g_2 = \frac{1}{2} q_x + \frac{1}{2}(1+\alpha) s_{1,x}, \\ e_2 = \frac{1}{2}\{[q + (1+\alpha)s_1]p + (1+\alpha) q r_1 + \alpha(\alpha+1) r_1 s_1\}; \end{cases}$$

and

$$\begin{cases} f_3 = -\frac{1}{2}[q + (1+\alpha)s_1]p^2 - (1+\alpha)(q + \alpha s_1) p r_1 \\ \qquad - \frac{1}{2}\alpha(1+\alpha) q r_1^2 - \frac{1}{2}\alpha^2(1+\alpha) r_1^2 s_1 \\ \qquad + \frac{1}{4} p_{xx} + \frac{1}{4}(1+\alpha) r_{1,xx}, \\ g_3 = -\frac{1}{2}[p + (1+\alpha)r_1]q^2 - (1+\alpha)(p + \alpha r_1) q s_1 \\ \qquad - \frac{1}{2}\alpha(1+\alpha) p s_1^2 - \frac{1}{2}\alpha^2(1+\alpha) r_1 s_1^2 \\ \qquad + \frac{1}{4} q_{xx} + \frac{1}{4}(1+\alpha) s_{1,xx}, \\ e_3 = \frac{1}{4}\{[q_x + (1+\alpha)s_{1,x}]p - [p_x + (1+\alpha) r_{1,x}]q \\ \qquad + (1+\alpha)(q_x + \alpha s_{1,x}) r_1 - (1+\alpha)(p_x + \alpha r_{1,x}) s_1\}. \end{cases}$$

In the second sequence, the first three sets of functions are as follows:

$$\begin{cases} f'_1 = p + \alpha r_1 + (\alpha+\mu+1) r_2, \\ g'_1 = q + \alpha s_1 + (\alpha+\mu+1) s_2, \\ e'_1 = 0; \end{cases}$$

$$\begin{cases} f'_2 = -\frac{1}{2} p_x - \frac{1}{2}\alpha r_{1,x} - \frac{1}{2}(\alpha+\mu+1) r_{2,x}, \\ g'_2 = \frac{1}{2} q_x + \frac{1}{2}\alpha s_{1,x} + \frac{1}{2}(\alpha+\mu+1) s_{2,x}, \\ e'_2 = \frac{1}{2}\{pq + [\alpha s_1 + (\alpha+\mu+1)s_2]p \\ \qquad + [\alpha r_1 + (\alpha+\mu+1) r_2] q \\ \qquad + \alpha[\alpha s_1 + (\alpha+\mu+1) s_2] r_1 \\ \qquad + (\alpha+\mu+1)(\alpha s_1 + \mu s_2) r_2\}; \end{cases}$$

and

$$\begin{aligned} f'_3 = &-\tfrac{1}{2}[q + \alpha s_1 + (\alpha+\mu+1)s_2]p^2 \\ &- \{\alpha(q r_1 + \alpha r_1 s_1) + (\alpha+\mu+1) \\ &\times [q r_2 + \alpha(r_2 s_1 + r_1 s_2) + \mu r_2 s_2]\} p \\ &- \left[\tfrac{1}{2}\alpha^2 r_1^2 + \alpha(\alpha+\mu+1) r_1 r_2 + \tfrac{1}{2}\mu(\alpha+\mu+1) r_2^2\right] q \\ &- \tfrac{1}{2}\alpha^2[\alpha s_1 + (\alpha+\mu+1)s_2] r_1^2 \\ &- \alpha(\alpha+\mu+1)(\alpha s_1 + \mu s_2) r_1 r_2 \\ &- \tfrac{1}{2}\mu(\alpha+\mu+1)(\alpha s_1 + \mu s_2) r_2^2 \\ &+ \tfrac{1}{4} p_{xx} + \tfrac{1}{4}\alpha r_{1,xx} + \tfrac{1}{4}(\alpha+\mu+1) r_{2,xx}, \\ g'_3 = &-\tfrac{1}{2}[p + \alpha r_1 + (\alpha+\mu+1) r_2] q^2 \\ &- \{\alpha(p s_1 + \alpha r_1 s_1) + (\alpha+\mu+1) \\ &\times [p s_2 + \alpha(r_1 s_2 + r_2 s_1) + \mu r_2 s_2]\} q \\ &- \left[\tfrac{1}{2}\alpha^2 s_1^2 + \alpha(\alpha+\mu+1) s_1 s_2 + \tfrac{1}{2}\mu(\alpha+\mu+1) s_2^2\right] p \\ &- \tfrac{1}{2}\alpha^2[\alpha r_1 + (\alpha+\mu+1) r_2] s_1^2 \\ &- \alpha(\alpha+\mu+1)(\alpha r_1 + \mu r_2) s_1 s_2 \\ &- \tfrac{1}{2}\mu(\alpha+\mu+1)(\alpha r_1 + \mu r_2) s_2^2 \\ &+ \tfrac{1}{4} q_{xx} + \tfrac{1}{4}\alpha s_{1,xx} + \tfrac{1}{4}(\alpha+\mu+1) s_{2,xx}, \\ e'_3 = &\tfrac{1}{4}\{[q_x + \alpha s_{1,x} + (\alpha+\mu+1) s_{2,x}] p \\ &- [p_x + \alpha r_{1,x} + (\alpha+\mu+1) r_{2,x}] q \\ &+ \alpha[q_x + \alpha s_{1,x} + (\alpha+\mu+1) s_{2,x}] r_1 \\ &- \alpha[p_x + \alpha r_{1,x} + (\alpha+\mu+1) r_{2,x}] s_1 \\ &+ (\alpha+\mu+1)(q_x + \alpha s_{1,x} + \mu s_{2,x}) r_2 \\ &- (\alpha+\mu+1)(p_x - \alpha r_{1,x} - \mu r_{2,x}) s_2\}. \end{aligned}$$

