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Application of a new hybrid method for solving singular fractional Lane–Emden-type equations in astrophysics

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In this research, a hybrid numerical method combining cosine and sine (CAS) wavelets and Green's function approach is created to acquire the arrangements of fractional Lane–Emden Problem. The suggested methodology for detecting the solution of nonlinear equations dependent on variations of germinal algorithms is applied on nonlinear fractional Lane–Emden Problem under some smooth conditions and results in an iterative scheme of nonlinear equations Because of its efficiency, this technique is applied on a large variety of equations from the boundary value problems to the optimization. This paper is extending the suggested methodology technique for fractional Lane–Emden Problem. Moreover, the feature of the present novel method is utilized to convert the problem under observance into a system of algebraic equations that can be illuminated by suitable algorithms. A rapprochement of results has likewise been obtained using the

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present strategy and those reported using other techniques seem to indicate the precision and computational efficiency to establish the suitability of the Green-CAS wavelet method.

Keywords: New hybrid method; Lane–Emden type equations; nonlinear ODE; collocation method; Newton's iterative techniques.

1. Introduction

The Lane–Emden Problem (LEP) is a common differential equation in scientific material science. In astronomy, LEP is a dimensionless type of Poisson's equation for the gravitational capability of basic models of a star.[1] Because of its peculiari conduct at the beginning, it is numerically tested to take care of the Lane–Emden issue, similar to the distinct, direct and nonlinear introductory esteemed issues in quantum mechanics and astronomy. This paper manages the numerical solution for the singular fractional LEP utilizing Green's function-CAS wavelet method. A specified function can be spoken to insofar as several fundamental functions are limited both in area and scale utilizing wavelets. Favorable circumstances of wavelet examination incorporate the limit of completing nearby investigation to reveal flag highlights, for example, patterns, limits, and discontinuities which distinct examination strategies may neglect to recognize.[2] The multiresolution examination of wavelet change makes it a useful, accurate asset for an assortment of capacities and administrators. Wavelet investigation results in inadequate frameworks which thus empower quick calculation. Such beneficial parts of wavelet investigation make it an incredible logical procedure to manage differential conditions in which the arrangement may progress toward becoming misshaped because of singularities or sharp changes.

There has been much research on wavelet-based technique for the integer order LEP lately. Disbanding the LEP utilizing Chebyshev wavelet operational matrix was contemplated in Ref. 3. Reference 4 got the arrangements for the popularize LEP utilizing Haar wavelet estimation. Moreover, Ref. 5, utilized ultraspherical wavelets for unraveling such equations. Kazemi Nasab *et al.*[6] proposed Chebyshev wavelet finite difference technique for unraveling nonlinear Lane–Emden conditions emerging in astronomy. Laguerre wavelets was utilized in Ref. 7. Sacrificial arrangement strategies for fractional differential equations (FDE) developed over the last decade include HAM,[8] VIM,[9] FDM for FPDE,[10] ADM,[11] FDTM[12] and BPOM.[13] Bhrawy *et al.*[14] employed HAM to find nominal sacrificial arrangements for FDE. For fractional LEP, the LSM was applied in Ref. 6. In Ref. 15, distributing arrangements of morse index of the fractional LEP are given. Laplace transform with Chebyshev collocation method to solve FPDE is given in Ref. 16. Latterly, MDTM was suggested in Ref. 2 to acquire the numerical arrangements of the singular fractional LEP. Akgul *et al.*[17] suggested the kernel method to investigate the singular fractional LEP. The operational matrices of fractional order integration for the Basis function were investigated in Ref. 18 to disband the differential equations of fractional order.

In this paper, we deem the singular fractional LEP shaped by

$$D^{\alpha}\nu(x)+\frac{L}{x^{\alpha-\beta}}D^{\beta}\nu(x)+g(x,\nu(x))=h(x), \tag{1.1}$$

submissive to the attaching types of initial or boundary conditions:

$$\mathrm{I}:\ \nu(0)=A_1,\quad \nu(1)=B_1, \tag{1.2}$$

$$\mathrm{II}:\ \nu(0)=A_2,\quad \nu'(1)=B_2. \tag{1.3}$$

The organization of this paper is as follows. In Sec. 2, some basic definitions of fractional calculus are given. In Sec. 3, we present the Orthonormal CAS wavelets scheme. The procedure of representation of Green-CAS method to discuss the numerical arrangements to the FLE equation with boundary conditions is demonstrated in Sec. 4 and finally in Sec. 5, the numerical experiments are presented that illustrate the efficiency of the proposed method.

