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Tri-integrable couplings of the KdV hierarchy associated with a non-semisimple Lie algebra

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ABSTRACT: We explore the possibility of creating non-semisimple matrix loop algebras which lead to tri-integrable couplings containing two known integrable couplings. A semi-direct sum of Lie algebras consisting of specific 4×4 block matrices is found to form the base of such integrable couplings. An application to the KdV equations is made as an illustrative example, and the resulting tri-integrable couplings are proved to possess bi-Hamiltonian structures, which implies that there are infinitely many common commuting symmetries and conserved functionals determined by a hereditary recursion operator.

Key words: Tri-integrable coupling, non-semisimple Lie algebra, Lax pair, Hamiltonian structure, Recursion operator
MSC codes: 37K05, 37K10, 35Q53

1 Introduction

Integrable couplings are a pretty new area of research in the field of integrable systems. The concept of integrable couplings was introduced, based on studies of τ-symmetries, and an integrable theory for the perturbation equations was developed by the perturbation bundle [1, 2, 3]. A general structure of integrable couplings connected with semi-direct sums of Lie algebras was recognized recently [4, 5]. The Levi-Mal'tsev theorem states that an arbitrary Lie algebra over a field of characteristic zero has a semi-direct sum structure of a solvable Lie algebra and a semisimple Lie algebra [6]. Therefore, semi-direct sums of Lie algebras, i.e., non-semisimple Lie algebras, lay the foundation for studying integrable couplings, which produce triangular integrable systems with multi-components (see, e.g., [4, 5, 7] for details).

Assume that an integrable system

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \cdots), \tag{1.1}$$

where u denotes a column vector of dependent variables, has two integrable couplings:

$$\bar{u}_{1,t} = \bar{K}_1(\bar{u}_1) = \begin{bmatrix} K(u) \\ S_1(u, u_1) \end{bmatrix}, \quad \bar{u}_1 = \begin{bmatrix} u \\ u_1 \end{bmatrix}, \tag{1.2}$$

and

$$\bar{u}_{2,t} = \bar{K}_2(\bar{u}_2) = \begin{bmatrix} K(u) \\ S_2(u, u_1) \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} u \\ u_2 \end{bmatrix}, \tag{1.3}$$

where u_1 and u_2 are two column vectors of additional dependent variables. A natural question is whether we can put them together to form a new larger integrable coupling which possesses a bi-Hamiltonian structure. The simplest such coupling system is

$$\bar{u}_t = \bar{K}(\bar{u}) = \begin{bmatrix} K(u) \\ S(u, u_1) \\ T(u, u_2) \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}. \tag{1.4}$$

This is a kind of degenerate system in the sense that the two subsystems for the dependent variables u_1 and u_2 are separated. Moreover, no Hamiltonian structure has been found for this system [8].

On the other hand, let us recall the definition of tri-integrable couplings. A tri-integrable coupling of a given integrable system (1.1) is the following enlarged triangular integrable system [9]:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2) \\ u_{3,t} = S_3(u, u_1, u_2, u_3). \end{cases} \tag{1.5}$$

We call this system to be a nonlinear integrable coupling if at least one of $S_1(u, u_1)$, $S_2(u, u_1, u_2)$ and $S_3(u, u_1, u_2, u_3)$ are nonlinear with respect to any sub-vectors u_1, u_2, u_3 of new dependent variables.

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Though we are not able to show that the system (1.4) is Hamiltonian, we will, from a specific non-semisimple matrix Lie algebra, postulate an even larger Lax pair and present a kind of bi-Hamiltonian tri-integrable couplings, which contains the system (1.4) as a sub-system. The zero curvature equation and the variational identity will be the basic tools we will adopt [13].

The manuscript is structured as follows. In the next section, we will propose a non-semisimple matrix Lie algebra consisting of 4×4 block matrices, and construct a kind of tri-integrable couplings by using Lax pairs from this specific Lie algebra. An application to the KdV soliton hierarchy will be made as an illustrative example, and all resulting tri-integrable couplings will be shown to be bi-Hamiltonian by the variational identity.

2 Matrix Lie algebras generating tri-integrable couplings

2.1 Soliton hierarchy

Assume that a soliton hierarchy is associated with a spectral problem

$$\phi_x = U\phi,\ U = U(u,\lambda) \in g, \tag{2.1}$$

where g is often a semisimple matrix Lie algebra.

The zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{2.2}$$

with the Lax matrices $V^{[m]} = V^{[m]}(u,\lambda) \in g, m \geq 0$, are the compatibility conditions between the spectral problem (2.1) and the auxiliary eigenvalue problems

$$\phi_x = V^{[m]}\phi,\ m \geq 0. \tag{2.3}$$

In order to determine suitable Lax matrices $V^{[m]},\ m \geq 0$, we first solve the stationary zero curvature equation

$$W_x = [U, W] \tag{2.4}$$

by assuming

$$W = W(u,\lambda) = \sum_{i\geq 0} W_{0,i}\lambda^{-i}, \tag{2.5}$$

where $W_{0,i} \in g,\ i \geq 0$. Then we define the Lax matrices $V^{[m]}$ by

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \Delta_m \in g,\ m \geq 0, \tag{2.6}$$

where P_+ denotes the polynomial part of P in λ, and the modification terms Δ_m are chosen to guarantee that zero curvature equations (2.2) yield a soliton hierarchy with a Hamiltonian structure:

