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Research Article

A Study on Lump Solutions to a Generalized Hirota-Satsuma-Ito Equation in (2+1)-Dimensions

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The Hirota-Satsuma-Ito equation in (2+1)-dimensions passes the three-soliton test. This paper aims to generalize this equation to a new one which still has abundant interesting solution structures. Based on the Hirota bilinear formulation, a symbolic computation with a new class of Hirota-Satsuma-Ito type equations involving general second-order derivative terms is conducted to require having lump solutions. Explicit expressions for lump solutions are successfully presented in terms of coefficients in a generalized Hirota-Satsuma-Ito equation. Three-dimensional plots and contour plots of a special presented lump solution are made to shed light on the characteristic of the resulting lump solutions.

1. Introduction

In the classical theory of differential equations, the main question is to study the existence of solutions to given equations, including many nonlinear equations describing real-world problems. Cauchy problems are to deal with the existence, uniqueness, and stability of solutions satisfying initial data. Laplace's method is developed for solving Cauchy problems for linear ordinary differential equations and the Fourier transform method for linear partial differential equations. In modern soliton theory, the isomonodromic transform method and the inverse scattering transform method have been designed to solve Cauchy problems for nonlinear ordinary and partial differential equations [1–3]. Explicitly solvable differential equations include various constant-coefficient and linear differential equations, but it is extremely difficult to compute exact solutions to variable-coefficient or nonlinear equations.

However, the Hirota bilinear method provides us with a working approach to soliton solutions, historically found for nonlinear integrable equations [4, 5]. Soliton solutions are analytic ones exponentially localized in all directions in space and time. Let a polynomial P determine a Hirota bilinear differential equation:

$$P\left(D_x, D_y, D_t\right) f \cdot f = 0, \tag{1}$$

in (2+1)-dimensions, where D_x, D_y, and D_t are Hirota's bilinear derivatives. The corresponding partial differential equation with a dependent variable u is determined usually by one of the logarithmic transformations: $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$. Within the Hirota bilinear formulation, soliton solutions are expressed through

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N}\mu_i\xi_i + \sum_{i<j}\mu_i\mu_j a_{ij}\right), \tag{2}$$

where $\sum_{\mu=0,1}$ means the sum over all possibilities for μ_1, $\mu_2, \ldots, \mu_N$ taking either 0 or 1, and the wave variables and the phase shifts are defined by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \quad 1 \le i \le N, \tag{3}$$

and

$$e^{a_{ij}} = -\frac{P\left(k_i - k_j, l_i - l_j, \omega_j - \omega_i\right)}{P\left(k_i + k_j, l_i + l_j, \omega_j + \omega_i\right)}, \quad 1 \le i < j \le N, \tag{4}$$

in which k_i, l_i, and ω_i, $1 \le i \le N$, satisfy the corresponding dispersion relation and $\xi_{i,0}$, $1 \le i \le N$, are arbitrary phase shifts.

Lump solutions are a class of analytic rational solutions which are localized in all directions in space, originated from solving integrable equations in (2+1)-dimensions (see, e.g., [6–8]). Taking long wave limits of N-soliton solutions can generate special lumps [9]. Many integrable equations in (2+1)-dimensions exhibit the remarkable richness of lump solutions (see, e.g., [6, 7]). Such equations contain the KPI equation [10], whose special lump solutions have been derived from N-soliton solutions [11], the three-dimensional three-wave resonant interaction [12], the BKP equation [13, 14], the Davey-Stewartson equation II [9], the Ishimori-I equation [15], and the KP equation with a self-consistent source [16]. An important step in the process of getting lumps is to determine positive quadratic function solutions to bilinear equations [6]. Then, through the mentioned logarithmic transformations, we present lump solutions to nonlinear equations (see, e.g., [6] for the case of Hirota bilinear equations and [7] for the case of generalized bilinear equations).

