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A second Wronskian formulation of the Boussinesq equation

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ABSTRACT

A Wronskian formulation leading to rational solutions is presented for the Boussinesq equation. It involves third-order linear partial differential equations, whose representative systems are systematically solved. The resulting solutions formulas provide a direct but powerful approach for constructing rational solutions, positon solutions and complexiton solutions to the Boussinesq equation. Various examples of exact solutions of those three kinds are computed. The newly presented Wronskian formulation is different from the one previously presented by Li et al., which does not yield rational solutions.

1. Introduction

Wronskian formulations are a common feature for soliton equations, and lead to a powerful tool to construct exact solutions to soliton equations [1–4]. The resulting technique has been applied to many soliton equations such as the KdV, MKdV, NLS, derivative NLS, KP, sine-Gordon and sinh-Gordon equations. Within Wronskian formulations, soliton solutions and rational solutions are usually expressed as some kind of logarithmic derivatives of Wronskian type determinants and the determinants involved are made of eigenfunctions satisfying linear differential equations. Clearly, Wronskian formulations connect nonlinear problems with linear problems, and thus, soliton equations can be solved by means of linear theories. There is also a discrete version of Wronskian formulations, called Casoratian formulations, for integrable lattice equations such as the Volttera, nonlinear electrical network, and Toda lattice equations [4–6]. Besides soliton solutions and rational solutions, the Wronskian and Casoratian techniques can be used to construct positon solutions [7–10], i.e., solutions involving one kind of transcendental functions: trigonometric functions. More generally, a novel kind of solutions called compelxiton solutions has been introduced and generated using such techniques for soliton equations [11,4] and soliton equations with sources [12]. Those solutions contain two kinds of transcendental functions: exponential functions and trigonometric functions, and they correspond to complex eigenvalues of at least one of associated characteristic problems and usually not traveling waves [13].

One of the physically significant soliton equations is the Boussinesq equation

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx}/3 = 0, \tag{1.1}$$

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which describes shallow-water waves moving in both directions [14]. It also appears in a wide variety of physical systems [15–22], for example, nonlinear waves in the evolution of perturbations with the dispersion relation close to that for the sound waves [15], electromagnetic waves interacting with transversal optical phonons in nonlinear dielectrics [16], magnetosound waves in plasmas [17], magnetoelastic waves in antiferromagnets [18], often observed occurrences of thin turbulent layers in the middle atmosphere [19,20] and Rayleigh-Bénard convection between two horizontal plates [21, 22]. For the transonic speed perturbations, by neglecting the interaction of waves moving in the opposite directions, the Boussinesq equation (1.1) can be reduced to the KdV equation. The Eq. (1.1) itself is also a dimensional reduction of the KP equation in the moving frame. Moreover, the Boussinesq equation (1.1) is integrable by the inverse scattering transformation [23,24], and can be studied by the Hamiltonian method [23], the algebro-geometric methods [25] and $\bar{\partial}$-dressing method [26]. The Boussinesq equation attracts much attention of researchers in both fields of mathematics and physics owing to its profound importance and nice mathematical properties, though its solitons may decay under perturbations and form a singularity in a finite time [27].

We will consider the Boussinesq equation of the form

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0, \tag{1.2}$$

and call it the Boussinesq equation I (also good Boussinesq equation). The problem here in this paper is to construct its real solutions. Obviously, a general Boussinesq equation

$$v_{tt} + a_1 v_{xx} + a_2 (v^2)_{xx} + a_3 v_{xxxx} = 0, \tag{1.3}$$

where a_i, $1 \le i \le 3$, are real numbers and $a_2 a_3 \neq 0$, is equivalent to the Boussinesq equation I in (1.2), under the transformation

$$v(x,t) = -\frac{a_1}{2a_2} + \frac{a_3}{a_2} u(x, \sqrt{a_3}\, t),$$

in the case of $a_3 > 0$. Similarly, in the case of $a_3 < 0$, the general Boussinesq equation (1.3) is equivalent to

$$u_{tt} + (u^2)_{xx} - u_{xxxx} = 0, \tag{1.4}$$

under the transformation

$$v(x,t) = -\frac{a_1}{2a_2} - \frac{a_3}{a_2} u(x, \sqrt{-a_3}\, t).$$

We call the Eq. (1.4) the Boussinesq equation II (also bad Boussinesq equation).