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{2.7}$$

of which the first system $u_{t_1} = K_1$ is usually the original integrable system (1.1) with

$$V = V^{[1]} = V^{[1]}(u,\lambda). \tag{2.8}$$

The above Hamiltonian functionals $\mathcal{H}_m$ are usually generated via the trace identity [10, 11] or more generally via the variational identity [12, 13]:

$$\frac{\delta}{\delta u}\int \langle \frac{\partial U}{\partial \lambda}, W\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle \frac{\partial U}{\partial u}, W\rangle, \tag{2.9}$$

where γ is a constant, $\langle \cdot,\cdot\rangle$ is a bilinear form on the Lie algebra g, which is non-degenerate, symmetric and ad-invariant [13], and W is a solution of the stationary zero curvature equation (2.4).

2.2 Matrix Lie algebras

A desired tri-integrable coupling, which contains (1.2) and (1.3) as subsystems, is of the form

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_2), \\ u_{3,t} = S_3(u,u_1,u_2,u_3). \end{cases} \tag{2.10}$$

It was shown [8] that the coupled system (1.4) of two integrable couplings (1.2) and (1.3) has an enlarged zero curvature representation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0,$$

where the Lax pair of $\bar{U}$ and $\bar{V}$ is defined by

$$\bar{U} = \begin{bmatrix} U & U_1 & U_2 \\ 0 & U & 0 \\ 0 & 0 & U \end{bmatrix},\ \bar{V} = \begin{bmatrix} V & V_1 & V_2 \\ 0 & V & 0 \\ 0 & 0 & V \end{bmatrix}.$$

Motivated by this statement, we expect that an ideal Lax pair of $\bar{U}$ and $\bar{V}$ for constructing a tri-integrable coupling (2.10) can be triangular block matrices of the following type:

$$M_g(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & 0 & \sum_{i=1}^4 \alpha_{1,i}\, A_i \\ 0 & 0 & A_1 & \sum_{i=1}^4 \alpha_{2,i}\, A_i \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{2.11}$$

where A_1, A_2, A_3 and A_4 are square submatrices of the same order, and $\{\alpha_{1,i},\ \alpha_{2,i}\}_{1\le i\le 4}$ are real constants to be determined. The submatrices should be chosen as Jordan blocks, because Jordan blocks build the canonical formulation of matrices under similarity transformations. Within the canonical formulation, the blocks can represent integrable couplings which can not be separated.

All block matrices of the form (2.11) need to constitute a Lie subalgebra of the 4×4 block matrix Lie algebra under the matrix commutator. By direct computation, we see that block matrices forming a Lie subalgebra must be of the following specific type:

$$M_s(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & 0 & \alpha\, A_2 + \beta\, A_3 \\ 0 & 0 & A_1 & \zeta\, A_2 + \mu\, A_3 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{2.12}$$

where α, β, ζ and μ are four arbitrary constants. For the sake of computational simplicity, we set $\zeta = \beta$ and use the following subclass of block matrices:

$$M(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & 0 & \alpha\, A_2 + \beta\, A_3 \\ 0 & 0 & A_1 & \beta\, A_2 + \mu\, A_3 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{2.13}$$

as a concrete example.

It is obvious to find that all block matrices defined by (2.13) constitute a matrix Lie algebra for fixed constants, α, β and μ, under the matrix commutator

$$[M_1, M_2] = M_1 M_2 - M_2 M_1,$$

since we have the closure property

$$[M(A_1, A_2, A_3, A_4), M(B_1, B_2, B_3, B_4)] = M(C_1, C_2, C_3, C_4), \tag{2.14}$$

where

$$\begin{cases} C_1 \;=\; [A_1, B_1], \\ C_2 \;=\; [A_1, B_2] + [A_2, B_1], \\ C_3 \;=\; [A_1, B_3] + [A_3, B_1], \\ C_4 \;=\; [A_1, B_4] + \alpha\,[A_2, B_2] + \beta\,[A_2, B_3] + \beta\,[A_3, B_3] + \mu\,[A_3, B_4] + [A_4, B_1]. \end{cases} \tag{2.15}$$

The resulting Lie algebra has a semi-direct sum decomposition of a semisimple subalgebra g and a solvable subalgebra g_c:

$$\bar{g} = g \Subset g_c, \tag{2.16}$$

where

$$g = \{M(A_1, 0, 0, 0)|\, A_1 \text{ - arbitrary}\},\ g_c = \{M(0, A_2, A_3, A_4)|\, A_i \text{ - arbitrary}\}, \tag{2.17}$$

and thus, it must be non-semisimple, because obviously one of nontrivial ideals of $\bar{g}$ is g_c. Such a Lie algebra $\bar{g}$ creates a basis for us to generate Hamiltonian tri-integrable couplings like the other presented Lie algebras in the literature (see, e.g., [14, 15, 16]). The block A_1 corresponds to the original integrable system, and the other three blocks A_2, A_3 and A_4 are used to generate the supplementary vector fields S_1, S_2 and S_3. We remark that the commutators $[A_2, B_2]$ and $[A_3, B_3]$ will lead to nonlinear terms in the resulting tri-integrable couplings.