In this paper, we would like to generalize the Hirota-Satsuma-Ito (HSI) equation in (2+1)-dimensions to a new one which still has abundant interesting solution structures. Hirota bilinear forms are the starting point for our discussion (see, e.g., [6, 7, 17, 18] for other equations). We will consider a general class of HSI type equations while keeping the existence of lump solutions. A general such generalized HSI equation in (2+1)-dimensions and its lump solutions will be determined through symbolic computations with Maple. For a special presented lump solution, three-dimensional plots and contour plots will be made via the Maple plot tool, to shed light on the characteristic of the presented lump solutions. A few concluding remarks will be given in the last section.

2. Lump Solutions

It is known that the Hirota-Satsuma shallow water wave equation [4],

$$\begin{aligned} u_t &= u_{xxt} + 3uu_t - 3u_x v_t - u_x, \\ v_x &= -u, \end{aligned} \tag{5}$$

has a bilinear form,

$$\left(D_t D_x^3 - D_t D_x - D_x^2\right) f \cdot f = 0, \tag{6}$$

under the logarithmic transformations $u = 2(\ln f)_{xx}$ and $v = 2(\ln f)_x$. An integrable $(2+1)$-dimensional extension of the Hirota-Satsuma equation reads

$$3\left(u_x u_t\right)_x + u_{xxxt} + u_{yt} + u_{xx} = 0, \tag{7}$$

which passes the Hirota three-soliton test [19], and has a bilinear form under the logarithmic transformation $u = 2(\ln f)_x$:

$$\left(D_x^3 D_t + D_y D_t + D_x^2\right) f \cdot f = 0. \tag{8}$$

Equation (7) is called the Hirota-Satsuma-Ito (HSI) equation in (2+1)-dimensions [19]. We would like to add three terms to generalize the abovementioned HSI equation to a new one which still possesses abundant interesting solution structures:

$$\begin{aligned} P(u) = {} & 3\left(u_x u_t\right)_x + u_{xxxt} + \delta_1 u_{yt} + \delta_2 u_{xx} + \delta_3 u_{xy} \\ & + \delta_4 u_{xt} + \delta_5 u_{yy} = 0. \end{aligned} \tag{9}$$

This generalized HSI equation has a bilinear form under the logarithmic transformation $u = 2(\ln f)_x$:

$$\begin{aligned} B(f) = {} & \left(D_x^3 D_t + \delta_1 D_y D_t + \delta_2 D_x^2 + \delta_3 D_x D_y \right. \\ & \left. + \delta_4 D_x D_t + \delta_5 D_y^2\right) f \cdot f = 0. \end{aligned} \tag{10}$$

Precisely, under $u = 2(\ln f)_x$, we have the relation $P(u) = (B(f)/f^2)_x$. In what follows, we would like to determine lump solutions to the gHSI equation in (2+1)-dimensions (9), through symbolic computations with Maple.

We start to search for positive quadratic solutions to the gHSI bilinear equation (10) to generate lump solutions to the gHSI equation (9):

$$\begin{aligned} f = {} & \left(a_1 x + a_2 y + a_3 t + a_4\right)^2 + \left(a_5 x + a_6 y + a_7 t + a_8\right)^2 \\ & + a_9. \end{aligned} \tag{11}$$

Plugging this function into the gHSI bilinear equation (10) generates a system of nonlinear algebraic equations on the parameters a_i, $1 \le i \le 9$. Conducting direct symbolic computation to solve this system gives a set of solutions for the parameters where

$$\begin{aligned} a_3 &= -\frac{b_1}{\left(a_1\delta_4 + a_2\delta_1\right)^2 + \left(a_5\delta_4 + a_6\delta_1\right)^2}, \\ a_7 &= -\frac{b_2}{\left(a_1\delta_4 + a_2\delta_1\right)^2 + \left(a_5\delta_4 + a_6\delta_1\right)^2}, \\ a_9 &= \frac{3\left(a_1^2 + a_5^2\right) b_3}{\left(a_1 a_6 - a_2 a_5\right)^2 \left(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4 + \delta_4^2\delta_5\right)}, \end{aligned} \tag{12}$$

and all other a_i's are arbitrary. The involved three constants are defined as follows:

$$\begin{aligned}
b_1 &= \left[\left(a_1^2 a_2 + 2a_1a_5a_6 - a_2a_5^2\right)\delta_2 + a_1\left(a_2^2 + a_6^2\right)\delta_3 \right.\\
&\quad \left. + a_2\left(a_2^2 + a_6^2\right)\delta_5\right]\delta_1 + \left[a_1\left(a_1^2 + a_5^2\right)\delta_2\right.\\
&\quad \left. + a_2\left(a_1^2 + a_5^2\right)\delta_3 + \left(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2\right)\delta_5\right]\delta_4,\\
b_2 &= \left[\left(-a_1^2 a_6 + 2a_1a_2a_5 + a_5^2a_6\right)\delta_2 + a_5\left(a_2^2 + a_6^2\right)\delta_3\right.\\
&\quad \left. + a_6\left(a_2^2 + a_6^2\right)\delta_5\right]\delta_1 + \left[a_5\left(a_1^2 + a_5^2\right)\delta_2\right.\\
&\quad \left. + a_6\left(a_1^2 + a_5^2\right)\delta_3 + \left(-a_2^2a_5 + 2a_1a_2a_6 + a_5a_6^2\right)\delta_5\right]\\
&\quad \cdot \delta_4,\\
b_3 &= \left(a_1^2 + a_5^2\right)\left(a_1a_2 + a_5a_6\right)\left(\delta_1\delta_2 + \delta_3\delta_4\right) + \left(a_1^2\right.\\
&\quad \left. + a_5^2\right)\left(a_2^2 + a_6^2\right)\delta_1\delta_3 + \left(a_1^2 + a_5^2\right)^2\delta_2\delta_4 + \left(a_2^2 + a_6^2\right)\\
&\quad \cdot \left(a_1a_2 + a_5a_6\right)\delta_1\delta_5 + \left[\left(a_1a_2 + a_5a_6\right)^2\right.\\
&\quad \left. - \left(a_1a_6 - a_2a_5\right)^2\right]\delta_4\delta_5.
\end{aligned} \tag{13}$$

Those formulas in (12) and (13) were obtained under a simplification process with Maple.

From (12), we can easily see that it is sufficient to guarantee $f > 0$ if we require

$$\left(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4 + \delta_4^2\delta_5\right)b_3 > 0, \tag{14}$$

and, thus, the function f defined by (12) and (13) under the abovementioned condition and

$$a_1a_6 - a_2a_5 \neq 0 \tag{15}$$

leads to lump solutions

$$u = 2\left(\ln f\right)_x = \frac{2f_x}{f} \tag{16}$$

to the gHSI equation in (2+1)-dimensions (9).

When one takes

$$\begin{aligned}
\delta_1 &= 1,\\
\delta_2 &= 1,\\
\delta_3 &= \delta_4 = \delta_5 = 0,
\end{aligned} \tag{17}$$

one obtains the original HSI equation in (2+1)-dimensions (7), and the function f by (12) and (13) presents a class of lump solutions to the HSI equation (7):

$$\begin{aligned}
u &= 2\left(\ln f\right)_x,\\
f &= \left(a_1x + a_2y + a_3t + a_4\right)^2\\
&\quad + \left(a_5x + a_6y + a_7t + a_8\right)^2 + a_9,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
a_3 &= -\frac{a_1^2a_2 + 2a_1a_5a_6 - a_2a_5^2}{a_2^2 + a_6^2},\\
a_7 &= \frac{a_1^2a_6 - 2a_1a_2a_5 - a_5^2a_6}{a_2^2 + a_6^2},\\
a_9 &= \frac{3\left(a_1^2 + a_5^2\right)^2\left(a_1a_2 + a_5a_6\right)}{\left(a_1a_6 - a_2a_5\right)^2},
\end{aligned} \tag{19}$$

and all other a_i's are arbitrary. Solving the abovementioned parameter solutions on a_3 and a_7 for a_2 and a_6 and substituting the resulting expressions for a_2 and a_6 into the formula for a_9 in (19), we get

$$\begin{aligned}
a_2 &= -\frac{a_1^2a_3 + 2a_1a_5a_7 - a_3a_5^2}{a_3^2 + a_7^2},\\
a_6 &= \frac{a_1^2a_7 - 2a_1a_3a_5 - a_5^2a_7}{a_3^2 + a_7^2},\\
a_9 &= -\frac{3\left(a_1^2 + a_5^2\right)\left(a_3^2 + a_7^2\right)\left(a_1a_3 + a_5a_7\right)}{\left(a_1a_7 - a_3a_5\right)^2}.
\end{aligned} \tag{20}$$