2. Definitions and Bases

There exist more definitions for fraction derivatives — the more pulper definition is the Riemann–Liouville (RL) approach. The (RL) approach to fractional calculus, the concept of fractional integral of order $\alpha(\alpha>0)$, is an indigenous effect of Cauchy's formula for regenerate integrals, $f(t)$ to an unpretentious convolution. This integral can be considered like[19]

$$J^{n}f(t)=\frac{1}{(n-1)!}\int_0^t(t-\tau)^{n-1}f(\tau)d\tau,\quad t>0,\quad n\in Z^{+}, \tag{2.1}$$

where n is positive integer number and inserting the positive real number α, the Riemann–Liouville fractional-order integral is calculated as

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\quad t>0,\quad \alpha\in R^{+}, \tag{2.2}$$

the inverse of our factor D^{α} is

$$J^{\alpha}t^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}t^{\beta+\alpha},\quad \gamma>-1,\quad t>0. \tag{2.3}$$

Figure 1 shows the effect of the variety of the values for β and α of (2.3). By using (2.3) and Taylor series, the fractional-order integral (J^{α}) for $\cos(t)$ and $\sin(t)$ is presented by

$$J^{\alpha}\cos(t)=\sum_{\beta=0}^{\infty}(-1)^{\gamma}\frac{t^{2\gamma+\alpha}}{\Gamma(2\beta+\alpha+1)}\quad J^{\alpha}\sin(t)=\sum_{\beta=0}^{\infty}(-1)^{\beta}\frac{t^{2\beta+1+\alpha}}{\Gamma(2\beta+\alpha+2)}. \tag{2.4}$$

Figure 2 shows the effect of fractional-order integral (J^{α}) on $\cos(t)$ for distinct estimates of fractional order α.

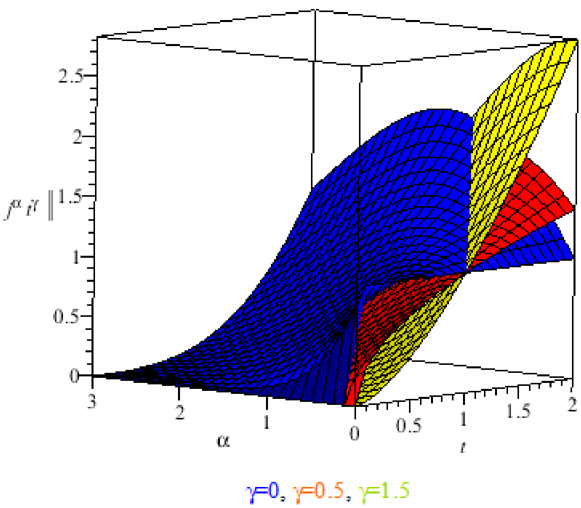

Fig. 1. (Color online) Distinguish assessments of γ for $J^{\alpha}t^{\gamma}$.

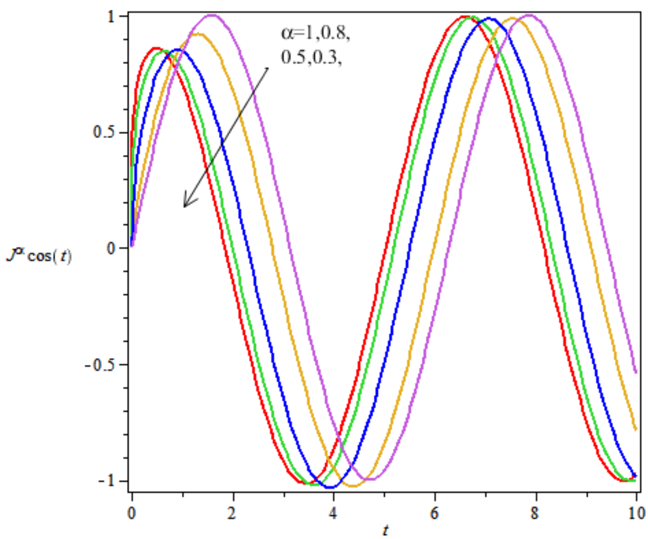

Fig. 2. (Color online) Distinguish assessments of α for $J^{\alpha}\cos t$.

Definition 2.1. Using (2.1) and (2.2), we present some preliminary definitions for Riemann–Liouville (RL) fractional derivative which can be considered as follows[19–21]:

$$D_t^{\alpha} f = \begin{cases} \dfrac{d^n f(t)}{dt^n} & \text{if } n = \alpha, \quad n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} I^{n-\alpha} f(t) & \text{if } 0 \leq n-1 < \alpha < n, \quad n \in \mathbb{N}. \end{cases} \tag{2.5}$$

If it exists, D_t^α is the total derivative of integer order $\alpha, (\alpha > 0)$ and $I^n f(t)$ is the RL fractional integral of order n which is defined as

$$I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad n > 0, \tag{2.6}$$

where $\Gamma(n-\alpha)$ is the Gamma function.