2.3 Tri-integrable couplings

Let $M(A_1, A_2, A_3, A_4)$ be defined by (2.13). We take the corresponding enlarged spectral matrix as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{g}, \tag{2.18}$$

and the corresponding enlarged Lax matrix as

$$\bar{V} = \bar{V}(\bar{u}, \lambda) = M(V, V_1, V_2, V_3) \in \bar{g}, \tag{2.19}$$

where $\bar{u} = (u^T, u_1^T, u_2^T, u_3^T)^T$, λ is the spectral parameter and

$$U_i = U_i(u_i, \lambda),\ V_i = V_i(u, u_1, \cdots, u_i, \lambda),\ 1 \le i \le 3. \tag{2.20}$$

Theorem 2.1. *Let U and V be a Lax pair of a given integrable system* (1.1). *If two integrable couplings* (1.2) *and* (1.3) *of* (1.1) *have the zero curvature equations*

$$U_{i,t} - V_{i,x} + [U, V_i] + [U_i, V] = 0,\ i = 1, 2,$$

then the enlarged zero curvature equation, associated with the new enlarged Lax pair of $\bar{U}$ and $\bar{V}$ defined in (2.18) *and* (2.19),

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0, \tag{2.21}$$

is equivalent to the following triangle system

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{1,t} - V_{1,x} + [U, V_1] + [U_1, V] = 0, \\ U_{2,t} - V_{2,x} + [U, V_2] + [U_2, V] = 0, \\ U_{3,t} - V_{3,x} + [U, V_3] + \alpha\,[U_1, V_1] \\ \quad +\beta\,([U_1, V_2] + [U_2, V_2]) + \mu\,[U_2, V_3] + [U_3, V] = 0. \end{cases} \tag{2.22}$$

The first equation in (2.22) precisely engenders the given integrable system (1.1), and thus, the whole system (2.22) is a coupling system of (1.1). This is the basic idea of enlarging given integrable systems by using the presented matrix Lie algebra $\bar{g}$.

Following the traditional scheme for constructing soliton hierarchies [10, 17], we solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{2.23}$$

by taking

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2, W_3) \in \bar{g}, \tag{2.24}$$

where W is defined by (2.5), and

$$W_i = W_i(u, u_1, \cdots, u_i, \lambda) = \sum_{j \ge 0} W_{i,j} \lambda^{-j},\ 1 \le i \le 3. \tag{2.25}$$

Then we define the enlarged Lax matrices $\bar{V}^{[m]}$ as

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{g},\ m \ge 0, \tag{2.26}$$

with the submatrices $V^{[m]}$ being defined by (2.6) and

$$V_i^{[m]} = (\lambda^m W_i)_+ + \Delta_{m,i},\ 1 \le i \le 3,\ m \ge 0, \tag{2.27}$$

where P_+ denotes the polynomial part of P in λ again. An important step to construct a hierarchy of triangular integrable couplings is to choose the modification terms $\Delta_{m,i}$ such that the enlarged zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \ge 0,$$

yield a hierarchy of enlarged soliton equations

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}),\ m \ge 0. \tag{2.28}$$

This hierarchy provides tri-integrable couplings for the given hierarchy (2.7):

$$\bar{u}_{t_m} = \begin{bmatrix} u_{t_m} \\ u_{1,t_m} \\ u_{2,t_m} \\ u_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_2) \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix}, \quad m \ge 0. \tag{2.29}$$

Hamiltonian structures of those tri-integrable couplings can be constructed through using the associated variational identities [12, 13], which contain the trace identities as particular examples [10, 11].

3 Application to the KdV hierarchy

3.1 The KdV equations

Let us recall the KdV soliton hierarchy [13, 18]. The typical spectral problem for the KdV hierarchy is given by

$$\phi_x = U\phi,\ U = U(u,\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda - u & 0 \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{3.1}$$

The stationary zero curvature equation

$$W_x = [U, W] \tag{3.2}$$

gives rise to

$$\begin{cases} a_x = (-\lambda + u)b + c, \\ b_x = -2\,a, \\ c_x = -2\,(-\lambda + u)a. \end{cases} \tag{3.3}$$

If we assume that W is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0} W_{0,i}\lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{3.4}$$

the systems (3.3) equivalently yields

$$\begin{cases} b_{i+1} = \frac{1}{4}\,b_{i,xx} + ub_i - \frac{1}{2}\,\partial^{-1}u_x b_i, \\ c_i = -\frac{1}{2}\,b_{i,xx} + b_{i+1} - ub_i, \qquad\qquad i \geq 0. \\ a_i = -\frac{1}{2}\,b_{i,x}, \end{cases} \tag{3.5}$$

Setting the initial values

$$b_0 = 0,\ b_1 = 1, \tag{3.6}$$

and selecting the constants of integration as zero, the recursion relation (3.5) uniquely determines W in (3.4). Further, a direct computation tells

$$b_2 = \frac{1}{2}\,u,\ b_3 = \frac{1}{8}\,u_{xx} + \frac{3}{8}\,u^2,\ b_4 = \frac{1}{32}\,u_{xxxx} + \frac{5}{32}\,u_x^2 + \frac{5}{16}\,uu_{xx} + \frac{5}{16}\,u^3.$$