In the third sequence, the first three sets of functions are as follows:

$$\begin{cases} f_1'' = (\alpha+\beta)r_1 + p + \nu r_2 + (\alpha+1)r_3, \\ g_1'' = (\alpha+\beta)s_1 + q + \nu s_2 + (\alpha+1)s_3, \\ e_1'' = 0; \end{cases}$$

$$\begin{cases} f_2'' = -\frac{1}{2}p_x - \frac{1}{2}(\alpha+\beta)r_{1x} - \frac{1}{2}\nu r_{2,x} - \frac{1}{2}(\alpha+1)r_{3,x}, \\ g_2'' = \frac{1}{2}q_x + \frac{1}{2}(\alpha+\beta)s_{1x} + \frac{1}{2}\nu s_{2,x} + \frac{1}{2}(\alpha+1)s_{3,x}, \\ e_2'' = \frac{1}{2}\{[q + (\alpha+\beta)s_1 + \nu s_2 + (\alpha+1)s_3]p \\ \qquad + [(\alpha+\beta)r_1 + \nu r_2 + (\alpha+1)r_3]q \\ \qquad + [(\alpha^2+2\alpha\beta+\beta)s_1 + \alpha\nu s_2 + \alpha(\alpha+1)s_3]r_1 \\ \qquad + \alpha[\nu r_2 + (\alpha+1)r_3]s_1 + \nu(\alpha+\mu+1)r_2 s_2\}; \end{cases}$$

and

$$\begin{aligned} f_3'' = &-\tfrac{1}{2}[q + (\alpha+\beta)s_1 + \nu s_2 + (\alpha+1)s_3]p^2 \\ &- \{[(\alpha+\beta)r_1 + \nu r_2 - (\alpha+1)r_3]q \\ &+ [(\alpha^2+2\alpha\beta+\beta)s_1 + \alpha\nu s_2 + \alpha(\alpha+1)s_3]r_1 \\ &- \alpha[\nu r_2 + (\alpha+1)r_3]s_1 - \nu(\alpha+\mu+1)r_2 s_2\}p \\ &- \{\tfrac{1}{2}(\alpha^2+\alpha\beta+\beta)r_1^2 + \alpha[\nu r_2 + (\alpha+1)r_3]r_1 \\ &+ \tfrac{1}{2}\nu(\alpha+\mu+1)r_2^2\}q \\ &- \alpha\big[\big(\beta + \tfrac{3}{2}\alpha\beta + \tfrac{1}{2}\alpha^2\big)s_1 + \tfrac{1}{2}\alpha(\alpha+1)s_3 + \tfrac{1}{2}\alpha\nu s_2\big]r_1^2 \\ &- \alpha\{\alpha[\nu r_2 + (\alpha+1)r_3]s_1 + \nu(\alpha+\mu+1)r_2 s_2\}r_1 \\ &- \tfrac{1}{2}\nu(\alpha+\mu+1)(\alpha s_1 + \mu s_2)r_2^2 + \tfrac{1}{4}p_{xx} \\ &+ \tfrac{1}{4}(\alpha+\beta)r_{1,xx} + \tfrac{1}{4}\nu r_{2,xx} + \tfrac{1}{4}(\alpha+1)r_{3,xx}, \end{aligned}$$