If we take the transformation

$$u = 6(\ln f)_{xx} = \frac{6(ff_{xx} - f_x^2)}{f^2}, \tag{1.5}$$

then the Boussinesq equation I in (1.2) becomes a bilinear differential equation

$$(D_t^2 + D_x^4) f \bullet f = 2(ff_{tt} - f_t^2 + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2) = 0, \tag{1.6}$$

where D_x and D_t are the Hirota operators [28]. Actually, we have

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = \left[\frac{3(D_t^2 + D_x^4) f \bullet f}{f^2} \right]_{xx}.$$

Therefore, if f solves the bilinear Boussinesq equation (1.6), then $u = 6(\ln f)_{xx}$ solves the Boussinesq equation I in (1.2).

Similarly, we have

$$u_{tt} + (u^2)_{xx} - u_{xxxx} = \left[\frac{-3(D_t^2 - D_x^4) f \bullet f}{f^2} \right]_{xx},$$

under the transformation

$$u = -6(\ln f)_{xx} = -\frac{6(ff_{xx} - f_x^2)}{f^2}. \tag{1.7}$$

Therefore, under (1.7), the Boussinesq equation II in (1.4) becomes

$$(D_t^2 - D_x^4) f \bullet f = 2(ff_{tt} - f_t^2 - ff_{xxxx} + 4f_x f_{xxx} - 3f_{xx}^2) = 0. \tag{1.8}$$

Such Hirota bilinear equations play an extremely important role in the field of integrable systems and solitons.

Very recently, a Wronskian formulation has been presented [29] for the Boussinesq equation (1.1). Based on the presented Wronskian formulation, solitons, negatons, positons and complexitons including many new solutions were generated in a direct way, along with plots of interactions of solitons moving in the same and opposite directions [29]. Interestingly, rational solutions do not come but we know they exists at least for the Boussinesq equation II [30]. In this paper, we aim to present another Wronskian formulation for solutions of the above Boussinesq equation I, which particularly leads to an approach for constructing rational solutions to the Boussinesq equation I. Our results will also show the richness and diversity of solution structures of the considered Boussinesq equation.

The paper is organized as follows. In Section 2, a new Wronskian formulation, different from the one in [29], is presented for the bilinear Boussinesq equation I and thus the Boussinesq equation I. In Section 3, the representative systems of differential equations in the proposed linear conditions are solved by applying the method of variation of parameters. In Section 4, an approach for constructing exact solutions including rational solutions is furnished, and many examples of solutions such as rational solutions, positons and complexitons are provided. Concluding remarks are given in Section 5.

2. A Wronskian formulation

To use the Wronskian technique, we adopt the compact notation introduced by Freeman and Nimmo [1,31]:

$$\mathrm{W}(\phi_1,\phi_2,\ldots,\phi_N)=(\widehat{N-1};\Phi)=(\widehat{N-1})=\begin{vmatrix}\phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)}\\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)}\\ \vdots & \vdots & \ddots & \vdots\\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)}\end{vmatrix}, \tag{2.1}$$

where

$$\Phi=(\phi_1,\ldots,\phi_N)^{\mathrm{T}},\quad \phi_i^{(0)}=\phi_i,\qquad \phi_i^{(j)}=\frac{\partial^j}{\partial x^j}\phi_i,\quad j\ge 1,\ 1\le i\le N. \tag{2.2}$$

Solutions determined by $u=6(\ln f)_{xx}$ with $f=(\widehat{N-1})$ to the Boussinesq equation (1.2) are called Wronskian solutions.

Theorem 2.1. *Let $\varepsilon=\pm1$. If a group of functions $\phi_i=\phi_i(x,t),\ 1\le i\le N$, satisfies the following linear conditions*

$$\phi_{i,xxx}=\sum_{j=1}^{N}\lambda_{ij}(t)\phi_j,\quad 1\le i\le N, \tag{2.3}$$

$$\phi_{i,t}=\varepsilon\sqrt{3}\,\phi_{i,xx},\quad 1\le i\le N, \tag{2.4}$$

where the λ_{ij}'s are arbitrary real functions of t, then $f=(\widehat{N-1})$ defined by (2.1) solves the bilinear Boussinesq equation (1.6).