It is easy to see that

$$a_1a_6 - a_2a_5 = \frac{\left(a_1^2 + a_5^2\right)\left(a_1a_7 - a_3a_5\right)}{a_3^2 + a_7^2}, \tag{21}$$

and, thus, the conditions of

$$\begin{aligned}
a_1a_3 + a_5a_7 &< 0,\\
a_1a_7 - a_3a_5 &\neq 0
\end{aligned} \tag{22}$$

guarantee that (16) with (11) and (20) will present lump solutions to the HSI equation (7) [20].

Particularly taking

$$\begin{aligned}
\delta_1 &= 1,\\
\delta_2 &= 1,\\
\delta_3 &= -1,\\
\delta_4 &= 1,\\
\delta_5 &= -1,
\end{aligned} \tag{23}$$

we obtain a special gHSI equation as follows:

$$u_{xxxt} + 3\left(u_xu_t\right)_x + u_{yt} + u_{xx} - u_{xy} + u_{xt} - u_{yy} = 0, \tag{24}$$

which has a Hirota bilinear form

$$\begin{aligned}
&\left(D_x^3D_t + D_yD_t + D_x^2 - D_xD_y + D_xD_t - D_y^2\right)f\cdot f\\
&= 0,
\end{aligned} \tag{25}$$

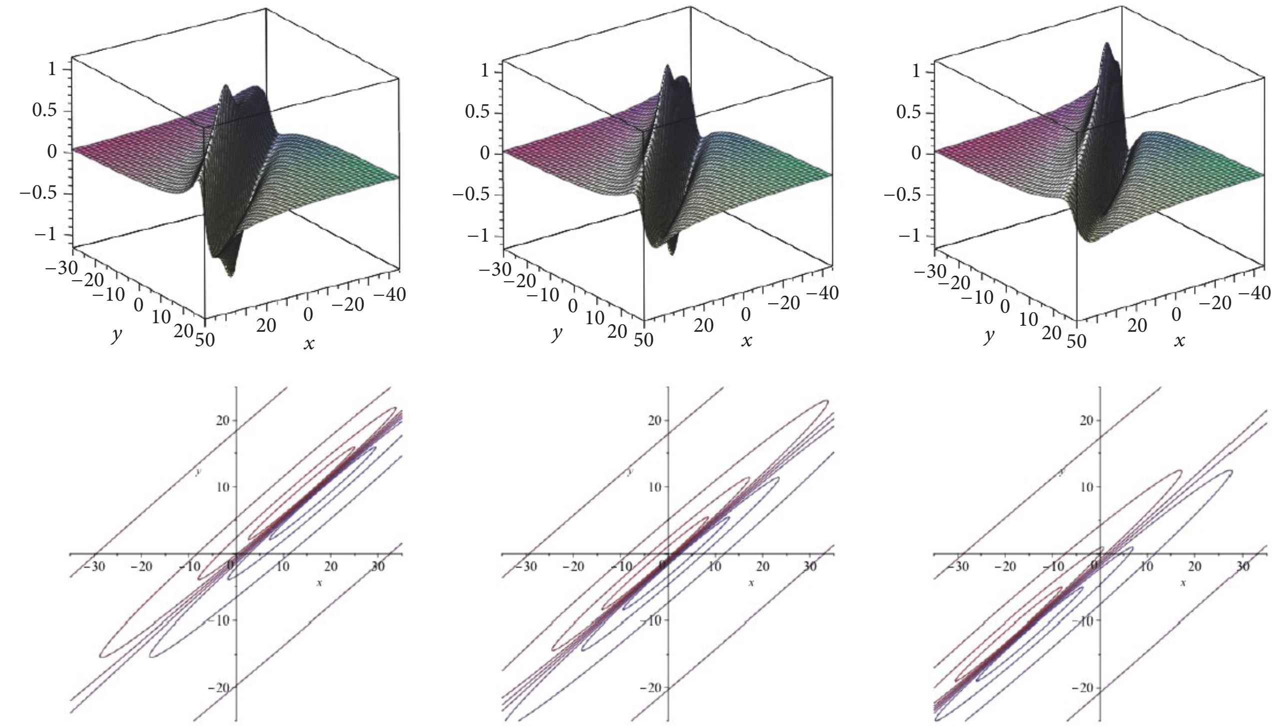

Figure 1: Profiles of u when $t = 0, 3, 6$: 3D plots (top) and contour plots (bottom).

under the logarithmic transformation (16). Associated with

$$a_1 = 1, \quad a_2 = -2, \quad a_4 = 2, \quad a_5 = 2, \quad a_6 = -3, \quad a_8 = -5, \tag{26}$$

(16) with (11) and (22) present the lump solution to the special gHSI equation (24):

$$u = \frac{2(10x - 16y - t - 16)}{(x - 2y - (3/2)t + 2)^2 + (2x - 3y + (1/2)t - 5)^2 + 15}. \tag{27}$$

Three three-dimensional plots and contour plots of this lump solution are made via Maple plot tools, to shed light on the characteristic of the presented lump solutions, in Figure 1.

All the exact solutions generated above add valuable insights into the existing theories on soliton solutions and dromion-type solutions, developed through various powerful solution techniques including the Hirota perturbation approach, the Riemann-Hilbert approach, the Wronskian technique, symmetry reductions, and symmetry constraints (see, e.g., [21–31]).

3. Concluding Remarks

We have studied a generalized (2+1)-dimensional Hirota-Satsuma-Ito (HSI) equation to explore different equations which possess lump solutions, through symbolic computations with Maple. The results enrich the theory of lumps and solitons, providing a new example of (2+1)-dimensional nonlinear equations, which possess beautiful lump structures. Three-dimensional plots and contour plots of a specially chosen lump solution were made by using the plot tool in Maple.