$$\begin{aligned} g_3'' = &-\tfrac{1}{2}[p + (\alpha+\beta)r_1 + \nu r_2 + (\alpha+1)r_3]q^2 \\ &- \{[(\alpha+\beta)s_1 + \nu s_2 - (\alpha+1)s_3]p \\ &+ [(\alpha^2+2\alpha\beta+\beta)r_1 + \alpha\nu r_2 + \alpha(\alpha+1)r_3]s_1 \\ &- \alpha[\nu s_2 + (\alpha+1)s_3]r_1 - \nu(\alpha+\mu+1)r_2 s_2\}q \\ &- \{\tfrac{1}{2}(\alpha^2+\alpha\beta+\beta)s_1^2 + \alpha[\nu s_2 + (\alpha+1)s_3]s_1 \\ &+ \tfrac{1}{2}\nu(\alpha+\mu+1)s_2^2\}p \\ &- \alpha\big[\big(\beta + \tfrac{3}{2}\alpha\beta + \tfrac{1}{2}\alpha^2\big)r_1 + \tfrac{1}{2}\alpha(\alpha+1)r_3 + \tfrac{1}{2}\alpha\nu r_2\big]s_1^2 \\ &- \alpha\{\alpha[\nu s_2 + (\alpha+1)s_3]r_1 + \nu(\alpha+\mu+1)r_2 s_2\}s_1 \\ &- \tfrac{1}{2}\nu(\alpha+\mu+1)(\alpha r_1 + \mu r_2)s_2^2 + \tfrac{1}{4}q_{xx} \\ &+ \tfrac{1}{4}(\alpha+\beta)s_{1,xx} + \tfrac{1}{4}\nu s_{2,xx} + \tfrac{1}{4}(\alpha+1)s_{3,xx}, \end{aligned}$$

$$\begin{aligned} e_3'' = &\tfrac{1}{4}\{[q_x + (\alpha+\beta)s_{1,x} + \nu s_{2,x} + (\alpha+1)s_{3,x}]p \\ &- [p_x + (\alpha+\beta)r_{1,x} + \nu r_{2,x} + (\alpha+1)r_{3,x}]q \\ &+ [(\alpha+\beta)q_x + (\alpha^2+2\alpha\beta+\beta)s_{1,x} \\ &+ \alpha\nu s_{2,x} + \alpha(\alpha+1)s_{3,x}]r_1 \\ &- [(\alpha+\beta)p_x + (\alpha^2+2\alpha\beta+\beta)r_{1,x} \\ &+ \alpha\nu r_{2,x} + \alpha(\alpha+1)r_{3,x}]s_1 \\ &+ \nu[q_x + \alpha s_{1,x} + (\alpha+\mu+1)s_{2,x}]r_2 \\ &- \nu[p_x + \alpha r_{1,x} + (\alpha+\mu+1)r_{2,x}]s_2 \\ &+ (\alpha+1)(q_x + \alpha s_{1,x})r_3 - (\alpha+1)(p_x + \alpha r_{1,x})s_3\}. \end{aligned}$$

Let us further adopt the enlarged Lax matrices

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{g}, \quad m \ge 0, \qquad (3.27)$$

where $V^{[m]}$ is defined as in (3.8) and

$$V_i^{[m]} = (\lambda^m W_i)_+, \quad m \ge 0, \qquad (3.28)$$

which means that $\Delta_{m,i}$ are chosen as zero.

Then, the m-th enlarged zero curvature equation

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} - [\bar{U}, \bar{V}^{[m]}] \qquad (3.29)$$

gives rise to

$$\begin{cases} U_{t_m} = V_x^{[m]} - [U, V^{[m]}], \\ U_{1,t_m} = V_{1,x}^{[m]} - [U, V_1^{[m]}] - [U_1, V^{[m]}] - \alpha[U_1, V_1^{[m]}], \\ U_{2,t_m} = V_{2,x}^{[m]} - [U, V_2^{[m]}] - [U_2, V^{[m]}] - \alpha[U_1, V_2^{[m]}] \\ \qquad - \alpha[U_2, V_1^{[m]}] - \mu[U_2, V_2^{[m]}], \\ U_{3,t_m} = V_{3,x}^{[m]} - [U, V_3^{[m]}] - \beta[U_1, V_1^{[m]}] - \alpha[U_1, V_3^{[m]}] \\ \qquad - \nu[U_2, V_2^{[m]}] - [U_3, V^{[m]}] - \alpha[U_3, V_1^{[m]}]. \end{cases} \qquad (3.30)$$