Proof. Obviously, we always have

$$\begin{aligned}
f_x&=(\widehat{N-2},N),\\
f_{xx}&=(\widehat{N-3},N-1,N)+(\widehat{N-2},N+1),\\
f_{xxx}&=(\widehat{N-4},N-2,N-1,N)+2(\widehat{N-3},N-1,N+1)+(\widehat{N-2},N+2),\\
f_{xxxx}&=(\widehat{N-5},N-3,N-2,N-1,N)+3(\widehat{N-4},N-2,N-1,N+1)\\
&\quad+2(\widehat{N-3},N,N+1)+3(\widehat{N-3},N-1,N+2)+(\widehat{N-2},N+3).
\end{aligned}$$

Using the conditions in (2.4) gives

$$\begin{aligned}
f_t&=\varepsilon[-\sqrt{3}\,(\widehat{N-3},N-1,N)+\sqrt{3}\,(\widehat{N-2},N+1)],\\
f_{tt}&=3(\widehat{N-5},N-3,N-2,N-1,N)+6(\widehat{N-3},N,N+1)-3(\widehat{N-3},N-1,N+2)\\
&\quad-3(\widehat{N-4},N-2,N-1,N+1)+3(\widehat{N-2},N+3).
\end{aligned}$$

Note that for a given matrix A, we have the following determinant equality:

$$\sum_{i=1}^{N}\begin{vmatrix}\mathrm{Row}(A,1)\\ \vdots\\ \mathrm{Row}(A,i)_{xxx}\\ \vdots\\ \mathrm{Row}(A,N)\end{vmatrix}=\sum_{j=1}^{N}\left|\,\mathrm{Col}(A,1),\ldots,\mathrm{Col}(A,j)_{xxx},\ldots,\mathrm{Col}(A,N)\,\right|, \tag{2.5}$$

where Row(A, i) and Col(A, j) denote the i-th row and the j-th column of A, respectively. Taking A as $(\widehat{N-1})$ and $(\widehat{N-2}, N)$ in the above equality, and then using the conditions in (2.3) result in

$$\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-1}) = (\widehat{N-4}, N-2, N-1, N) - (\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2),$$

$$\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-2}, N) = (\widehat{N-5}, N-3, N-2, N-1, N) - (\widehat{N-3}, N, N+1) + (\widehat{N-2}, N+3).$$

Now, we can compute that

$$\begin{aligned} f(f_{tt} + f_{xxxx}) &= (\widehat{N-1})\,[\,4(\widehat{N-5}, N-3, N-2, N-1, N) + 8(\widehat{N-3}, N, N+1) + 4(\widehat{N-2}, N+3)] \\ &= 12(\widehat{N-1})(\widehat{N-3}, N, N+1) + 4\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-1})(\widehat{N-2}, N), \end{aligned}$$

$$-4f_x f_{xxx} = -12(\widehat{N-2}, N)(\widehat{N-3}, N-1, N+1) - 4\sum_{i=1}^{N} \lambda_{ii}(t)(\widehat{N-2}, N)(\widehat{N-1}),$$

$$-f_t^2 + 3f_{xx}^2 = 12(\widehat{N-3}, N-1, N)(\widehat{N-2}, N+1).$$

It follows then that

$$\begin{aligned} &(D_t^2 + D_x^4)f \bullet f = 2(ff_{tt} - f_t^2 + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2) \\ &\quad = 24(\widehat{N-1})(\widehat{N-3}, N, N+1) - 24(\widehat{N-2}, N)(\widehat{N-3}, N-1, N+1) + 24(\widehat{N-3}, N-1, N)(\widehat{N-2}, N+1) \\ &\quad = 12\begin{vmatrix} \widehat{N-3} & 0 & N-2 & N-1 & N & N+1 \\ 0 & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix} = 0. \end{aligned}$$

This shows that $f = (\widehat{N-1})$ solves the bilinear Boussinesq equation (1.6). The proof is finished. □

Theorem 2.1 tells us that if a group of functions $\phi_i(x, t)$, $1 \le i \le N$, satisfies the linear conditions in (2.3) and (2.4), then we can get a solution $f = (\widehat{N-1})$ to the bilinear Boussinesq equation (1.6). Before we proceed to solve (2.3) and (2.4), let us observe how the Wronskian formulation generates solutions more carefully.

Observation I. From the compatibility conditions $\phi_{i,xxxt} = \phi_{i,txxx}$, $1 \le i \le N$, of the conditions (2.3) and (2.4), we have the equalities

$$\sum_{j=1}^{N} \lambda_{ij,t}\phi_j = 0, \quad 1 \le i \le N, \tag{2.6}$$

and thus we see that the Wronskian determinant $W(\phi_1, \phi_2, \ldots, \phi_N)$ becomes zero, if the coefficient matrix $\Lambda = (\lambda_{ij})$ is dependent on t, i.e., $\Lambda_t \neq 0$.