Definition 2.2. The RL fractional partial derivative is given by

$$\partial_t^\alpha = \begin{cases} \dfrac{\partial^n f}{\partial t^n}, & n = \alpha, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} f(s,x) ds, & 0 \le n-1 < \alpha < n, \end{cases} \tag{2.7}$$

where ∂_t^α is the partial derivative of integer order n.

The RL fractional derivative (D^α) for basic assignments is as follows:

$$D^\alpha \cos(t) = \sum_{\beta=0}^{\infty} (-1)^\beta \frac{t^{2\beta-\alpha}}{\Gamma(2\beta-\alpha+1)}, \tag{2.8}$$

$$D^\alpha \sin(t) = \sum_{\beta=0}^{\infty} (-1)^\beta \frac{t^{2\beta+1-\alpha}}{\Gamma(2\beta-\alpha+2)}. \tag{2.9}$$

Figure 3 shows the effect of RL fractional derivative (D^α) on $\sin(t)$ with the fraction order for different values for α.

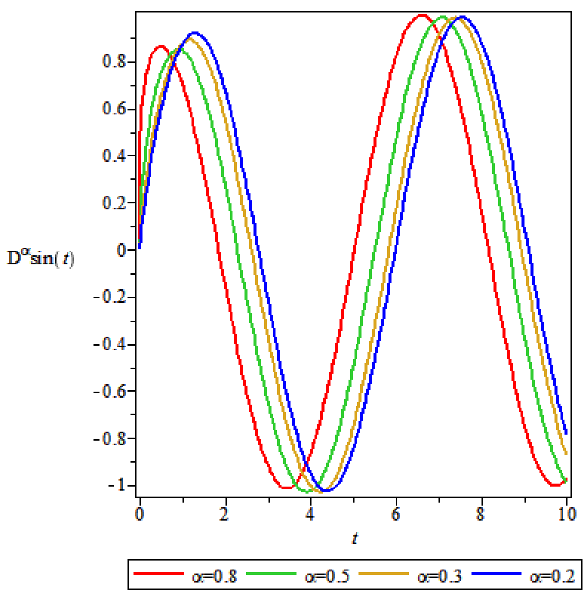

Fig. 3. (Color online) The RL fractional derivative $D^\alpha \cos t$ for different values of α.

3. CAS Wavelet

Wavelet permission is a new area in applied mathematics. Wavelets are individual sort of family of functions structured from specie transformations, namely translation and dilation of a single function called the mother wavelet, which can be adequate for the CAS wavelet form:

$$\Psi_{a,b}(t) = |a|^{\frac{-1}{2}}\Psi\left(\frac{t-b}{a}\right), \quad a, b \in R, \quad a \neq 0\,. \tag{3.1}$$

If the parameter $|a| < 1$, the wavelet (3.1) corresponds to higher frequencies having smaller backup in time domain and becomes a compressed form of mother wavelet. On the contrary, when $|a| > 1$, the wavelet has larger backup in time domain and corresponds to lower frequencies. Discretizing the parameters via $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 1$

$$\Psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\Psi(a_0^k t - nb_0)\,, \tag{3.2}$$

where $\Psi_{k,n}(t)$ yields a subsistence for $L^2(R)$. If $a_0 = 2$ and $b_0 = 1$, then these wavelets for all integers k and n yield an orthonormal basis. The orthonormal CAS wavelets are illustrated on the range $[0, 1]$ through

$$\Psi_{k,n}(x) = \begin{cases} 2^{\frac{k}{2}}\mathrm{CAS}_m(2^k x - n + 1), & x \in \left[\dfrac{n-1}{2^k}, \dfrac{n}{2^k}\right], \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

where $\mathrm{CAS}_m(x) = \cos(2m\varphi x) + \sin(2m\varphi x)$ and $n = 0, 1, 2, 3, \ldots, 2k - 1$, is the translation parameter. The non-negative integer k is the level of resolution and m is any integer.