Now, the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \tag{3.7}$$

with

$$V^{[m]} = (\lambda^{m+1}W)_+ + \Delta_{m,0},\ \ \Delta_{m,0} = \begin{bmatrix} 0 & 0 \\ -b_{m+2} & 0 \end{bmatrix},\ \ m \geq 0, \tag{3.8}$$

generate the KdV hierarchy of soliton equations:

$$u_{t_m} = K_m = 2\,b_{m+2,x},\ m \geq 0, \tag{3.9}$$

which satisfies

$$K_m = \Phi K_{m-1},\ \ \Phi = \frac{1}{4}\partial^2 + u + \frac{1}{2}u_x\partial^{-1},\ \ m \geq 1. \tag{3.10}$$

Furthermore, the KdV hierarchy has a bi-Hamiltonian structure

$$u_{t_m} = J\frac{\delta\mathcal{H}_m}{\delta u} = M\frac{\delta\mathcal{H}_{m-1}}{\delta u},\ m \geq 1, \tag{3.11}$$

with the Hamiltonian pair defined by

$$J = \partial,\ M = \Phi J = \frac{1}{4}\,\partial^3 + u\partial + \frac{1}{2}\,u_x, \tag{3.12}$$

where $\partial = \frac{\partial}{\partial x}$, and Hamiltonian functionals, by

$$\mathcal{H}_m = \int \frac{4b_{m+3}}{2m+3}\,dx,\ m \geq 0. \tag{3.13}$$

The operator Φ defined in (3.10) is a hereditary recursion operator for the KdV hierarchy (3.9).

3.2 Tri-integrable couplings of the KdV equations

We use the specific non-semisimple Lie algebra

$$\bar{g} = g \Subset g_c \tag{3.14}$$

with

$$g = \{\, M(A_1, 0, 0, 0)|\ A_1 \in \mathrm{sl}(2, \mathbb{R}),\ \text{entries of } A_1 \text{ - Laurent series in } \lambda\}\,, \tag{3.15}$$

and

$$g_c = \{\, M(0, A_2, A_3, A_4)|\ A_i \in \mathrm{sl}(2, \mathbb{R}),\ \text{entries of } A_i \text{ - Laurent series in } \lambda,\ 2 \le i \le 4\}\,, \tag{3.16}$$

where $M(A_1, A_2, A_3)$ is defined by (2.13).

To construct tri-integrable couplings for the KdV equations, we introduce the corresponding enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{g}, \tag{3.17}$$

with $U = U(u, \lambda)$ be defined by (3.1) and

$$U_i = U_i(u_i) = \begin{bmatrix} 0 & 0 \\ -u_i & 0 \end{bmatrix},\ \ 1 \le i \le 3, \tag{3.18}$$

where $\bar{u} = (u, u_1, u_2, u_3)^T$, and u_1, u_2 and u_3 are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{3.19}$$

we search for a solution of the following form

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2, W_3) \in \bar{g}, \tag{3.20}$$

where W is given by (3.4). Assume that

$$W_1, W_2, W_3 \in \widetilde{\mathrm{sl}}(2, \mathbb{R}) = \{A \in \mathrm{sl}(2, \mathbb{R})|\, \text{entries of } A \text{ - Laurent series in } \lambda^{-1}\}$$

are of the form

$$W_1 = \begin{bmatrix} e & f \\ g & -e \end{bmatrix},\ W_2 = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix},\ W_3 = \begin{bmatrix} e'' & f'' \\ g'' & -e'' \end{bmatrix}. \tag{3.21}$$

Plugging (3.21) in the enlarged stationary zero curvature equation (3.19), we get

$$\begin{cases} e_x = u_1\, b + (u - \lambda)\, f + g, \\ f_x = -2\, e, \\ g_x = -2\, u_1\, a + 2\, (\lambda - u)\, e; \end{cases} \tag{3.22}$$

$$\begin{cases} e'_x = u_2\, b + (u - \lambda)\, f' + g', \\ f'_x = -2\, e', \\ g'_x = -2\, u_2 a + 2\, (\lambda - u)\, e'; \end{cases} \tag{3.23}$$

and

$$\begin{cases} e''_x \ =\ u_3 b + (\alpha\, u_1 + \beta\, u_2) f + (\beta\, u_1 + \mu\, u_2) f' + (u - \lambda) f'' + g'', \\ f''_x \ =\ -2\, e'', \\ g''_x \ =\ -2\, u_3 a - 2\, (\alpha\, u_1 + \beta\, u_2) e - 2\, (\beta\, u_1 + \mu\, u_2) e' + 2\, (\lambda - u) e''. \end{cases} \tag{3.24}$$

Trying a solution $\bar{W}$ with

$$\begin{cases} e = \sum_{i\ge 0} e_i \lambda^{-i},\ f = \sum_{i\ge 0} f_i \lambda^{-i},\ g = \sum_{i\ge 0} g_i \lambda^{-i}, \\ e' = \sum_{i\ge 0} e'_i \lambda^{-i},\ f' = \sum_{i\ge 0} f'_i \lambda^{-i},\ g' = \sum_{i\ge 0} g'_i \lambda^{-i}, \\ e'' = \sum_{i\ge 0} e''_i \lambda^{-i},\ f'' = \sum_{i\ge 0} f''_i \lambda^{-i},\ g'' = \sum_{i\ge 0} g''_i \lambda^{-i}, \end{cases} \tag{3.25}$$