Observation II. If the coefficient matrix Λ is similar to another matrix M under an invertible constant matrix P, i.e., we have $\Lambda = P^{-1}MP$, then $\tilde{\Phi} = P\Phi$ solves

$$\tilde{\Phi}_{xxx} = M\tilde{\Phi}, \qquad \tilde{\Phi}_t = \varepsilon\sqrt{3}\,\tilde{\Phi}_{xx}, \quad \varepsilon = \pm 1,$$

and the resulting Wronskian solutions to the Boussinesq equation (1.2) are the same:

$$\begin{aligned} u(\Lambda) &= 2\partial_x^2 \ln|\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(N-1)}| \\ &= 2\partial_x^2 \ln|P\Phi^{(0)}, P\Phi^{(1)}, \ldots, P\Phi^{(N-1)}| = u(M). \end{aligned}$$

Based on Observation I, we only need to consider the reduced case of (2.3) and (2.4) under $\mathrm{d}\Lambda/\mathrm{d}t = 0$, i.e., the following conditions:

$$\phi_{i,xxx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \qquad \phi_{i,t} = \varepsilon\sqrt{3}\,\phi_{i,xx}, \quad 1 \le i \le N, \tag{2.7}$$

where $\varepsilon = \pm 1$ and the λ_{ij}'s are arbitrary real constants. Moreover, Observation II tells us that an invertible constant linear transformation on Φ in the Wronskian determinant does not change the corresponding Wronskian solution, and thus, we only have to solve (2.3) and (2.4) under the Jordan form of Λ.

3. Solving the representative systems

Note that the Jordan form of a real matrix Λ has two types of blocks. Therefore, in order to construct Wronskian solutions as in the case of the KdV equation [32], we need to solve the following two representative systems:

$$\phi_{xxx} = \lambda\phi + h, \qquad \phi_t = \pm\sqrt{3}\,\phi_{xx}, \tag{3.1}$$

and

$$\phi_{1,xxx} = \alpha\phi_1 - \beta\phi_2 + h_1, \qquad \phi_{2,xxx} = \beta\phi_1 + \alpha\phi_2 + h_2, \tag{3.2}$$

$$\phi_{1,t} = \pm\sqrt{3}\,\phi_{1,xx}, \qquad \phi_{2,t} = \pm\sqrt{3}\,\phi_{2,xx}, \tag{3.3}$$

where λ, α and $\beta > 0$ are real constants, and $h = h(x,t), h_1 = h_1(x,t)$ and $h_2 = h_2(x,t)$ are three given functions satisfying the compatibility condition $g_t = \pm\sqrt{3}\,g_{xx}$. The first system corresponds to real eigenvalues of the coefficient matrix Λ, while the second system corresponds to complex eigenvalues of the coefficient matrix Λ. In what follows, we will consider both homogenous and non-homogenous equations in all cases of real and complex eigenvalues.

3.1. The case of real eigenvalues

First, let us consider the first representative system (3.1). In terms of the eigenvalue λ, we will establish solution formulae for two situations of the representative system (3.1).

3.1.1. The sub-case of $\lambda = 0$

Obviously, the first differential equation of the representative system (3.1) has a general solution

$$\phi = c_1(t)x^2 + c_2(t)x + c_3(t) + \int_0^x\int_0^\xi\int_0^\eta h(\zeta,t)\,d\zeta\,d\eta\,d\xi. \tag{3.4}$$

This makes it possible to break down the second differential equation of the representative system (3.1) into

$$c_{1t} = \pm\frac{\sqrt{3}}{2}h_x(0,t), \qquad c_{2t} = \pm\sqrt{3}\,h(0,t), \qquad c_{3t} = \pm 2\sqrt{3}\,c_1.$$

Now, integrating these equations with respect to t yields the solution formula of the representative system (3.1):

$$\begin{aligned}\phi = {} & c_{1,0}x^2 + c_{2,0}x + c_{3,0} \pm 2\sqrt{3}\,c_{1,0}t \pm \frac{\sqrt{3}}{2}x^2\int_0^t h_x(0,t')\,dt' \\ & \pm\sqrt{3}\,x\int_0^t h(0,t')\,dt' + 3\int_0^t\int_0^{t'} h_x(0,t'')\,dt''\,dt' + \int_0^x\int_0^\xi\int_0^\eta h(\zeta,t)\,d\zeta\,d\eta\,d\xi,\end{aligned} \tag{3.5}$$

where $c_{1,0}, c_{2,0}$ and $c_{3,0}$ are arbitrary real constants.