$$\nu(x) = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} c_{nm}\psi_{nm}(x) = A^T\Psi(x)\,, \tag{3.4}$$

where A and Ψ are two vectors given as

$$A = [a_{0,-M},\ a_{0,-M+1}, \ldots,\ a_{0,M},\ a_{1,-M},\ a_{1,-M+1}, \ldots,\ a_{1,M}, \ldots,\ a_{2^k-1,-M},\ a_{2^k-1,-M+1}, \ldots, a_{2^k-1,M}]^T, \tag{3.5}$$

$$\Psi(x) = [\psi_{0,-M},\ \psi_{0,-M+1}, \ldots,\ \psi_{0,M},\ \psi_{1,-M},\ \psi_{1,-M+1}, \ldots,\ \psi_{1,M}, \ldots,\ \psi_{2^k-1,-M},\ \psi_{2^k-1,-M+1}, \ldots,\ \psi_{2^k-1,M}]^T\,. \tag{3.6}$$

4. Numerical Steps to FLEP

The fractional Green's function is exploited in Ref. 23 design to FDE consisting of derivatives of order $k\alpha$ only, where $k \in z$. We propose a new method for disbanding nonlinear ordinary fractional boundary value problems numerically, called the Green-CAS technique. In general, this method does not require the use of operational matrix $P^{\alpha}_{m,n}$ and the operational matrix related to boundary value problems for FLEP with Dirichlet boundary conditions and for mixed boundary conditions. However, for some cases, Green-CAS is utilized along with operational matrices. The investigation shows that the method is even more efficacious against some of the relevant numerical methods discussed in previous studies. Interestingly, accuracy is not compromised, rather enhanced by using the Green-CAS method for disbanding fractional boundary value problems.

In this part, we described the procedure of implementation of the proposed method to converge the numerical arrangements to the FLE equations with boundary conditions. But, in some FDEs, we have utilized the proposed method along with the CAS wavelet operational matrix.

Status 1: Using Eq. (1.1) with the boundary condition (1.2), we have

$$D^{\alpha}\nu(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}\Psi_{nm}(x)\,. \tag{4.1}$$

Stratifying the integral operator on the two parts of (4.1), we have

$$\nu(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}(I^{\alpha}\Psi_{nm}(x)) + xS_1 + S_2\,. \tag{4.2}$$

Stratifying the boundary conditions (1.2), we have

$$S_2 = A_1, \quad S_1 = B_1 - A_1 - \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}(I^{\alpha}\Psi_{nm}(1))\,. \tag{4.3}$$

So,

$$\begin{aligned}\nu(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} & c_{nm} \\ & \times \left(\begin{cases} \displaystyle\int_0^1 \frac{1}{\Gamma(\alpha)}((x-\zeta)^{\alpha-1} - x(1-\zeta)^{\alpha-1})\Psi_{nm}(\zeta)d\zeta, & 0 \le \zeta \le x \\ \displaystyle\int_0^1 -\frac{x}{\Gamma(\alpha)}((1-\zeta)^{\alpha-1})\Psi_{nm}(\zeta)d\zeta, & x \le \zeta \le 1 \end{cases}\right) \\ & + xB_1 - xA_1 + A_1\,. \end{aligned} \tag{4.4}$$

And, the fractional derivative of $\nu(x)$ with the mixed boundary conditions can be composed as

$$D^{\beta}\nu(x) = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} c_{nm} \times \left(\begin{cases}\left(\int_0^1 \frac{1}{\Gamma(\alpha-\beta)}\left((x-\zeta)^{\alpha-\beta-1} - \frac{x^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}(1-c)^{\alpha-1}\right)\right)\Psi_{nm}(\zeta)d\zeta, & 0 \le \zeta \le x \\ \left(\int_0^1 -\frac{x^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}(1-\zeta)^{\alpha-1}\right)\Psi_{nm}(\zeta)d\zeta, & x \le \zeta \le 1\end{cases}\right) + \frac{x^{1-\beta}}{\Gamma(2-\beta)}B_1 - \frac{x^{1-\beta}}{\Gamma(2-\beta)}A_1. \tag{4.5}$$