All above systems of equations determine a hierarchy of coupling systems for the AKNS equations:

$$\bar{u}_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{1,t_m} \\ s_{1,t_m} \\ r_{2,t_m} \\ s_{2,t_m} \\ r_{3,t_m} \\ s_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u,u_1) \\ S_{2,m}(u,u_1,u_2) \\ S_{3,m}(u,u_1,u_2,u_3) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \\ -2f'_{m+1} \\ 2g'_{m+1} \\ -2f''_{m+1} \\ 2g''_{m+1} \end{bmatrix}, \quad m \ge 0. \qquad (3.31)$$

It is direct to check that all members in (3.31) with $m \ge 2$ provide nonlinear tri-integrable couplings for the AKNS equations. The first nonlinear tri-integrable coupling system having four subsystems reads

$$\begin{cases} p_{t_2} = -2b_3, \quad q_{t_2} = 2c_3; \\ r_{1,t_2} = -2f_3, \quad s_{1,t_2} = 2g_3; \\ r_{2,t_2} = -2f_3', \quad s_{2,t_2} = 2g_3'; \\ r_{3,t_2} = -2f_3'', \quad s_{3,t_2} = 2g_3'', \end{cases} \qquad (3.32)$$

where b_3, c_3, f_3, g_3, f_3', g_3', f_3'' and g_3'' are defined as before.

3.3 Hamiltonian structures

In order to establish Hamiltonian structures for the presented tri-integrable couplings in (3.31), we apply the variational identity [12, 24]:

$$\frac{\delta}{\delta \bar{u}} \int \left\langle \frac{\partial \bar{U}}{\partial \lambda}, \bar{W} \right\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left\langle \frac{\partial \bar{U}}{\partial \bar{u}}, \bar{W} \right\rangle. \tag{3.33}$$

To construct symmetric and ad-invariant bilinear forms on $\bar{g}$ conveniently, we first transform the semi-direct sum $\bar{g}$ into a vector form. Define a mapping

$$\sigma : \bar{g} \to \mathbb{R}^{12}, \quad A \mapsto (a_1, \dots, a_{12})^T, \tag{3.34}$$

where

$$A = M(A_1, A_2, A_3, A_4) \in \bar{g}, \\ A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix}, \quad 1 \le i \le 4. \tag{3.35}$$

This mapping σ induces a Lie algebraic structure on $\mathbb{R}^{12}$, isomorphic to the matrix Lie algebra $\bar{g}$. The corresponding Lie bracket $[\cdot, \cdot]$ on $\mathbb{R}^{12}$ can be computed as follows

$$[a, b]^T = a^T R(b), \quad a = (a_1, \dots, a_{12})^T, \\ b = (b_1, \dots, b_{12})^T \in \mathbb{R}^{12}, \tag{3.36}$$

where

$$R(b) = M(R_1, R_2, R_3, R_4), \\ R_i = \begin{bmatrix} 0 & 2b_{3i-1} & -2b_{3i} \\ b_{3i} & -2b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2b_{3i-2} \end{bmatrix}, \quad 1 \le i \le 4. \tag{3.37}$$

This Lie algebra $(\mathbb{R}^{12}, [\cdot, \cdot])$ is isomorphic to the matrix Lie algebra $\bar{g}$, and the mapping σ, defined by (3.34), is a Lie algebra isomorphism between the two Lie algebras.

A bilinear form on $\mathbb{R}^{12}$ can be defined by

$$\langle a, b \rangle = a^T F b, \tag{3.38}$$

where F is a constant matrix (actually, $F = (\langle \mathbf{e}_i, \mathbf{e}_j \rangle)_{12\times 12}$, where $\mathbf{e}_1, \dots, \mathbf{e}_{12}$ are the standard basis of $\mathbb{R}^{12}$). The symmetric property

$$\langle a, b \rangle = \langle b, a \rangle \tag{3.39}$$

requires that

$$F^T = F. \tag{3.40}$$

Under this symmetric condition, the ad-invariance property

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle \tag{3.41}$$

equivalently requires that

$$F(R(b))^T = -R(b)F, \quad b \in \mathbb{R}^{12}. \tag{3.42}$$

This matrix equation with an arbitrary b leads to a linear system of equations on the elements of the matrix F. Solving the resulting system tells

