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Lump solutions to a (2 + 1)-dimensional fourth-order nonlinear PDE possessing a Hirota bilinear form

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Through the Hirota bilinear formulation, a (2 + 1)-dimensional combined fourth-order nonlinear equation is proposed, which possesses lump solutions. Two classes of lump solutions are presented explicitly in terms of the coefficients in the combined nonlinear equation. A set of examples of equations is provided to show the diversity of the considered combined nonlinear equation, together with three-dimensional plots, x-curves and y-curves of two specific lump solutions in two cases of the combined equation.

Keywords: (2+1)-dimensional combined fourth-order nonlinear equation; Hirota bilinear formulation; lump solutions.

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1. Introduction

The description of the laws of physics changing over time and space is usually expressed in terms of partial differential equations (PDEs). For the vast majority of nonlinear problems in mathematics and physical sciences, the involved PDEs cannot be solved through analytical methods. Exactly solvable PDEs are often constant-coefficient and linear. Nevertheless, soliton theory provides a few working

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approaches to nonlinear PDEs,[1–3] and one of them is the Hirota bilinear approach to soliton solutions, historically developed for integrable equations.[4,5]

Soliton solutions are analytic and exponentially localized in all directions in time and space. Within the Hirota bilinear formulation, a $(2+1)$-dimensional PDE with a dependent variable u is connected with a Hirota bilinear differential equation

$$P(D_x, D_y, D_t) f \cdot f = 0\,,$$

where P is a polynomial and D_x, D_y and D_t are Hirota's bilinear derivatives. The link is often taken as one of the logarithmic derivative transformations:

$$u = 2(\ln f)_x, \quad u = 2(\ln f)_{xx}\,.$$

Soliton solutions can then be formulated as follows:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\right),$$

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \;\; 1 \le i \le N\,,$$

where $\sum_{\mu=0,1}$ denotes the sum over all possibilities for $\mu_1, \mu_2, \ldots, \mu_N$ taking either 0 or 1, the phase shifts are defined by

$$\mathrm{e}^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, \quad 1 \le i < j \le N\,,$$

with k_i, l_i and ω_i, $1 \le i \le N$, satisfying the corresponding dispersion relation, and $\xi_{i,0}$, $1 \le i \le N$, in the wave variables being arbitrary phase shifts.

Lump solutions are a class of rational function solutions which are localized in all directions in space, which originated from solving $(2+1)$-dimensional integrable equations (see, e.g. Refs. 6, 7, 8). Long wave limits of N-soliton solutions can produce special lumps as envelope solutions.[9] Many studies on $(2+1)$-dimensional integrable equations domenstrate the remarkable richness of lump solutions (see, e.g. Refs. 6 and 7), which can be used to describe diverse nonlinear wave phenomena in engineering and physical sciences. The KPI equation possesses diverse lump solutions (see, e.g. Ref. 10), among which are specific lump solutions derived from N-soliton solutions.[11] Other such integrable equations possessing lump solutions contain the three-dimensional three-wave resonant interaction,[12] the BKP equation,[13,14] the Davey–Stewartson equation II,[9] the Ishimori-I equation,[15] the Kadomtsev–Petviashvili (KP) equation with a self-consistent source[16] and the second KP equation.[17] A crucial step in the process of constructing lump solutions is to determine positive quadratic function solutions to Hirota bilinear equations.[6] One then gets lump solutions to nonlinear equations by the logarithmic derivative transformations.

In this paper, we would like to consider a $(2+1)$-dimensional combined fourth-order nonlinear equation which possesses diverse lump solutions. The Hirota bilinear form is the starting point for our construction (see, e.g. Refs. 6, 7 and 18, 19, 20 for other equations). We will propose a $(2+1)$-dimensional combined fourth-order nonlinear equation which includes all linear second-order derivative terms and possesses lump solutions. We will concisely present the expressions for the parameters involved in lump solutions with Maple symbolic computations. For two specially chosen nonlinear equations, three-dimensional plots, x-curves and y-curves will be made for two specific lump solutions via Maple plot tools, to shed light on the presented lump solutions. Together with conclusions, a few concluding remarks will be given in Sec. 3.