we have

$$\begin{cases} f_{i+1} \ =\ u_1\, b_i - \frac{1}{2}\, \partial^{-1} u_{1,x} b_i + \frac{1}{4}\, f_{i,xx} + u\, f_i - \frac{1}{2}\, \partial^{-1} u_x f_i, \\ e_i \ \ =\ -\frac{1}{2}\, f_{i,x}, \\ g_i \ \ =\ -\frac{1}{2}\, f_{i,xx} - u_1\, b_i + f_{i+1} - u\, f_i; \end{cases} \tag{3.26}$$

$$\begin{cases} f'_{i+1} &= u_2\, b_i - \frac{1}{2}\,\partial^{-1}u_{2,x}\, b_i + \frac{1}{4}\, f'_{i,xx} + u\, f'_i - \frac{1}{2}\,\partial^{-1}u_x\, f'_i, \\ e'_i &= -\frac{1}{2}\, f'_{i,x}, \\ g'_i &= -\frac{1}{2}\, f'_{i,xx} - u_2\, b_i + f'_{i+1} - u\, f'_i; \end{cases} \tag{3.27}$$

and

$$\begin{cases} f''_{i+1} &= u_3\, b_i - \frac{1}{2}\,\partial^{-1}u_{3,x}\, b_i + (\alpha\, u_1 + \beta\, u_2)\, f_i - \frac{1}{2}\,\partial^{-1}(\alpha\, u_{1,x} + \beta\, u_{2,x})\, f_i \\ &\quad +(\beta\, u_1 + \mu\, u_2)\, f'_i - \frac{1}{2}\,\partial^{-1}(\beta\, u_{1,x} + \mu\, u_{2,x})\, f'_i + \frac{1}{4}\, f''_{i,xx} + u\, f''_i - \frac{1}{2}\,\partial^{-1}u_x\, f''_i, \\ e''_i &= -\frac{1}{2}\, f''_{i,x}, \\ g''_i &= -\frac{1}{2}\, f''_{i,xx} - u_3\, b_i - (\alpha\, u_1 + \beta\, u_2)\, f_i - (\beta\, u_1 + \mu\, u_2)\, f'_i + f''_{i+1} - u\, f''_i. \end{cases} \tag{3.28}$$

Further taking

$$f_0 = f'_0 = f''_0 = 0,\ f_1 = f'_1 = f''_1 = 1, \tag{3.29}$$

and selecting the constants of integration as zero, the recursion relations (3.26), (3.27) and (3.28) uniquely generate three sequences of $\{e_i, f_i, g_i\}_{i\geq 1}$, $\{e'_i, f'_i, g'_i\}_{i\geq 1}$ and $\{e''_i, f''_i, g''_i\}_{i\geq 1}$, respectively. We list the first three sets of selected functions as follows:

$$f_2 = \tfrac{1}{2}\, u_1,\ f'_2 = \tfrac{1}{2}\, u_2,\ f''_2 = \tfrac{1}{2}\, u_3;$$

$$\begin{cases} f_3 &= \frac{1}{8}\, u_{1,xx} + \frac{3}{4}\, uu_1, \\ f'_3 &= \frac{1}{8}\, u_{2,xx} + \frac{3}{4}\, uu_2, \\ f''_3 &= \frac{1}{8}\, u_{3,xx} + \frac{3}{4}\, uu_3 + \frac{3}{4}\beta\, u_1u_2 + \frac{3}{8}\,\alpha\, u_1^2 + \frac{3}{8}\,\mu\, u_2^2; \end{cases}$$

$$\begin{cases} f_4 &= \frac{5}{16}\, u_{xx}u_1 + \frac{15}{16}\, u^2u_1 + \frac{5}{16}\, u_{1,x}u_x + \frac{5}{16}\, u_{1,xx}u + \frac{1}{32}\, u_{1,xxxx}, \\ f'_4 &= \frac{5}{16}\, u_{xx}u_2 + \frac{15}{16}\, u^2u_2 + \frac{5}{16}\, u_{2,x}u_x + \frac{5}{16}\, u_{2,xx}u + \frac{1}{32}\, u_{2,xxxx}, \\ f''_4 &= \frac{15}{16}\, u^2u_3 + \frac{1}{32}\, u_{3,xxxx} + \frac{15}{16}\,(\alpha\, u_1^2 + \mu\, u_2^2 + 2\,\beta\, u_1u_2 + \frac{1}{3}\, u_{3,xx})u \\ &\quad + \frac{5}{16}\,(\alpha\, u_{1,xx} + \beta\, u_{2,xx})u_1 + \frac{5}{16}\,(\beta\, u_{1,xx} + \mu\, u_{2,xx})u_2 \\ &\quad + \frac{5}{16}\,(\beta\, u_{1,x}u_{2,x} + u_3u_{xx} + u_{3,x}u_x) + \frac{5}{32}\,(\alpha\, u_{1,x}^2 + \mu\, u_{2,x}^2). \end{cases}$$

Let us now introduce the enlarged Lax matrices

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{g},\ m \geq 0, \tag{3.30}$$

where $V^{[m]}$ is defined as in (3.8) and

$$V_i^{[m]} = (\lambda^{m+1}V_i)_+ + \Delta_{m,i},\ 1 \leq i \leq 3,\ m \geq 0, \tag{3.31}$$

and we choose

$$\Delta_{m,1} = \begin{bmatrix} 0 & 0 \\ -f_{m+2} & 0 \end{bmatrix},\quad \Delta_{m,2} = \begin{bmatrix} 0 & 0 \\ -f'_{m+2} & 0 \end{bmatrix},\quad \Delta_{m,3} = \begin{bmatrix} 0 & 0 \\ -f''_{m+2} & 0 \end{bmatrix}. \tag{3.32}$$