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\eta_2 + \beta\eta_4 & \alpha\eta_3 & \alpha\eta_4 \\ \eta_3 & \alpha\eta_3 & \mu\eta_3 + \nu\eta_4 & 0 \\ \eta_4 & \alpha\eta_4 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.43}$$

where η_i, $1 \le i \le 4$, are arbitrary constants and $\otimes$ is the Kronecker product. Now, the corresponding bilinear form on the semi-direct sum $\bar{g}$ is given as follows:

$$\begin{aligned} \langle A, B \rangle_{\bar{g}} &= \langle \sigma(A), \sigma(B) \rangle_{\mathbb{R}^{12}} = (a_1, \dots, a_{12}) F (b_1, \dots, b_{12})^T \\ &= (2a_1b_1 + a_2b_3 + a_3b_2)\eta_1 \\ &\quad + (2a_1b_4 + a_2b_6 + a_3b_5 + 2a_4b_1 + 2\alpha a_4b_4 \\ &\quad + a_5b_3 + \alpha a_5b_6 + a_6b_2 + \alpha a_6b_5)\eta_2 \\ &\quad + (2a_1b_7 + a_2b_9 + a_3b_8 + 2\alpha a_4b_7 + \alpha a_5b_9 \\ &\quad + \alpha a_6b_8 + 2a_7b_1 + 2\alpha a_7b_4 + 2\mu a_7b_7 + a_8b_3 \\ &\quad + \alpha a_8b_6 + \mu a_8b_9 + a_9b_2 + \alpha a_9b_5 + \mu a_9b_8)\eta_3 \\ &\quad + (2a_1b_{10} + a_2b_{12} + a_3b_{11} + 2\alpha a_4b_{10} + 2\beta a_4b_4 \\ &\quad + \beta a_5b_6 + \alpha a_5b_{12} + \beta a_6b_5 + \alpha a_6b_{11} + 2\nu a_7b_7 \\ &\quad + \nu a_8b_9 + \nu a_9b_8 + 2\alpha a_{10}b_4 + 2a_{10}b_1 \\ &\quad + \alpha a_{11}b_6 + a_{11}b_3 + \alpha a_{12}b_5 + a_{12}b_2)\eta_4, \end{aligned} \tag{3.44}$$

where

$$A = \sigma^{-1}((a_1, \dots, a_{12}))^T \in \bar{g}, \\ B = \sigma^{-1}((b_1, \dots, b_{12}))^T \in \bar{g}. \tag{3.45}$$

Owing to the isomorphism of σ, the bilinear form (3.44) is also symmetric and ad-invariant:

$$\langle A, B \rangle_{\bar{g}} = \langle B, A \rangle_{\bar{g}}, \quad \langle A, [B, C] \rangle_{\bar{g}} = \langle [A, B], C \rangle_{\bar{g}}, \\ A, B, C \in \bar{g}. \tag{3.46}$$

But this kind of bilinear forms is not of Killing type, since the matrix Lie algebra $\bar{g}$ is not semisimple.

A bilinear form, defined by (3.44), is non-degenerate if and only if the determinant of F is not zero, i.e.,

$$\det(F) = -16{\eta_4}^6(\alpha^2\eta_1 - \alpha\eta_2 + \beta\eta_4)^3 \times (\mu\eta_3 + \nu\eta_4)^3 \neq 0. \quad (3.47)$$

Therefore, we can choose η_1, η_2, η_3 and η_4 such that $\det(F)$ is non-zero to get non-degenerate bilinear forms over $\bar{g}$.

It is now direct to compute that

$$\langle \overline{W}, \bar{U}_\lambda \rangle_{\bar{g}} = -2a\eta_1 - 2e\eta_2 - 2e'\eta_3 - 2e''\eta_4,$$

and

$$\langle \overline{W}, \bar{U}_{\bar{u}} \rangle_{\bar{g}} = \begin{bmatrix} c\eta_1 + g\eta_2 + g'\eta_3 + g''\eta_4 \\ b\eta_1 + f\eta_2 + f'\eta_3 + f''\eta_4 \\ \alpha g'\eta_3 + c\eta_2 + (\alpha\eta_2 + \beta\eta_4)g + \alpha g''\eta_4 \\ b\eta_2 + \alpha f'\eta_3 + \alpha f''\eta_4 + (\alpha\eta_2 + \beta\eta_4)f \\ c\eta_3 + (\mu\eta_3 + \nu\eta_4)g' + \alpha g\eta_3 \\ b\eta_3 + \alpha\eta_3 f + (\mu\eta_3 + \nu\eta_4)f' \\ \alpha g\eta_4 + c\eta_4 \\ b\eta_4 + \alpha f\eta_4 \end{bmatrix}.$$