2. A Fourth-Order Nonlinear PDE and Its Lump Solutions

2.1. *A fourth-order nonlinear PDE in* $(2+1)$*-dimensions*

We would like to consider a general combined fourth-order nonlinear equation in $(2+1)$-dimensions as follows:

$$\begin{aligned} P(u) &= (6u_x u_{xx} + u_{xxxx}) + \alpha[3(u_x u_t)_x + u_{xxxt}] + \beta[3(u_x u_y)_x + u_{xxxy}] \\ &\quad + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} + \gamma_6 u_{tt} = 0\,, \end{aligned} \tag{2.1}$$

where the constants α, β and $\gamma_i, 1 \le i \le 6$, are arbitrary. The equation contains three combinations of fourth-order derivative terms and all linear second-order derivative terms. It is straightforward to see that it has a Hirota bilinear form[21]

$$\begin{aligned} B(f) &= (D_x^4 + \alpha D_x^3 D_t + \beta D_x^3 D_y + \gamma_1 D_y D_t \\ &\quad + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2) f \cdot f = 0\,, \end{aligned} \tag{2.2}$$

under the logarithmic derivative transformation

$$u = 2(\ln f)_x = \frac{2f_x}{f}\,. \tag{2.3}$$

Precisely, we have the relation $P(u) = (\frac{B(f)}{f^2})_x$, where u and f satisfy the link (2.3). Therefore, if f solves (2.2), then u determined by (2.3) will present a solution to (2.1).

Particularly, on the one hand, when $\alpha = \beta = 0$, $\gamma_3 = -\gamma_5 = 1$ and the other γs are zero, we obtain the potential KP equation in $(2+1)$ dimensions

$$6u_x u_{xx} + u_{xxxx} + u_{xt} - u_{yy} = 0\,, \tag{2.4}$$

which possesses lump solutions[10] and is equivalent to the bilinear KP equation

$$(D_x^4 + D_x D_t - D_y^2) f \cdot f = 0\,, \tag{2.5}$$

under the logarithmic derivative transformation (2.3).

On the other hand, when $\alpha = 0$, $\beta = 1$, $\gamma_3 = \gamma_5 = 1$ and the other γs are zero, we obtain a generalized Bogoyavlensky–Konopelchenko equation:

$$6u_x u_{xx} + u_{xxxx} + 3(u_x u_y)_x + u_{xxxy} + u_{xt} + u_{yy} = 0\,, \tag{2.6}$$

which possesses a Hirota bilinear form

$$(D_x^4 + D_x^3 D_y + D_x D_t + D_y^2) f \cdot f = 0\,, \tag{2.7}$$

under (2.3), and has lump solutions.[22]

2.2. *Lump solutions*

In this section, we are going to construct lump solutions to the $(2+1)$-dimensional combined fourth-order nonlinear equation (2.1), through conducting symbolic computations.

We begin with a search for positive quadratic solutions to the combined bilinear equation (2.2):

$$f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9, \tag{2.8}$$

where $a_i, 1 \le i \le 9$, are constant parameters to be determined, to determine lump solutions for the combined nonlinear equation (2.1). There are two solutions situations for the combined nonlinear equation (2.1), which we can deal with by making symbolic computations.

The first solution situation is associated with $\gamma_6 = 0$. A direct symbolic computation provides us with a set of solutions for the parameters, which tells

$$\begin{cases} a_3 = -\dfrac{b_1}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}\,, \\ a_7 = -\dfrac{b_2}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}\,, \\ a_9 = \dfrac{3(a_1^2 + a_5^2)(\alpha b_3 - \beta b_4 - b_5)}{(a_1 a_6 - a_2 a_5)^2 (\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_3^2\gamma_5)}\,, \end{cases} \tag{2.9}$$