Posture (4.1)–(4.5) in (1.1) result,

$$\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} c_{nm}\Psi_{nm}(x) + \frac{L}{\frac{(2i-1)^{\alpha-\beta}}{2m}} \times \left(\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} c_{nm}\left(\begin{cases}\left(\int_0^1 \frac{1}{\Gamma(\alpha-\beta)}\left((x-\zeta)^{\alpha-\beta-1} - \frac{x^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}(1-\zeta)^{\alpha-1}\right)\right. \\ \quad \Psi_{nm}(\zeta)d\zeta\Big), \quad 0 \le \zeta \le x \\ \left(\int_0^1 -\frac{x^{1-\beta}}{\Gamma(\alpha)\Gamma(2-\beta)}(1-\zeta)^{\alpha-1}\Psi_{nm}(\zeta)d\zeta\right), \quad x \le \zeta \le 1\end{cases}\right)\right.$$
$$\left. + \frac{x^{1-\beta}}{\Gamma(2-\beta)}B_1 - \frac{x^{1-\beta}}{\Gamma(2-\beta)}A_1\right) + g\left(x, \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} c_{nm} \times \left(\begin{cases}\int_0^1 \frac{1}{\Gamma(\alpha)}((x-\zeta)^{\alpha-1} - x(1-\zeta)^{\alpha-1})\Psi_{nm}(\zeta)d\zeta, & 0 \le \zeta \le x \\ \int_0^1 -\frac{x}{\Gamma(\alpha)}((1-\zeta)^{\alpha-1})\Psi_{nm}(\zeta)d\zeta, & x \le \zeta \le 1\end{cases}\right)\right)$$
$$+ xB_2 - xA_2 + A_2 = \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} h_{nm}\Psi_{nm}(x). \tag{4.6}$$

Disband (4.6) at the collocation point $x_i = \frac{(2i-1)}{2m}$ where $i = 1, 2, 3, \ldots, m$. Then we can solve the system of nonlinear equations to obtain the unknown constant coefficients.

Status 2: In fact, we can use Eq. (1.1) with mixed boundary condition (1.3), we have

$$S_2 = A_2, \quad S_1 = B_2 - \sum_{n=0}^{2^k-1}\sum_{m=-M}^{M} c_{nm}(I^{\alpha-1}\Psi_{nm}(1)).$$

Then,

$$\nu(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}$$
$$\times \left(\begin{cases} \int_0^1 \frac{1}{\Gamma(\alpha)} \left((x-\zeta)^{\alpha-1} - \frac{x}{\Gamma(\alpha-1)}(1-\zeta)^{\alpha-2} \right) \Psi_{nm}(\zeta) d\zeta, & 0 \le \zeta \le x \\ \left(\int_0^1 -\frac{x}{\Gamma(\alpha-1)}(1-\zeta)^{\alpha-2} d\zeta \right), & x \le \zeta \le 1 \end{cases} \right)$$
$$+ xB_2 + A_2 \,. \tag{4.7}$$

And, the fractional derivative of $\nu(x)$ with the mixed boundary conditions can be composed as

$$D^\beta \nu(x) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}$$
$$\times \left(\begin{cases} \left(\int_0^1 \frac{1}{\Gamma(\alpha-\beta)} \left((x-\zeta)^{\alpha-\beta-1} - \frac{x^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}(1-\zeta)^{\alpha-2} \right) \right) \Psi_{nm}(\zeta) d\zeta \,, \\ \qquad 0 \le \zeta \le x \,, \\ \left(\int_0^1 -\frac{x^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}(1-\zeta)^{\alpha-2} \right) \Psi_{nm}(\zeta)\, d\zeta, \quad x \le \zeta \le 1 \end{cases} \right)$$
$$+ \frac{x^{1-\beta}}{\Gamma(2-\beta)} B_2 \,. \tag{4.8}$$

Posture (4.7)–(4.8) in (1.1) result,

$$\sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm} \Psi_{nm}(x) + \frac{L}{\frac{(2i-1)^{\alpha-\beta}}{2m}}$$
$$\times \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}$$
$$\times \left(\begin{cases} \left(\int_0^1 \frac{1}{\Gamma(\alpha-\beta)} \left((x-\zeta)^{\alpha-\beta-1} - \frac{x^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}(1-\zeta)^{\alpha-2} \right) \right) \Psi_{nm}(\zeta)\, d\zeta \,, \\ \qquad 0 \le \zeta \le x \,, \\ \left(\int_0^1 -\frac{x^{1-\beta}}{\Gamma(\alpha-1)\Gamma(2-\beta)}(1-\zeta)^{\alpha-2} \right) \Psi_{nm}(\zeta)\, d\zeta, \quad x \le \zeta \le 1 \end{cases} \right)$$
$$+ \frac{x^{1-\beta}}{\Gamma(2-\beta)} B_2 + g(x \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{nm}$$

$$\times\left(\begin{cases}\displaystyle\int_0^1 \frac{1}{\Gamma(\alpha)}\left((x-\zeta)^{\alpha-1}-\frac{x}{\Gamma(\alpha-1)}(1-\zeta)^{\alpha-2}\right)\Psi_{nm}(\zeta)\,d\zeta, & 0\le\zeta\le x\\ \displaystyle\int_0^1 -\frac{x}{\Gamma(\alpha-1)}((1-\zeta)^{\alpha-2})d\zeta, & x\le\zeta\le 1\end{cases}\right)$$

$$+xB_2+A_2)=\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}h_{nm}\Psi_{nm}(x)\,. \tag{4.9}$$

Disband (4.9) at the collocation point $x_i=\frac{(2i-1)}{2m}$, where $i=1,2,3,\ldots,m$. Then we can solve the system of nonlinear equations to obtain the unknown constant coefficients.