Furthermore, since we have [11]:

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \overline{W}, \overline{W} \rangle|,$$

we can obtain $\gamma = 0$. Thus, the corresponding variational identity becomes

$$\frac{\delta}{\delta \bar{u}}\int \frac{2a_{m+1}\eta_1 + 2e_{m+1}\eta_2 + 2e'_{m+1}\eta_3 + 2e''_{m+1}\eta_4}{m}\,dx = \begin{bmatrix} c_m\eta_1 + g_m\eta_2 + g'_m\eta_3 + g''_m\eta_4 \\ b_m\eta_1 + f_m\eta_2 + f'_m\eta_3 + f''_m\eta_4 \\ \alpha g'_m\eta_3 + c_m\eta_2 + (\alpha\eta_2 + \beta\eta_4)g_m + \alpha g''_m\eta_4 \\ b_m\eta_2 + \alpha f'_m\eta_3 + \alpha f''_m\eta_4 + (\alpha\eta_2 + \beta\eta_4)f_m \\ c_m\eta_3 + (\mu\eta_3 + \nu\eta_4)g'_m + \alpha g_m\eta_3 \\ b_m\eta_3 + \alpha f_m\eta_3 + (\mu\eta_3 + \nu\eta_4)f'_m \\ \alpha g_m\eta_4 + c_m\eta_4 \\ b_m\eta_4 + \alpha f_m\eta_4 \end{bmatrix}.$$

Consequently, we obtain a Hamiltonian structure for the hierarchy (3.31) of tri-integrable couplings:

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathscr{H}}_m}{\delta \bar{u}}, \quad m \geq 0, \quad (3.48)$$

with the Hamiltonian operator

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\eta_2 + \beta\eta_4 & \alpha\eta_3 & \alpha\eta_4 \\ \eta_3 & \alpha\eta_3 & \mu\eta_3 + \nu\eta_4 & 0 \\ \eta_4 & \alpha\eta_4 & 0 & 0 \end{bmatrix}^{-1} \otimes \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \quad (3.49)$$

and the Hamiltonian functionals

$$\bar{\mathscr{H}}_m = \int \frac{2a_{m+2}\eta_1 + 2e_{m+2}\eta_2 + 2e'_{m+2}\eta_3 + 2e''_{m+2}\eta_4}{m+1}\,dx, \quad m \geq 0. \quad (3.50)$$

3.4 Liouville integrability

The recursion relation

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1}, \quad m \geq 1, \quad (3.51)$$

generated from (3.6), (3.23), (3.24) and (3.25), tells that the recursion operator $\bar{\Phi}$ (see [25] for definition) reads

$$\bar{\Phi} = \bar{\Phi}(\bar{u}) = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ \Phi_1 & \Phi + \alpha\Phi_1 & 0 & 0 \\ \Phi_2 & \alpha\Phi_2 & \Phi + \alpha\Phi_1 + \mu\Phi_2 & 0 \\ \Phi_3 & \beta\Phi_1 + \alpha\Phi_3 & \nu\Phi_2 & \Phi + \alpha\Phi_1 \end{bmatrix}, \quad (3.52)$$

where Φ is given as in (3.10) and

$$\Phi_1 = \begin{bmatrix} r_1\partial^{-1}q + (p + \alpha r_1)\partial^{-1}s_1 & r_1\partial^{-1}p + (p + \alpha r_1)\partial^{-1}r_1 \\ -s_1\partial^{-1}q - (q + \alpha s_1)\partial^{-1}s_1 & -s_1\partial^{-1}p - (q + \alpha s_1)\partial^{-1}r_1 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} (\theta_1 + \mu r_2)\partial^{-1}s_2 + r_2\partial^{-1}(q + \alpha s_1) & (\theta_1 + \mu r_2)\partial^{-1}r_2 + r_2\partial^{-1}(p + \alpha r_1) \\ -(\theta_2 + \mu s_2)\partial^{-1}s_2 - s_2\partial^{-1}(q + \alpha s_1) & -(\theta_2 + \mu s_2)\partial^{-1}r_2 - s_2\partial^{-1}(p + \alpha r_1) \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} \theta_1\partial^{-1}s_3 + \theta_3\partial^{-1}s_1 + \nu r_2\partial^{-1}s_2 + r_3\partial^{-1}q, & \theta_1\partial^{-1}r_3 + \theta_3\partial^{-1}r_1 + \nu r_2\partial^{-1}r_2 + r_3\partial^{-1}p \\ -\theta_2\partial^{-1}s_3 - \theta_4\partial^{-1}s_1 - \nu s_2\partial^{-1}s_2 - s_3\partial^{-1}q, & -\theta_2\partial^{-1}r_3 - \theta_4\partial^{-1}r_1 - \nu s_2\partial^{-1}r_2 - s_3\partial^{-1}p \end{bmatrix},$$