3.1.2. The sub-case of $\lambda \neq 0$

In this sub-case, the characteristic equation $\mu^3 = \lambda$ of the first differential equation of the representative system (3.1) has one real root and two conjugate complex roots:

$$\mu_1 = -2a, \qquad \mu_2 = a + bI, \qquad \mu_3 = a - bI, \quad I = \sqrt{-1}, \tag{3.6}$$

where

$$a = -\frac{1}{2}\sqrt[3]{\lambda}, \qquad b = \frac{\sqrt{3}}{2}\sqrt[3]{\lambda}. \tag{3.7}$$

Therefore, the homogeneous equation $\phi_{xxx} = \lambda\phi$ has three fundamental solutions e^{-2ax}, $e^{ax}\sin(bx)$ and $e^{ax}\cos(bx)$, and further, an application of variation of parameters leads to the general solution of the non-homogeneous equation $\phi_{xxx} = \lambda\phi + h$:

$$\begin{aligned}\phi = {} & c_1(t)e^{-2ax} + c_2(t)e^{ax}\cos bx + c_3(t)e^{ax}\sin bx \\ & + \frac{1}{4b^2}\int_0^x \{e^{-2a(x-\xi)} - e^{a(x-\xi)}[\sqrt{3}\,\sin b(x-\xi) + \cos b(x-\xi)]\}\,h(\xi,t)\,d\xi.\end{aligned} \tag{3.8}$$

Owing to $h_t = \pm\sqrt{3}\,h_{xx}$, upon setting

$$g = e^{-2a(x-\xi)} - e^{a(x-\xi)}[\sqrt{3}\,\sin b(x-\xi) + \cos b(x-\xi)], \tag{3.9}$$

we have

$$\begin{aligned}\phi_t \;=\; & c_{1,t}\, \mathrm{e}^{-2ax} + c_{2,t}\, \mathrm{e}^{ax} \cos bx + c_{3,t}\, \mathrm{e}^{ax} \sin bx \pm \frac{\sqrt{3}}{4b^2} \left\{ [-\,\mathrm{e}^{-2ax} + \mathrm{e}^{ax}(\sqrt{3} \sin bx + \cos bx)] h_x(0,t) \right. \\ & \left. + [2a\mathrm{e}^{-2ax} + (\sqrt{3}\, a - b)\mathrm{e}^{ax} \sin bx + (a + \sqrt{3}\, b)\mathrm{e}^{ax} \cos bx] h(0,t) + \int_0^x g_{\xi\xi}\, h(\xi,t)\, \mathrm{d}\xi \right\},\end{aligned}$$

$$\phi_{xx} = 4a^2 c_1 \mathrm{e}^{-2ax} + (a^2 c_2 + 2abc_3 - b^2 c_2)\mathrm{e}^{ax} \cos bx + (a^2 c_3 - 2abc_2 - b^2 c_3)\mathrm{e}^{ax} \sin bx + \frac{1}{4b^2} \int_0^x g_{xx} h(\xi,t)\, \mathrm{d}\xi.$$

Obviously, the second partial derivatives of g are equal to

$$\begin{aligned} g_{\xi\xi} = g_{xx} \;&=\; 4a^2 \mathrm{e}^{-2a(x-\xi)} + (-\sqrt{3}\, a^2 + 2ab + \sqrt{3}\, b^2)\mathrm{e}^{a(x-\xi)} \sin b(x-\xi) \\ &\quad + (-a^2 - 2\sqrt{3}\, ab + b^2)\mathrm{e}^{a(x-\xi)} \cos b(x-\xi) \\ &=\; 4a^2 \mathrm{e}^{-2a(x-\xi)} + (-a^2 - 2\sqrt{3}\, ab + b^2)\mathrm{e}^{a(x-\xi)} \cos b(x-\xi), \end{aligned}$$

where the last equality is due to (3.7). Then, the second differential equation $\phi_t = \pm\sqrt{3}\, \phi_{xx}$ of the representative system (3.1) equivalently requires

$$c_{1,t} \mp 4\sqrt{3}\, a^2 c_1 \pm \frac{\sqrt{3}}{4b^2} [-h_x(0,t) + 2ah(0,t)] = 0, \tag{3.10}$$

$$c_{2,t} \mp \sqrt{3}\, [(a^2 - b^2)c_2 + 2abc_3] \pm \frac{\sqrt{3}}{4b^2} [h_x(0,t) + (a + \sqrt{3}\, b)h(0,t)] = 0, \tag{3.11}$$

$$c_{3,t} \mp \sqrt{3}\, [(a^2 - b^2)c_3 - 2abc_2] \pm \frac{\sqrt{3}}{4b^2} [\sqrt{3}\, h_x(0,t) + (\sqrt{3}\, a - b)h(0,t)] = 0. \tag{3.12}$$