Next, we arrange Eqs. (4.6) and (4.9) that yield $\hat{m}$ nonlinear equations which can be illuminated for the obscure vector C by Newton's iterative procedure. Due to their significant importance, numerous numerical and analytical strategies have been created for these issues hence it is not sensible to deduce its exact solution by an algebraic operation, for instance, iterative numerical solvers dependent on Newton's method. It is notable that the underlying estimates for Newton's iterative system are imperative. A strategy can be utilized for picking the underlying estimates.

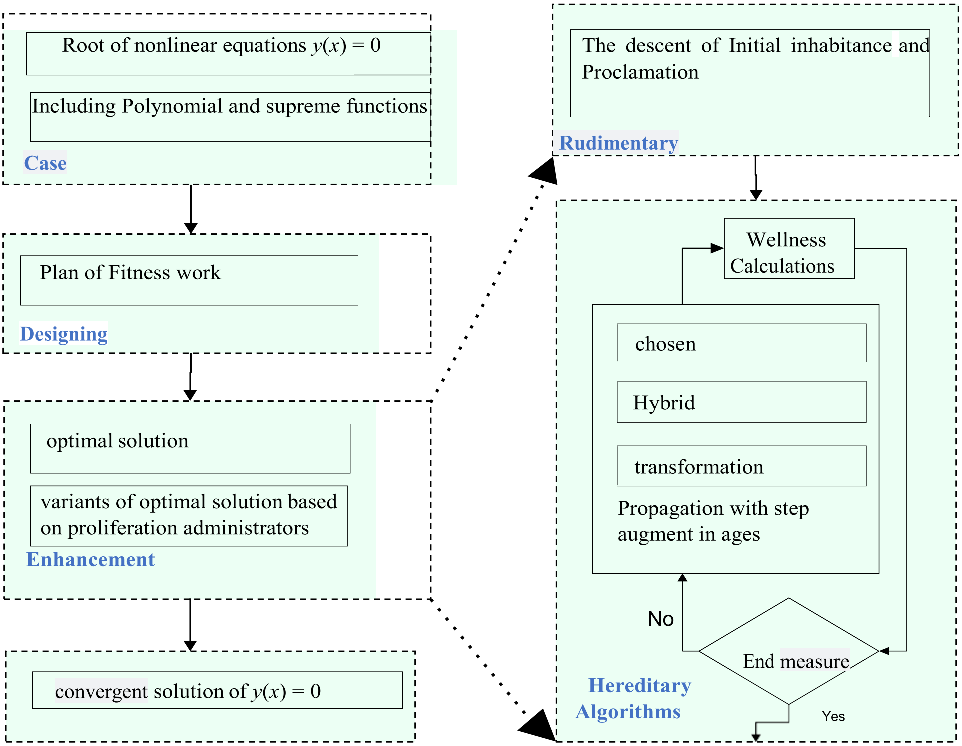

Fig. 4. (Color online) Schematic depiction of the suggested methodology for detecting the solution of nonlinear equations dependent on variations of germinal algorithms.

$$y_n = x_n - \frac{2y(x_n)y'(x_n)}{2y'^2(x_n) - y(x_n)y'(x_n)}, \tag{4.10}$$

$$x_{n+1} = x_n - \frac{2(y(x_n)+y(y_n))\, y'(x_n))}{2y'^2(x_n) - (y(x_n) + y(y_n))y'(x_n)}. \tag{4.11}$$

The nonexclusive flow outline strategy is given in Fig. 4.

Since the correct arrangement of this issue isn't known, from the figure, it manifests that we get accurate results by Green CAS at $k = 5$, $M = 5$ in contrast with different techniques. We will get a more accurate result by increasing the value of k, M.

5. Demonstrative Models

In this branch, fractional LEP of several modality is disbanded to establish the relevance and accuracy of the chosen method in settling singular fractional LEP cases.

Example 5.1. Suppose the subsequent nonlinear fractional spherical isothermal LEP-type equation

$$D^\alpha \nu(x) + \frac{2}{x^{\alpha-\beta}} D^\beta \nu(x) = e^{-\nu(x)}, \quad 0 \le x \le 1, \tag{5.1}$$

is submissive to the boundary conditions

$$u(0) = 1, \quad u'(0) = 0. \tag{5.2}$$

We utilized the technique in Ref. 24 to acquire the exact solution of this problem when $\alpha = 1.5$, $\beta = 0.75$ as $\nu(x) = 0.2368456765x^{1.5} - 0.02568610429x^3 + 0.005004915254x^{4.5} - 0.001292847099x^6 + 0{:}0004026307151x^{7.5}$. We employ the Green-CAS technique with distinct estimates of M and k for disbanding this problem. In Table 1, we compare the absolute errors of $\nu(x)$ at distinct points.