Many nonlinear equations possess lump solutions, which include (2+1)-dimensional generalized KP, BKP, KP-Boussinesq, Sawada-Kotera, and Bogoyavlensky-Konopelchenko equations [32–36]. Some recent studies also demonstrate the strikingly high richness of lump solutions to linear partial differential equations [37] and nonlinear partial differential equations in (2+1)-dimensions (see, e.g., [38–41]) and (3+1)-dimensions (see, e.g., [42–48]). Diversity of lump solutions supplements exact solutions generated from different kinds of combinations (see, e.g., [49–52]) and can yield various Lie-Bäcklund symmetries, which can be used to determine conservation laws by symmetries and adjoint symmetries [53–55]. Moreover, diverse interaction solutions [35] have been exhibited for many integrable equations in (2+1)-dimensions, including lump-soliton interaction solutions (see, e.g., [56–58]) and lump-kink interaction solutions (see, e.g., [59–62]).

We finally remark that we could add one more term to the gHSI equation (9) to formulate a more generalized HSI bilinear equation,

$$\left(D_x^3D_t + \delta_1 D_yD_t + \delta_2 D_x^2 + \delta_3 D_xD_y + \delta_4 D_xD_t + \delta_5 D_y^2 + \delta_6 D_t^2\right) f\cdot f = 0, \tag{28}$$

where δ_i, $1 \le i \le 6$, are all constants, but we failed to drive any lump solution to the corresponding nonlinear equation on $u = 2(\ln f)_x$. The first term in the abovementioned bilinear equation is crucial in determining lump solutions but the last term brings the difficulty to work out lump solutions. There is no hint on how to solve any big system of resulting nonlinear algebraic equations. Nevertheless, some general considerations on the existence of lumps have been made for the Hirota bilinear case [6] and the generalized bilinear cases [7].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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