with all arbitrary other as. The above five constants are determined by

$$\begin{cases} b_1 = [(a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2)\gamma_2 + a_1(a_2^2 + a_6^2)\gamma_4 + a_2(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ \qquad + [a_1(a_1^2 + a_5^2)\gamma_2 + a_2(a_1^2 + a_5^2)\gamma_4 + (a_1 a_2^2 + 2a_2 a_5 a_6 - a_1 a_6^2)\gamma_5]\gamma_3\,, \\ b_2 = [(-a_1^2 a_6 + 2a_1 a_2 a_5 + a_5^2 a_6)\gamma_2 + a_5(a_2^2 + a_6^2)\gamma_4 + a_6(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ \qquad + [a_5(a_1^2 + a_5^2)\gamma_2 + a_6(a_1^2 + a_5^2)\gamma_4 + (-a_2^2 a_5 + 2a_1 a_2 a_6 + a_5 a_6^2)\gamma_5]\gamma_3\,, \\ b_3 = (a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_5^2)(a_2^2 + a_6^2)\gamma_1\gamma_4 \\ \qquad + (a_1^2 + a_5^2)^2\gamma_2\gamma_3 + (a_2^2 + a_6^2)(a_1 a_2 + a_5 a_6)\gamma_1\gamma_5 + [(a_1 a_2 + a_5 a_6)^2 \\ \qquad - (a_1 a_6 - a_2 a_5)^2]\gamma_3\gamma_5\,, \\ b_4 = (a_1 a_2 + a_5 a_6)[(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2]\,, \\ b_5 = (a_1^2 + a_5^2)[(a_2^2 + a_6^2)\gamma_1^2 + 2(a_1 a_2 + a_5 a_6)\gamma_1\gamma_3 + (a_1^2 + a_5^2)\gamma_3^2]\,. \end{cases} \tag{2.10}$$

The second solution situation is associated with $\gamma_5 = 0$. Similarly, a direct symbolic computation presents a set of solutions for the parameters, which tells

$$\begin{cases} a_2 = -\dfrac{c_1}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}\,, \\ a_6 = -\dfrac{c_2}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}\,, \\ a_9 = -\dfrac{3(a_1^2 + a_5^2)(\alpha c_3 - \beta c_4 + c_5)}{(a_1a_7 - a_3a_5)^2(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_4^2\gamma_6)}\,, \end{cases} \tag{2.11}$$

and all arbitrary other as. The above five constants are determined by

$$\begin{cases} c_1 = [(a_1^2a_3 + 2a_1a_5a_7 - a_3a_5^2)\gamma_2 + a_1(a_3^2 + a_7^2)\gamma_3 + a_3(a_3^2 + a_7^2)\gamma_6]\gamma_1 \\ \qquad + [a_1(a_1^2 + a_5^2)\gamma_2 + a_3(a_1^2 + a_5^2)\gamma_3 + (a_1a_3^2 + 2a_3a_5a_7 - a_1a_7^2)\gamma_6]\gamma_4\,, \\ c_2 = [(-a_1^2a_7 + 2\,a_1a_3a_5 + a_5^2a_7)\gamma_2 + a_5(a_3^2 + a_7^2)\gamma_3 + a_7(a_3^2 + a_7^2)\gamma_6]\gamma_1 \\ \qquad + [a_5(a_1^2 + a_5^2)\gamma_2 + a_7(a_1^2 + a_5^2)\gamma_3 + (-a_3^2a_5 + 2\,a_1a_3a_7 + a_5a_7^2)\gamma_6]\gamma_4\,, \\ c_3 = (a_1a_3 + a_5a_7)[(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2]\,, \\ c_4 = (a_1^2 + a_5^2)(a_1a_3 + a_5a_7)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_5^2)(a_3^2 + a_7^2)\gamma_1\gamma_3 \\ \qquad + (a_1^2 + a_5^2)^2\gamma_2\gamma_4 + (a_3^2 + a_7^2)(a_1a_3 + a_5a_7)\gamma_1\gamma_6 \\ \qquad + [(a_1a_3 + a_5a_7)^2 - (a_1a_7 - a_3a_5)^2]\gamma_4\gamma_6\,, \\ c_5 = (a_1^2 + a_5^2)[(a_3^2 + a_7^2)\gamma_1^2 + 2(a_1a_3 + a_5a_7)\gamma_1\gamma_4 + (a_1^2 + a_5^2)\gamma_4^2]\,. \end{cases} \tag{2.12}$$

All those expressions in the above formulas (2.9)–(2.12) have been presented through some simplification with the computer algebra system Maple.