Then, all enlarged zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \geq 0, \tag{3.33}$$

determine a hierarchy of coupling systems for the KdV equations:

$$\bar{u}_{t_m} = \begin{bmatrix} u_{t_m} \\ u_{1,t_m} \\ u_{2,t_m} \\ u_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ K'_m(u)[u_1] \\ K'_m(u)[u_2] \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix} = \begin{bmatrix} 2\, b_{m+2,x} \\ 2\, f_{m+2,x} \\ 2\, f'_{m+2,x} \\ 2\, f''_{m+2,x} \end{bmatrix},\quad m \geq 0, \tag{3.34}$$

where $K'(u)[S]$ is the Gateaux derivative

$$K'(u)[S] = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} K(u + \varepsilon S).$$

Obviously, the tri-integrable couplings of the KdV equation and the fifth-order KdV equation read

$$\begin{cases} u_{t_1} &= \frac{3}{2}\, uu_x + \frac{1}{4}\, u_{xxx}, \\ u_{1,t_1} &= \frac{3}{2}\, uu_{1,x} + \frac{3}{2}\, u_xu_1 + \frac{1}{4}\, u_{1,xxx}, \\ u_{2,t_1} &= \frac{3}{2}\, uu_{2,x} + \frac{3}{2}\, u_xu_2 + \frac{1}{4}\, u_{2,xxx}, \\ u_{3,t_1} &= \frac{3}{2}\,[u_xu_3 + uu_{3,x} + (\alpha\, u_{1,x} + \beta\, u_{2,x})u_1 + (\beta\, u_{1,x} + \mu\, u_{2,x})u_2] + \frac{1}{4}\, u_{3,xxx}; \end{cases}$$

$$\begin{cases} u_{t_2} &= \frac{15}{8}\, u^2 u_x + \frac{5}{8}\, u u_{xxx} + \frac{1}{16}\, u_{xxxxx} + \frac{5}{4}\, u_x u_{xx}, \\ u_{1,t_2} &= \frac{15}{8}\, u_{1,x} u^2 + \frac{1}{16}\, u_{1,xxxxx} + (\frac{15}{4}\, u_x u_1 + \frac{5}{8}\, u_{1,xxx})\, u \\ &\quad + \frac{5}{8}\, u_{xxx} u_1 + \frac{5}{4}\, (u_{1,xx} u_x + u_{xx} u_{1,x}), \\ u_{2,t_2} &= \frac{15}{8}\, u_{2,x} u^2 + \frac{1}{16}\, u_{2,xxxxx} + (\frac{15}{4}\, u_x u_2 + \frac{5}{8}\, u_{2,xxx})\, u \\ &\quad + \frac{5}{8}\, u_{xxx} u_2 + \frac{5}{4}\, (u_{2,xx} u_x + u_{xx} u_{2,x}), \\ u_{3,t_2} &= \frac{15}{8}\, u_{3,x} u^2 + [\frac{15}{4}\, (\alpha\, u_{1,x} + \beta\, u_{2,x}) u_1 + \frac{15}{4}\, (\beta\, u_{1,x} + \mu\, u_{2,x}) u_2 + \frac{15}{4}\, u_x u_3 + \frac{5}{8}\, u_{3,xxx}] u \\ &\quad + \frac{15}{8}\, \alpha\, u_x u_1^2 + (\frac{15}{4}\, \beta\, u_x u_2 + \frac{5}{8}\, \alpha\, u_{1,xxx} + \frac{5}{8}\, \beta\, u_{2,xxx}) u_1 + \frac{15}{8}\, \mu\, u_x u_2^2 \\ &\quad + \frac{5}{8}\, (\beta\, u_{1,xxx} + \mu\, u_{2,xxx}) u_2 + \frac{5}{8}\, u_{xxx} u_3 + \frac{5}{8}\, \mu\, u_{2,x} u_{2,xx} + \frac{5}{8}\, \alpha\, u_{1,x} u_{1,xx} \\ &\quad + \frac{5}{4}\, u_x u_{3,xx} + \frac{5}{4}\, u_{3,x} u_{xx} + \frac{5}{8}\, (\alpha\, u_{1,xx} + \beta\, u_{2,xx}) u_{1,x} \\ &\quad + \frac{5}{8}\, (\beta\, u_{1,xx} + \mu\, u_{2,xx}) u_{2,x} + \frac{1}{16}\, u_{3,xxxxx} + \frac{5}{8}\, \beta\, (u_{2,x} u_{1,xx} + u_{1,x} u_{2,xx}); \end{cases}$$

respectively.

3.3 Bi-Hamiltonian structure

A bi-Hamiltonian structure of the presented tri-integrable couplings in (3.34) can be generated by applying the variational identity [12, 13, 19]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \frac{\partial \bar{U}}{\partial \lambda}, \bar{W} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \frac{\partial \bar{U}}{\partial \bar{u}}, \bar{W} \rangle, \tag{3.35}$$

where $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric, and ad-invariant bilinear form on the non-semisimple Lie algebra $\bar{g}$.

Since the trace identity [10, 11] does not work with non-semisimple Lie algebras, we need to construct a specific non-degenerate bilinear form on $\bar{g}$ with the symmetric and ad-invariant properties.