in which

$$\theta_1 = p + \alpha r_1, \quad \theta_2 = q + \alpha s_1,$$
$$\theta_3 = \alpha r_3 + \beta r_1, \quad \theta_4 = \alpha s_3 + \beta s_1.$$

It is direct but lengthy to show by Maple that $\bar{\Phi}$ is hereditary [26], i.e., it satisfies

$$\begin{aligned}&\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{K}]\bar{S}-\bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{K}]\bar{S}\\&\quad=\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{S}]\bar{K}-\bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{S}]\bar{K}\end{aligned} \tag{3.53}$$

for all enlarged vector fields $\bar{K}$ and $\bar{S}$; and that $\bar{J}$ and $\bar{M}=\bar{\Phi}\bar{J}$ constitutes a Hamiltonian pair [27], i.e., any linear combination $\bar{N}$ of $\bar{J}$ and $\bar{M}$ satisfies

$$\int(\bar{K})^T\bar{N}'(\bar{u})[\bar{N}\bar{S}]\bar{T}dx+\text{cycle}(\bar{K},\bar{S},\bar{T})=0 \tag{3.54}$$

for all enlarged vector fields $\bar{K}$, $\bar{S}$ and $\bar{T}$. Therefore, the hierarchy (3.31) of tri-integrable couplings is bi-Hamiltonian (see, e.g., [27, 28]), and so, it is Liouville integrable. In particular, we have

$$[\bar{K}_m,\bar{K}_n]:=\bar{K}_m'(\bar{u})[\bar{K}_n]-\bar{K}_n'(\bar{u})[\bar{K}_m]=0,\ m,n\geq 0, \tag{3.55}$$

and

$$\{\bar{\mathcal{H}}_m,\bar{\mathcal{H}}_n\}_{\bar{J}}:=\int\left(\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\right)^T\bar{J}\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}dx=0,\quad m,n\geq 0. \tag{3.56}$$

3.5 Canonical forms of matrix Lie algebras

Let $M=M(A_1,A_2,A_3,A_4)$ be defined by (2.9) in Section 2.2. We would like to derive canonical forms of all matrix Lie algebras consisting of such matrices M by (2.9) associated with four arbitrary constants α, β, μ and ν, which determine equivalent classes of integrable couplings.

Case 1: $\alpha\neq 0$ and $\mu\neq 0$:

Under a similarity transformation

$$P=\begin{bmatrix}1&-\frac{1}{\alpha}&0&\frac{\beta}{\alpha^2}\\0&\mu&-\alpha&-\frac{\alpha\nu}{\mu}\\0&0&0&1\\0&0&1&\frac{\nu}{\mu}\end{bmatrix}, \tag{3.57}$$

we can simplify M as follows:

$$PMP^{-1}=\begin{bmatrix}A_1&0&0&0\\0&A_1+\alpha A_2&\beta\mu A_2-\alpha\nu A_3+\alpha\mu A_4&0\\0&0&A_1+\alpha A_2&0\\0&0&0&A_1+\alpha A_2+\mu A_3\end{bmatrix}. \tag{3.58}$$

Upon resetting entries, we can show that this kind of enlarged spectral matrices yields tri-integrable couplings of the following type

$$\begin{cases}u_t=K(u),\\u_{1,t}=S_1(u_1),\\u_{2,t}=S_2(u_1,u_2),\\u_{3,t}=S_3(u_3),\end{cases} \tag{3.59}$$

of which the first and fourth subsystems are separated.