Table 1. Arbitrage of absolute errors in Example 5.1 for $\alpha = 1.5, \beta = 0.75$.

x	Green's function — CAS at $M = 7, k = 2$	CWFD in Ref. 26 at $M = 7, k = 2$	Green's function — CAS at $M = 10, k = 2$	CWFD in Ref. 26 at $M = 10, k = 2$
0.1	1.3×10^{-4}	1.3×10^{-4}	3.5×10^{-7}	4.3×10^{-5}
0.2	1.3×10^{-4}	1.3×10^{-4}	4.4×10^{-6}	3.3×10^{-5}
0.3	9.8×10^{-5}	9.8×10^{-5}	2.1×10^{-7}	3.9×10^{-6}
0.4	6.4×10^{-5}	6.4×10^{-5}	1.4×10^{-6}	3.4×10^{-5}
0.5	2.8×10^{-5}	2.8×10^{-5}	2.6×10^{-6}	3.3×10^{-5}
0.6	6.3×10^{-4}	6.3×10^{-4}	7.1×10^{-5}	1.6×10^{-4}
0.7	1.0×10^{-3}	1.0×10^{-3}	4.2×10^{-5}	2.1×10^{-4}
0.8	1.4×10^{-3}	1.4×10^{-3}	3.4×10^{-5}	1.5×10^{-4}
0.9	1.9×10^{-3}	1.9×10^{-3}	6.5×10^{-6}	8.2×10^{-6}

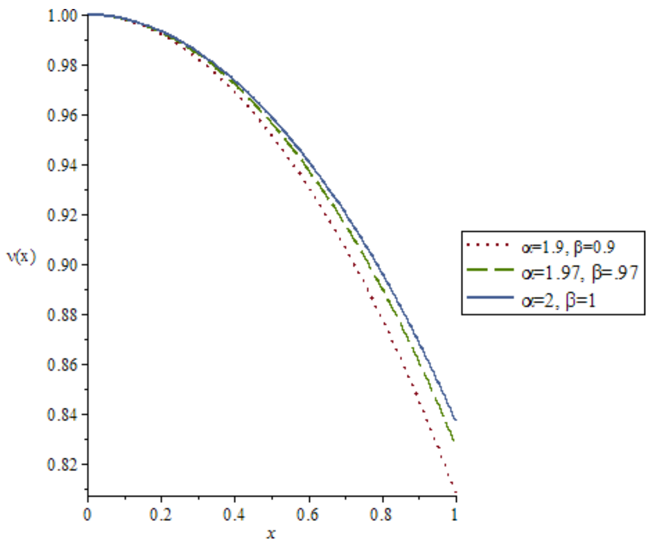

Fig. 5. (Color online) Arbitrage of $\nu(x)$ for $M = 12$, $k = 2$ and different values of α and β for Example 5.2.

Example 5.2. Suppose the attaching nonlinear fractional LEP

$$D^{\alpha}\nu(x) + \frac{2}{x^{\alpha-\beta}} D^{\beta}\nu(x) + \sinh(\nu(x)) = 0, \quad 0 \le x \le 1\,, \tag{5.3}$$

is submissive to the boundary conditions

$$\nu(0) = 1, \quad \nu'(0) = 0\,. \tag{5.4}$$

Wazwaz *et al.*[25] employed ADM to acquire a series solution of (5.3), when $\alpha = 2$, as follows:

$$\nu(x) = 1 - \frac{(e^2-1)x^2}{12e} + \frac{(e^4-1)x^4}{480e^2} - \frac{(2e^6+3e^2-3e^4-2)x^6}{30240e^3} + \frac{(61e^8-104e^6+104e^2-61)x^8}{26127360e^4}\,.$$

We apply the novel method introduced in the previous section with $M = 12, k = 2$ and solve this problem with distinct estimates of α and β. It can be seen from Fig. 5 that the solution of the fractional differential equation approaches that of the integer-order differential equation.