To do this, we first transform the semi-direct sum $\bar{g}$ into a vector form by defining mapping

$$\sigma : \bar{g} \to \mathbb{R}^{12},\ A \mapsto (a_1, \cdots, a_{12})^T, \tag{3.36}$$

where

$$A = M(A_1, A_2, A_3, A_4) \in \bar{g},\ A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ 1 \le i \le 4. \tag{3.37}$$

This mapping σ induces a Lie algebra structure on $\mathbb{R}^{12}$, with which $\mathbb{R}^{12}$ is isomorphic to the matrix Lie algebra $\bar{g}$. The corresponding Lie bracket $[\cdot, \cdot]$ on $\mathbb{R}^{12}$ can be computed as follows

$$[a, b]^T = a^T R(b),\ a = (a_1, \cdots, a_{12})^T,\ b = (b_1, \cdots, b_{12})^T \in \mathbb{R}^{12}, \tag{3.38}$$

where

$$R(b) = M(R_1, R_2, R_2, R_3),\ R_i = \begin{bmatrix} 0 & 2\, b_{3i-1} & -2\, b_{3i} \\ b_{3i} & -2\, b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2\, b_{3i-2} \end{bmatrix},\ 1 \le i \le 4. \tag{3.39}$$

This Lie algebra $(\mathbb{R}^{12}, [\cdot, \cdot])$ is isomorphic to the matrix Lie algebra $\bar{g}$, and the mapping σ, defined by (3.36), is a Lie algebra isomorphism between the two Lie algebras.

Instead of defining a bilinear form on $\bar{g}$, we define a bilinear form on $\mathbb{R}^{12}$ by setting

$$\langle a, b \rangle = a^T F b, \tag{3.40}$$

where F is a constant matrix. It follows from the symmetric property $\langle a, b \rangle = \langle b, a \rangle$ that

$$F^T = F. \tag{3.41}$$

Together with this symmetric condition, the ad-invariant property $\langle a, [b, c] \rangle = \langle [a, b], c \rangle$ requires that

$$F (R(b))^T = -R(b) F,\ b \in \mathbb{R}^{12}. \tag{3.42}$$

This matrix equation with an arbitrary b yields to a linear system of equations for the entries of the matrix F to satisfy. By solving the resulting system, we obtain

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\, \eta_4 & \beta\, \eta_4 & 0 \\ \eta_3 & \beta\, \eta_4 & \mu\, \eta_4 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.43}$$

where η_i, $1 \le i \le 4$, are arbitrary constants, and $C \otimes D$ is the Kronecker product:

$$C \otimes D = \begin{bmatrix} c_{1,1}D & c_{1,2}D & \cdots & c_{1,p}D \\ c_{2,1}D & c_{2,2}D & \cdots & c_{2,p}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1}D & c_{n,2}D & \cdots & c_{n,p}D \end{bmatrix},\ C = (c_{ij}).$$

Since the mapping σ, defined by (3.36), is a Lie algebra isomorphism between $\bar{g}$ and $\mathbb{R}^{12}$, the corresponding bilinear form on the non-semisimple Lie algebra $\bar{g}$ can be worked out as follows:

$$\begin{aligned} \langle A, B\rangle_{\bar{g}} &= \langle \sigma(A), \sigma(B)\rangle_{\mathbb{R}^{12}} = (a_1, \cdots, a_{12})F(b_1, \cdots, b_{12})^T \\ &= (2\,a_1b_1 + a_2b_3 + a_3b_2)\eta_1 + (2\,a_1b_4 + a_2b_6 + a_3b_5 + 2\,a_4b_1 \\ &\quad + a_5b_3 + a_6b_2)\eta_2 + (2\,a_1b_7 + a_2b_9 + a_3b_8 + 2\,a_7b_1 + a_8b_3 + a_9b_2)\eta_3 \\ &\quad + (2\,a_1b_{10} + a_2b_{12} + a_3b_{11} + 2\,\alpha\,a_4b_4 + 2\,\beta\,a_4b_7 + \alpha\,a_5b_6 + \beta\,a_5b_9 \\ &\quad + \alpha\,a_6b_5 + \beta\,a_6b_8 + 2\,\beta\,a_7b_4 + 2\,\mu\,a_7b_7 + \beta\,a_8b_6 + \mu\,a_8b_9 \\ &\quad + \beta\,a_9b_5 + \mu\,a_9b_8 + 2\,a_{10}b_1 + a_{11}b_3 + a_{12}b_2)\eta_4, \end{aligned} \tag{3.44}$$

where

$$A = \sigma^{-1}((a_1, \cdots, a_{12})^T) \in \bar{g},\ B = \sigma^{-1}((b_1, \cdots, b_{12})^T) \in \bar{g}.$$

Again, because of the isomorphism σ, the bilinear form (3.44) is also symmetric and ad-invariant:

$$\langle A, B\rangle_{\bar{g}} = \langle B, A\rangle_{\bar{g}},\ \langle A, [B, C]\rangle_{\bar{g}} = \langle [A, B], C\rangle_{\bar{g}},\ A, B, C \in \bar{g}.$$

It should be noted that this kind of bilinear forms is not of Killing type, since the matrix Lie algebra $\bar{g}$ is non-semisimple.