For the second solution situation associated with $\gamma_5 = 0$, one needs to check the conditions on the parameters, under which the presented solutions become lumps. Obviously, one has

$$a_1a_6 - a_2a_5 = \frac{\begin{array}{l}(a_1a_7 - a_3a_5)[(a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_7^2)\gamma_1\gamma_6 \\ \quad - 2(a_1a_3 + a_5a_7)\gamma_4\gamma_6]\end{array}}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}\,.$$

Therefore, one knows that $a_1a_6 - a_2a_5 \neq 0$ if and only if

$$\begin{cases} a_1a_7 - a_3a_5 \neq 0, \quad \gamma_1^2 + \gamma_4^2 \neq 0\,, \\ (a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_7^2)\gamma_1\gamma_6 - 2(a_1a_3 + a_5a_7)\gamma_4\gamma_6 \neq 0\,, \end{cases} \tag{2.13}$$

which guarantees, together with $a_9 > 0$, that the corresponding set of the parameters in (2.11) will generate lump solutions to the combined nonlinear equation (2.1).

2.3. *Abundant examples of equations*

We would like to present abundant illustrative examples of the considered combined nonlinear equation (2.1), which possess lumps, according to three categories of combinations of fourth-order derivative terms.

2.3.1. *The case of* $\alpha = \beta = 0$

When $\alpha = \beta = 0$, the combined bilinear equation (2.2) reduces to

$$(D_x^4 + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2) f \cdot f = 0 \,. \qquad (2.14)$$

When $\gamma_1\gamma_2 \neq 0$, or $\gamma_3\gamma_5 \neq 0$, or $\gamma_4\gamma_6 \neq 0$, or $\gamma_1\gamma_3\gamma_4 \neq 0$, or $\gamma_1\gamma_3\gamma_5 \neq 0$, or $\gamma_1\gamma_4\gamma_6 \neq 0$, and the corresponding other γs are zero, the bilinear equation (2.14) gives six corresponding $(2+1)$-dimensional reduced nonlinear equations which possess lump solutions, based on the two presented classes of exact solutions by (2.9) and (2.11).

A special lump solution by (2.9) in the first subcase with $\gamma_1 = -\gamma_2 = 1$ will be plotted in the next subsection. The second subcase with $\gamma_3 = -\gamma_5 = 1$ is just the potential KP equation (2.4), as mentioned previously.

Generally, the second subcase and the third subcase, and the fifth subcase and the sixth subcase become each other, under an exchange of t and y, γ_3 and γ_4 and γ_5 and γ_6.

2.3.2. *The case of* $\alpha = \beta = 1$

When $\alpha = \beta = 1$, the combined bilinear equation (2.2) reduces to

$$\begin{aligned}(D_x^4 + D_x^3 D_t + D_x^3 D_y + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y \\ + \gamma_5 D_y^2 + \gamma_6 D_t^2) f \cdot f = 0 \,. \end{aligned} \qquad (2.15)$$

When $\gamma_1\gamma_2 \neq 0$, or $\gamma_3\gamma_5 \neq 0$, or $\gamma_4\gamma_6 \neq 0$, or $\gamma_1\gamma_3\gamma_4 \neq 0$, or $\gamma_1\gamma_3\gamma_5 \neq 0$, or $\gamma_1\gamma_4\gamma_6 \neq 0$, and the corresponding other γs are zero, the bilinear equation (2.15) similarly gives six corresponding $(2+1)$-dimensional reduced nonlinear equations which possess lump solutions, based on the two presented classes of exact solutions by (2.9) and (2.11).

Again, the second subcase and the third subcase, and the fifth subcase and the sixth subcase, become each other, under an exchange of t and y, γ_3 and γ_4 and γ_5 and γ_6.

A special lump solution by (2.11) in the fourth subcase with $\gamma_1 = -\gamma_3 = \gamma_4 = 1$ will be plotted in the next section.

2.3.3. *The case of* $\alpha = 1$ *and* $\beta = 0$

When $\alpha = 1$ and $\beta = 0$, the combined bilinear equation (2.2) reduces to

$$(D_x^4 + D_x^3 D_t + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2) f \cdot f = 0 \,. \qquad (2.16)$$

When $\gamma_1\gamma_2 \neq 0$, or $\gamma_3\gamma_5 \neq 0$, or $\gamma_4\gamma_6 \neq 0$, or $\gamma_1\gamma_3\gamma_4 \neq 0$, or $\gamma_1\gamma_3\gamma_5 \neq 0$, or $\gamma_1\gamma_4\gamma_6 \neq 0$, and the corresponding other γs are zero, the bilinear equation (2.16) also gives six corresponding $(2+1)$-dimensional reduced nonlinear equations which possess lump solutions, based on the two presented classes of exact solutions by (2.9) and (2.11).