Case 2: $\alpha\neq 0$ and $\mu=0$:

Under a similarity transformation

$$P=\begin{bmatrix}1&-\frac{1}{\alpha}&0&\frac{\beta}{\alpha^2}\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \tag{3.60}$$

we can simplify M as follows:

$$PMP^{-1}=\begin{bmatrix}A_1&0&0&0\\0&A_1+\alpha A_2&\alpha A_3&\beta A_2+\alpha A_4\\0&0&A_1+\alpha A_2&\nu A_3\\0&0&0&A_1+\alpha A_2\end{bmatrix}. \tag{3.61}$$

Similarly upon resetting entries, we can show that this kind of enlarged spectral matrices engenders tri-integrable couplings of the following type

$$\begin{cases}u_t=K(u),\\u_{1,t}=S_1(u_1),\\u_{2,t}=S_2(u_1,u_2),\\u_{3,t}=S_3(u_1,u_2,u_3),\end{cases} \tag{3.62}$$

of which the first subsystem is separated.

Case 3: $\alpha=0$ and $\mu\neq 0$:

Under a similarity transformation

$$P=\begin{bmatrix}-\mu&0&1&0\\0&1&0&0\\0&0&0&1\\0&0&\frac{1}{\mu}&\frac{\nu}{\mu^2}\end{bmatrix}, \tag{3.63}$$

we can simplify M as follows:

$$PMP^{-1}=\begin{bmatrix}A_1&-\mu A_2&\nu A_3-\mu A_4&0\\0&A_1&\beta A_2&0\\0&0&A_1&0\\0&0&0&A_1+\mu A_3\end{bmatrix}. \tag{3.64}$$

Upon resetting entries, we can show that this kind of enlarged spectral matrices yields tri-integrable couplings of the following type

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_1,u_2), \\ u_{3,t} = S_3(u_3), \end{cases} \tag{3.65}$$

of which the fourth subsystem is separated.

Case 4: $\alpha = \mu = 0$:

In this case, we have an interesting matrix Lie algebra consisting of the following block matrices

$$M = M(A_1,A_2,A_3,A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & 0 & \beta A_2 \\ 0 & 0 & A_1 & \nu A_3 \\ 0 & 0 & 0 & A_1 \end{bmatrix}. \tag{3.66}$$

Similarly upon resetting entries, we can show that this kind of enlarged spectral matrices generates tri-integrable couplings of the following type

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_2), \\ u_{3,t} = S_3(u,u_1,u_2,u_3), \end{cases} \tag{3.67}$$

of which the third subsystem does not depend on the second subvector u_1 of dependent variables.

Since the similarity transformation does not change zero curvature equations, the first three cases lead to some classes of tri-integrable couplings which can be decomposed into integrable couplings or bi-integrable couplings plus separated integrable systems. However, the fourth one yields a class of specific tri-integrable couplings, and provides an answer to a question about the coupling system

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_2), \end{cases} \tag{3.68}$$

raised in [29]: Is there any Hamiltonian structure behind this coupling system? Our results show that by adding one more subsystem, one can build a bigger integrable coupling being Hamiltonian as a whole.

4 Conclusions and remarks

We created a class of matrix loop algebras consisting of 4×4 block matrices to generate tri-integrable couplings, and successfully constructed a hierarchy of tri-integrable couplings for the AKNS equations, which possesses a bi-Hamiltonian structure. The presented matrix loop algebras served as a beginning point to construct tri-integrable couplings and their canonical forms were analyzed in detail. The whole construction scheme can be used to the other existing soliton hierarchies such as the KdV hierarchy, the Dirac hierarchy and the Kaup-Newell hierarchy.

Together with bi-integrable couplings, tri-integrable couplings provide us with insightful thoughts about general structures of integrable systems with multi-components. It will be very helpful in building an exhaustive list of integrable systems to collect more examples of integrable couplings. Multi-integrable couplings yield diverse recursion operators in block matrix form. The mathematical structures behind integrable couplings are rich and interesting [30].

There are many other interesting questions on integrable couplings. For instance, what other non-semisimple matrix loop algebras can one begin with, to generate integrable couplings? It is known that Hamiltonian structures exist for the perturbation systems [31, 32, 33, 34], but some enlarged spectral matrices do not yield any non-degenerate bilinear forms over the associated matrix loop algebras required in the variational identities [29, 35]. Are there any criteria which guarantee the existence of Hamiltonian structures for bi- or tri-integrable couplings? How can one compute solution groups for integrable couplings, either by symmetry constraints as did for the perturbation systems [36, 37] or by Darboux transformations engendered through moving frames [38]?

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