It is easy to see that the bilinear form, defined by (3.44), is non-degenerate if and only if the determinant of matrix F is non-zero, i.e.,

$$\det(F) = -16\,\eta_4^{12}(\alpha\,\mu - \beta^2)^3 \ne 0. \tag{3.45}$$

Therefore, we can choose suitable constants α, β, μ and η_4 such that $\det(F)$ is non-zero to get non-degenerate bilinear forms over $\bar{g}$.

It is now direct to compute that

$$\langle \bar{W}, \frac{\partial \bar{U}}{\partial \lambda}\rangle_{\bar{g}} = \eta_1 b + \eta_2 f + \eta_3 f' + \eta_4 f'',$$

and

$$\langle \bar{W}, \frac{\partial \bar{U}}{\partial \bar{u}}\rangle_{\bar{g}} = \begin{bmatrix} -\eta_1 b - \eta_2 f - \eta_3 f' - \eta_4 f'' \\ -\eta_2 b - \eta_4 \alpha\, f - \eta_4 \beta\, f' \\ -\eta_3 b - \eta_4 \beta\, f - \eta_4 \mu\, f' \\ -\eta_4 b \end{bmatrix}.$$

To calculate the parameter γ in the variational identity (3.35), we use the formula [12]:

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \bar{W}, \bar{W}\rangle|,$$

and find $\gamma = \frac{1}{2}$. Consequently, applying the corresponding variational identity, we obtain a Hamiltonian structure for the hierarchy of tri-integrable couplings (3.34):

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}},\ m \ge 0, \tag{3.46}$$

with the Hamiltonian operator

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\,\eta_4 & \beta\,\eta_4 & 0 \\ \eta_3 & \beta\,\eta_4 & \mu\,\eta_4 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix}^{-1} \otimes \partial, \tag{3.47}$$

and the Hamiltonian functionals

$$\mathcal{H}_m = \int \frac{4(\eta_1 b_{m+3} + \eta_2 f_{m+3} + \eta_3 f'_{m+3} + \eta_4 f''_{m+3})}{2m+3}dx,\ m \ge 0. \tag{3.48}$$

Checking the recursion relation

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1},\ m \geq 1, \tag{3.49}$$

shows that the recursion operator $\bar{\Phi}$ reads

$$\bar{\Phi} = \bar{\Phi}(\bar{u}) = M^T(\Phi, \Phi_1, \Phi_2, \Phi_3), \tag{3.50}$$

where M^T denotes the transpose of the matrix M defined in (2.13), and Φ is given by (3.10) and

$$\Phi_i = u_i + \frac{1}{2}\, u_{i,x}\partial^{-1},\ 1 \leq i \leq 3. \tag{3.51}$$

It can be directly verified that $\bar{\Phi}$ is hereditary [20], and $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ constitute a Hamiltonian pair [21]. Therefore, the resulting hierarchy of tri-integrable couplings (3.34) has a bi-Hamiltonian structure

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}} = \bar{M}\frac{\delta \bar{\mathcal{H}}_{m-1}}{\delta \bar{u}},\ m \geq 1. \tag{3.52}$$

This bi-Hamiltonian structure implies [20, 21] that

$$[\bar{K}_m, \bar{K}_n] := \bar{K}_m'(\bar{u})[\bar{K}_n] - \bar{K}_n'(\bar{u})[\bar{K}_m] = 0,\ m, n \geq 0,$$

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} := \int \left(\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}\right)^T \bar{J}\frac{\delta \bar{\mathcal{H}}_n}{\delta \bar{u}}\, dx = 0,\ m, n \geq 0.$$

That is, the hierarchy (3.34) possesses a common commuting symmetries $\{\bar{K}_n\}_{n=0}^{\infty}$ and a common commuting conserved functional $\{\bar{\mathcal{H}}_n\}_{n=0}^{\infty}$. It follows that every tri-integrable coupling in (3.34) is Liouville integrable.

4 Conclusion and remarks

We have successfully constructed a kind of tri-integrable couplings for the KdV hierarchy through a specific non-semisimple Lie algebra consisting of 4×4 block matrices. All resulting tri-integrable couplings are bi-Hamiltonian and Liouville integrable. The result also provides an affirmative answer to the question in the introduction: Can one put two integrable couplings together to form an even larger integrable coupling?

Due to rich structures of block matrices, non-semisimple matrix loop algebras can yield various integrable couplings with multiple components (see, e.g., [15, 22, 23, 24]), including bi-integrable couplings and tri-integrable couplings. Actually, once a generating scheme associated with a non-semisimple Lie algebra is established, it can be applied to different soliton hierarchies to engender integrable couplings.

We would like to emphasize that we are at the beginning of understanding multi-integrable couplings based on non-semisimple matrix Lie algebras. There are lots of questions on theories of integrable couplings. Can one have any criterion on non-semisimple Lie algebras to guarantee existence of Hamiltonian structures of associated integrable couplings? Note that the first order perturbation has been successfully used [25, 26, 27] to generate Hamiltonian integrable couplings, even local bi-Hamiltonian integrable systems in $(2+1)$-dimensions [26]. We conjecture that the second order perturbations may be good candidates for enlarging Lax matrices to get more diverse Hamiltonian integrable couplings. Another basic question is what kind of solution groups can be generated for integrable couplings by symmetry constraints as did for the perturbation systems [28, 29]? It is also an interesting question to develop bilinear theories for dealing with integrable couplings.

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