The third subcase with $\gamma_4 = \gamma_6 = 1$ produces the generalized Bogoyavlensky–Konopelchenko equation (2.6), after an exchange of t and y. Another example is the following $(2+1)$-dimensional combined nonlinear equation:

$$6u_x u_{xx} + u_{xxxx} + 3(u_x u_t)_x + u_{xxxt} + +u_{yt} + u_{xt} + u_{yy} = 0, \tag{2.17}$$

which corresponds to the fifth subcase with $\gamma_1 = \gamma_3 = \gamma_5 = 1$.

2.3.4. *The case of $\alpha = 0$ and $\beta = 1$*

This case is covered by the previous case under an exchange of t and y.

2.4. ***Plots of two specific lumps***

For the first case, particularly taking

$$\gamma_1 = -\gamma_2 = 1, \quad \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 0\,, \tag{2.18}$$

we obtain a special fourth-order nonlinear equation as follows:

$$6u_x u_{xx} + u_{xxxx} + u_{yt} - u_{xx} = 0\,, \tag{2.19}$$

which has a Hirota bilinear form

$$(D_x^4 + D_y D_t - D_x^2) f \cdot f = 0\,, \tag{2.20}$$

under the logarithmic derivative transformation (2.3). Associated with

$$a_1 = 1, \quad a_2 = -2, \quad a_4 = 10, \quad a_5 = 3, \quad a_6 = -1, \quad a_8 = 5\,, \tag{2.21}$$

which, together with (2.9), leads to

$$a_3 = 2, \quad a_7 = -4, \quad a_9 = 60\,, \tag{2.22}$$

the logarithmic derivative transformation (2.3) with f defined by (2.8) provides the following lump solution to the special fourth-order nonlinear equation (2.19):

$$u_1 = \frac{20(-2t + 2x - y + 5)}{(2t + x - 2y + 10)^2 + (-4t + 3x - y + 5)^2 + 60}\,. \tag{2.23}$$

A three-dimensional plot, x-curves and y-curves of this lump solution at $t = 0$ are made via Maple plot tools, to shed light on the characteristic of the presented lump solutions, as shown in Fig. 1.

For the second case, specially taking

$$\gamma_1 = \gamma_3 = \gamma_4 = 1, \quad \gamma_2 = \gamma_5 = \gamma_6 = 0\,, \tag{2.24}$$

we obtain another special fourth-order nonlinear equation as follows:

$$6u_x u_{xx} + u_{xxxx} + 3(u_x u_t)_x + u_{xxxt} + 3(u_x u_y)_x$$

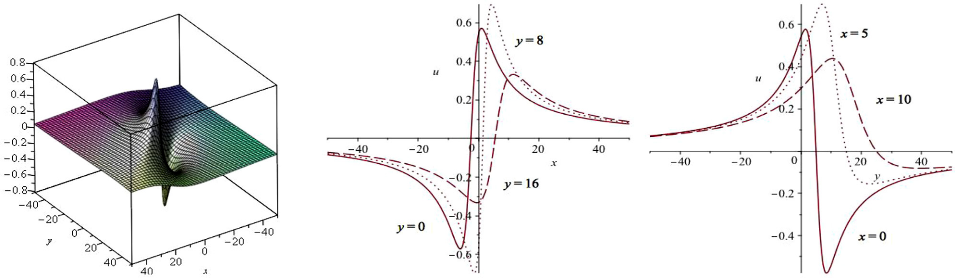

Fig. 1. Profiles of u_1 at $t = 0$: 3d plot (left), x-curves (middle) and y-curves (right).