Thus, the solution formula for c_1 is given by

$$c_1(t) = \mathrm{e}^{\pm 4\sqrt{3}\, a^2 t} \left\{ c_{1,0} \mp \frac{\sqrt{3}}{4b^2} \int_0^t \mathrm{e}^{\mp 4\sqrt{3}\, a^2 t'} [-h_x(0,t') + 2ah(0,t')]\, \mathrm{d}t' \right\}, \tag{3.13}$$

where $c_{1,0}$ is an arbitrary real constant. The solution to the system of (3.11) and (3.12) is given by

$$\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \mathrm{e}^{At} \begin{bmatrix} c_{2,0} \\ c_{3,0} \end{bmatrix} + \mathrm{e}^{At} \int_0^t \mathrm{e}^{-As} \begin{bmatrix} g_2(s) \\ g_3(s) \end{bmatrix} \mathrm{d}s, \tag{3.14}$$

where $c_{2,0}$ and $c_{3,0}$ are arbitrary real constants, the coefficient matrix A is defined by

$$A = \mp\sqrt{3} \begin{bmatrix} b^2 - a^2 & -2ab \\ 2ab & b^2 - a^2 \end{bmatrix}, \tag{3.15}$$

and g_2 and g_3 are defined by

$$\begin{bmatrix} g_2(s) \\ g_3(s) \end{bmatrix} = \mp \frac{\sqrt{3}}{4b^2} \begin{bmatrix} h_x(0,t) + (a + \sqrt{3}\, b)h(0,t) \\ \sqrt{3}\, h_x(0,t) + (\sqrt{3}\, a - b)h(0,t) \end{bmatrix}. \tag{3.16}$$

Evidently, the coefficient matrix A has a pair of conjugate eigenvalues

$$\tilde{a} + \tilde{b}I,\ \tilde{a} - \tilde{b}I \ \text{ where } \tilde{a} = \mp\sqrt{3}\,(b^2 - a^2), \qquad \tilde{b} = \mp 2\sqrt{3}\, ab. \tag{3.17}$$

It follows then that

$$\mathrm{e}^{At} = \mathrm{e}^{\tilde{a}t} \begin{bmatrix} \cos \tilde{b}t & -\sin \tilde{b}t \\ \sin \tilde{b}t & \cos \tilde{b}t \end{bmatrix}, \tag{3.18}$$

by which we can obtain the required general solution to the system of (3.11) and (3.12) explicitly.

3.2. The case of complex eigenvalues

Let us now consider the second representative system defined by (3.2) and (3.3). If we set

$$\phi = \phi_1 + \phi_2 \mathrm{I}, \qquad h = h_1 + h_2 \mathrm{I}, \qquad \lambda = \alpha + \beta \mathrm{I},$$

the system is transformed into a complex form of (3.1):

$$\phi_{xxx} = \lambda\phi + h, \qquad \phi_t = \pm\sqrt{3}\, \phi_{xx}. \tag{3.19}$$

The characteristic equation $\mu^3 = \lambda$ of the associated ordinary differential equation has three distinct complex roots

$$\mu_1 = \sqrt[3]{\lambda}, \qquad \mu_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} I\right)\sqrt[3]{\lambda}, \qquad \mu_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} I\right)\sqrt[3]{\lambda}. \tag{3.20}$$

It follows that the solution formula for the first differential equation of (3.19) is determined by

$$\phi = \nu_1(t)e^{\mu_1 x} + \nu_2(t)e^{\mu_2 x} + \nu_3(t)e^{\mu_3 x} + \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)(\mu_2-\mu_1)} \times \int_0^x [(\mu_3-\mu_2)e^{\mu_1(x-x')} - (\mu_3-\mu_1)e^{\mu_2(x-x')} + (\mu_2-\mu_1)e^{\mu_3(x-x')}]h(x',t)dx'. \tag{3.21}$$

Now, because of $h_t = \pm\sqrt{3}\,h_{xx}$, similarly to the sub-case of real eigenvalues $\lambda \neq 0$, we can see that the second differential equation of (3.19) equivalently requires

$$\nu_{1,t} = \pm\sqrt{3}\,\mu_1^2\nu_1 \pm \frac{\sqrt{3}}{(\mu_3-\mu_1)(\mu_2-\mu_1)}[h_x(0,t) + \mu_1 h(0,t)],$$
$$\nu_{2,t} = \pm\sqrt{3}\,\mu_2^2\nu_2 \mp \frac{\sqrt{3}}{(\mu_3-\mu_2)(\mu_2-\mu_1)}[h_x(0,t) + \mu_2 h(0,t)],$$
$$\nu_{3,t} = \pm\sqrt{3}\,\mu_3^2\nu_3 \pm \frac{\sqrt{3}}{(\mu_3-\mu_1)(\mu_3-\mu_2)}[h_x(0,t) + \mu_3 h(0,t)],$$