Example 5.3. Suppose the attaching linear fractional LEP

$$D^{\alpha}\nu(x) + \frac{2}{x^{\alpha-\beta}} D^{\beta}\nu(x) = \frac{0.76129\,u(x)}{\nu(x)+0.03119}, \quad 0 \le x \le 1\,, \tag{5.5}$$

is submissive to the boundary conditions

$$\nu'(0) = 0, \quad 5\nu(1) + \nu'(1) = 5\,. \tag{5.6}$$

Table 2. Comparison of approximate solutions $\alpha = 1.9$, 1.7 and $\beta = 0.9$, 0.7 for Example 5.3.

x	Green's function — CAS $\alpha = 1.9, \beta = 0.9$	CWFD at $\alpha = 1.9, \beta = 0.9$	Green's function — CAS $\alpha = 1.7, \beta = 0.7$	CWFD at $\alpha = 1.7, \beta = 0.7$
0.1	0.816514521141765	0.816514521141764	0.790555364088862	0.790555364088865
0.2	0.821123173757562	0.821123173757560	0.797659339431067	0.797659339431068
0.3	0.828429359689671	0.828429359689670	0.807825369576836	0.807825369576831
0.4	0.838326176512754	0.838326176512752	0.820716305422745	0.820716305422740
0.5	0.850740215946675	0.850740215946676	0.836100384725764	0.836100384725762
0.6	0.865614149890048	0.865614149890049	0.853839766598643	0.853839766598646
0.7	0.882901208768501	0.882901208768500	0.873779197287925	0.873779197287922
0.8	0.902571377092314	0.902571377092311	0.895829358309582	0.895829358309583
0.9	0.9245982832360296	0.924598283236026	0.919913436181024	0.919913436181026

There is no exact solution for this problem even for the integer case. We solve this problem using CWFD. In Table 2, the estimates for distinct fractional estimates of α and β at distinct points are tabulated. We make arbitrage between the results obtained by the CWFD method and Green's function — CAS method.

Table 2 demonstrates that our outcomes are near the outcomes revealed in other papers and show the materialness and precision of the proposed technique. It may very well be plainly seen from Table 2, as α and β approach 2 and 1, the arrangement of fragmentary differential condition ways to deal with that of the number request differential equation.

Example 5.4. Suppose the attaching nonlinear singular fractional two-point BVP[7]

$$D^\alpha \nu(x) + \frac{1}{x^{\alpha-\beta}} D^\beta \nu(x) + \frac{\nu^2(x)}{x(1-x)} = 4\arctan x + \frac{1+3x^2}{x(1+x^2)} + \frac{(1+x^2)^2 \arctan^2(x)}{x(1-x)}, \quad 0 < x < 1, \tag{5.7}$$

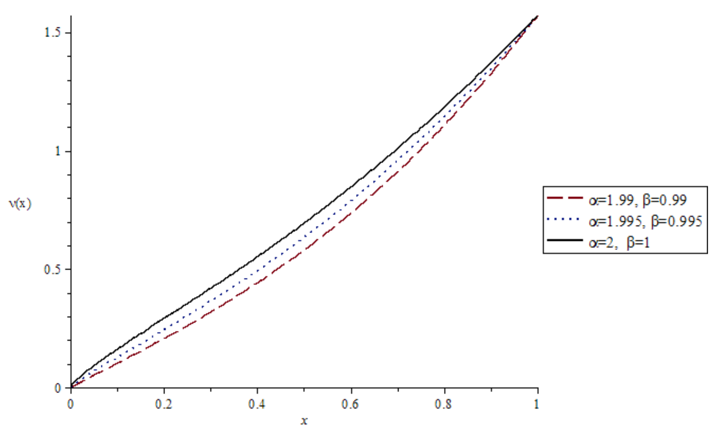

Fig. 6. The outline of $\nu(x)$ for distinct rate of α and β for Example 5.5.

with the BC.

$$\nu(0) + \nu'(0) = 1, \quad \nu(1) + \nu'(1) = 4.14159265358979. \tag{5.8}$$

The analytic solution of the above case at $\alpha = 2$ and $\beta = 1$ is specified as $\nu(x) = (1+x^2)\arctan(x)$. The schemes of $\nu(x)$ at several estimates of α and β are sketched in Fig. 6.

6. Conclusion

A novel method named Green-CAS method has been developed for disbanding fractional differential equations with boundary conditions. The method exploits CAS wavelets as primary underlying tool. Principally, the method is utilized to solve linear differential equations. The suggested methodology for detecting the solution of nonlinear equations dependent on variations of germinal algorithms has been employed to find results for otherwise nonlinear equations as well. The method has likewise been analyzed for convergence. Some numerical applications have also been documented. Their results and comparisons against previously employed methods have been inscribed in tables and graphs. It can be concluded that Green-CAS method is not just computationally efficient but relatively precise and accurate as well.

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