$$+ u_{xxxy} + u_{yt} + u_{xt} + u_{xy} = 0\,, \tag{2.25}$$

which has a Hirota bilinear form

$$(D_x^4 + D_x^3 D_t + D_x^3 D_y + D_y D_t + D_x D_t + D_x D_y) f \cdot f = 0\,, \tag{2.26}$$

under the logarithmic derivative transformation (2.3). Associated with

$$a_1 = 3, \quad a_3 = -1, \quad a_4 = 3, \quad a_5 = 1, \quad a_7 = 3, \quad a_8 = 6\,, \tag{2.27}$$

which, together with (2.11), leads to

$$a_2 = -1, \quad a_6 = -2, \quad a_9 = 30\,, \tag{2.28}$$

the logarithmic derivative transformation (2.3) with f defined by (2.8) provides the following lump solution to the special fourth-order nonlinear equation (2.25):

$$u_2 = \frac{20(2x - y + 3)}{(-t + 3x - y + 3)^2 + (3t + x - 2y + 6)^2 + 30}\,. \tag{2.29}$$

A three-dimensional plot, x-curves and y-curves of this lump solution at $t = 1$ are made via Maple plot tools, to shed light on the characteristic of the presented lump solutions, as shown in Fig. 2.

We point out that all the exact lump solutions generated above add valuable insights into the existing theories on soliton solutions and dromion-type solutions,

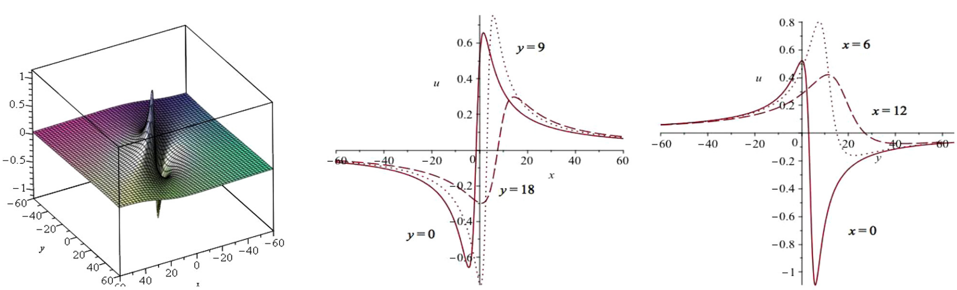

Fig. 2. Profiles of u_2 at $t = 1$: 3d plot (left), x-curves (middle) and y-curves (right).

established with various efficient techniques such as the Hirota perturbation approach, the Riemann–Hilbert approach, the Wronskian technique, symmetry reductions and symmetry constraints (see, e.g. Refs. 23–34).

3. Concluding Remarks

Within the Hirota bilinear formulation, a $(2+1)$-dimensional combined fourth-order nonlinear equation was proposed and its lump solutions were constructed explicitly via Maple symbolic computations. Three-dimensional plots, x-curves and y-curves of two special lump solutions in two cases of the combined nonlinear equation were also made by Maple plot tools to shed light on the presented lump solutions. Our results provide novel examples of $(2+1)$-dimensional nonlinear equations that possess lump solution structures, and enrich the existing theory of lumps and solitons.

We remark that the second and third combinations of fourth-order derivative terms can be emerged together into one new combination in the considered nonlinear model, but the transformed linear second-order derivative terms will be different as well as solvability situations will be changed. The combined nonlinear equation (2.1) has a symmetric characteristic for the independent variables t and y, and we have two situations of lump solutions. Indeed, the linear terms u_{tt} and u_{yy} have a serious effect on the determination of lump solutions with symbolic computations. Some general considerations have been made on the existence of lump solutions for the Hirota bilinear case[6] and the generalized bilinear cases.[7]

There is a large class of nonlinear equations which possess lump solutions, and it contains $(2+1)$-dimensional generalized KP, BKP, KP–Boussinesq and Sawada–Kotera equations.[35–38] Some recent studies also show the strikingly high richness of lump solutions to linear partial differential equations[39,40] and nonlinear partial differential equations in $(2+1)$-dimensions (see, e.g. Refs. 41–46) and $(3+1)$ dimensions (see, e.g. Refs. 47–50). Diverse lump solutions enrich the existing solution theories which originated from different kinds of combinations (see, e.g. Refs. 51–54), and can lead to abundant Lie–Bäcklund symmetries, which can also be used to determine conservation laws by symmetries and adjoint symmetries.[55–57] Moreover, diverse interaction solutions[38] have been reported for different integrable equations in $(2+1)$ dimensions, including lump–soliton interaction solutions (see, e.g. Refs. 58–60) and lump–kink interaction solutions (see, e.g. Refs. 61–63). All those show the diversity of exact solutions and the difficulty to get them for nonlinear PDEs.

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