which engender

$$\nu_1 = e^{\pm\sqrt{3}\mu_1^2 t}\left\{\nu_{1,0} \pm \frac{\sqrt{3}}{(\mu_3-\mu_1)(\mu_2-\mu_1)}\int_0^t e^{\mp\sqrt{3}\mu_1^2 t'}[h_x(0,t') + \mu_1 h(0,t')]\,dt'\right\}, \tag{3.22}$$

$$\nu_2 = e^{\pm\sqrt{3}\mu_2^2 t}\left\{\nu_{2,0} \mp \frac{\sqrt{3}}{(\mu_3-\mu_2)(\mu_2-\mu_1)}\int_0^t e^{\mp\sqrt{3}\mu_2^2 t'}[h_x(0,t') + \mu_2 h(0,t')]\,dt'\right\}, \tag{3.23}$$

$$\nu_3 = e^{\pm\sqrt{3}\mu_3^2 t}\left\{\nu_{3,0} \pm \frac{\sqrt{3}}{(\mu_3-\mu_1)(\mu_3-\mu_2)}\int_0^t e^{\mp\sqrt{3}\mu_3^2 t'}[h_x(0,t') + \mu_3 h(0,t')]\,dt'\right\}, \tag{3.24}$$

where $\nu_{1,0}$, $\nu_{2,0}$ and $\nu_{3,0}$ are arbitrary complex constants.

Therefore, the general solution of the second representative system of (3.2) and (3.3) is given by (3.21) with (3.22)–(3.24). We can obtain the real solution to the system of (3.2) and (3.3) by taking the real and imaginary parts of the above general complex solution. But by choosing conjugate pairs of complex eigenvalues, we can avoid computing real and imaginary parts of the above solutions while computing real solutions. Such application examples will be given in the next section.

4. Wronskian solutions

Sections 2 and 3 provide a general procedure for constructing Wronskian solutions associated with two types of Jordan blocks of the coefficient matrix Λ. In what follows, we will present a few specific procedures for constructing different kinds of Wronskian solutions to the Boussinesq equation I in (1.2), together with examples of exact solutions. We will only consider the case of $\varepsilon = -1$. The case of $\varepsilon = 1$ is just an action of replacing t with $-t$ in the obtained solutions.

4.1. Rational solutions

Consider the case of $\lambda = 0$ and define

$$\psi_{0,xxx} = 0, \qquad \psi_{i+1,xxx} = \psi_i, \qquad \psi_{i,t} = -\sqrt{3}\,\psi_{i,xx}, \quad i \geq 0. \tag{4.1}$$

It follows from Section 3 that such functions ψ_i, $i \geq 0$, are all polynomials in x and t, and a general Wronskian solution

$$u = 6\partial_x^2 \ln W(\psi_0, \psi_1, \ldots, \psi_{k-1})$$

corresponding to the following Jordan block:

$$\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{k\times k}, \tag{4.2}$$

is rational and is called a rational Wronskian solution of order $k-1$. Since ψ_0 has three linearly independent solutions, let us say $\psi_{1,0}$, $\psi_{2,0}$ and $\psi_{3,0}$, we can have two other general rational Wronskian solutions:

$$u = 6\partial_x^2 \ln W(\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,k_1-1}; \psi_{2,0}, \psi_{2,1}, \ldots, \psi_{2,k_2-1}),$$
$$u = 6\partial_x^2 \ln W(\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,k_1-1}; \psi_{2,0}, \psi_{2,1}, \ldots, \psi_{2,k_2-1}; \psi_{3,0}, \psi_{3,1}, \ldots, \psi_{3,k_3-1}),$$

where $\psi_{i,0}, \psi_{i,1}, \ldots, \psi_{i,k_i-1}$, $1 \le i \le 3$, are three sets of functions corresponding to three Jordan blocks of the above type. To reflect the size of their associated Jordan blocks, we call such two solutions the rational Wronskian solutions of orders (k_1-1, k_2-1) and (k_1-1, k_2-1, k_3-1), respectively.

(a) Zero-order: Taking $\psi_0 = c_1 + c_2 x + c_3(x^2 - 2\sqrt{3}\,t)$, the corresponding Wronskian determinant and the associated rational Wronskian solution of zero-order read

$$f = W(\psi_0) = c_1 + c_2 x + c_3(x^2 - 2\sqrt{3}\,t),$$
$$u = 6\partial_x^2 \ln W(\phi_0) = \frac{6(2\,c_1c_3 - c_2^2 - 2\,c_2c_3x - 2\,c_3^2x^2 - 4\sqrt{3}\,c_3{}^2t)}{[c_1 + c_2x + c_3(x^2 - 2\sqrt{3}\,t)]^2},$$

where c_1, c_2 and c_3 are arbitrary constants.

If taking $\psi_0 = x$ and $\psi_0 = x^2 - 2\sqrt{3}\,t$, we have

$$u = -\frac{1}{x^2}, \qquad u = -\frac{12(x^2 + 2\sqrt{3}\,t)}{(x^2 - 2\sqrt{3}\,t)^2}.$$

(b) First-order: Taking $\psi_0 = 1$, we can have $\psi_1 = \frac{1}{6}x^3 - \sqrt{3}\,xt$. Then, the corresponding Wronskian determinant and rational Wronskian solution of first-order are

$$f = W(\psi_0, \psi_1) = \frac{1}{2}x^2 - \sqrt{3}\,t,$$
$$u = 6\partial_x^2 \ln W(\psi_0, \psi_1) = -\frac{12(x^2 + 2\sqrt{3}\,t)}{(x^2 - 2\sqrt{3}\,t)^2}.$$

This solution is exactly the same as one of the previous solutions of zero-order.

Taking $\psi_0 = x$, we can have $\psi_1 = \frac{1}{24}x^4 - \frac{\sqrt{3}}{2}x^2t + \frac{3}{2}t^2$. In this case, the corresponding Wronskian determinant and rational Wronskian solution of first-order read

$$f = W(\psi_0, \psi_1) = \frac{1}{8}x^4 - \frac{\sqrt{3}}{2}x^2t - \frac{3}{2}t^2,$$
$$u = 6\partial_x^2 \ln W(\psi_0, \psi_1) = -\frac{24(x^6 - 2\sqrt{3}\,x^4t + 60x^2t^2 - 24\sqrt{3}\,t^3)}{(x^4 - 4\sqrt{3}\,x^2t - 12t^2)^2}.$$

Taking $\psi_0 = x^2 - 2\sqrt{3}\,t$, we can have $\psi_1 = \frac{1}{60}x^5 - \frac{\sqrt{3}}{3}x^3t + 3xt^2$. Then, we have the following Wronskian determinant and rational Wronskian solution of first-order:

$$f = W(\psi_0, \psi_1) = \frac{1}{20}x^6 - \frac{\sqrt{3}}{2}x^4t + 3x^2t^2 - 6\sqrt{3}\,t^3$$
$$u = 6\partial_x^2 \ln W(\psi_0, \psi_1) = -\frac{36(x^{10} - 10\sqrt{3}\,x^8t + 120x^6t^2 + 400\sqrt{3}\,x^4t^3 - 6000x^2t^4 + 2400\sqrt{3}\,t^5)}{(x^6 - 10\sqrt{3}\,x^4t + 60x^2t^2 - 120\sqrt{3}\,t^3)^2}.$$

(c) Second-order: Taking $\psi_0 = x$, we can have $\psi_1 = \frac{1}{24}x^4 - \frac{\sqrt{3}}{2}x^2t + \frac{3}{2}t^2$ and

$$\psi_2 = \frac{1}{5040}\,x^7 - \frac{\sqrt{3}}{120}x^5t + \frac{1}{4}x^3\,t^2 - \frac{\sqrt{3}}{2}xt^3.$$

A direct computation shows that the corresponding Wronskian determinant and rational Wronskian solution of third-order are given by

$$f = W(\psi_0, \psi_1, \psi_2) = \frac{1}{2240}\,x^9 - \frac{\sqrt{3}}{140}\,x^7t + \frac{3}{40}\,x^5t^2 - \frac{9}{4}\,xt^4,$$
$$u = 6\partial_x^2 \ln W(\psi_0, \psi_1, \psi_3) = -\frac{18p}{x^2q^2},$$

where

$$\begin{aligned} p &= 3x^{16} - 64\sqrt{3}\,x^{14}t + 1680\,x^{12}t^2 - 7168\sqrt{3}\,x^{10}t^3 + 137\,760\,x^8t^4 \\ &\quad - 752\,640\sqrt{3}\,x^6t^5 + 2\,822\,400\,x^4t^6 + 8\,467\,200\,t^8, \\ q &= -x^8 + 16\sqrt{3}\,x^6t - 168\,x^4t^2 + 5040\,t^4. \end{aligned}$$

(d) (1, 1)-order: Taking $\psi_{1,0} = 1,\ \psi_{1,1} = \frac{1}{6}x^3 - \sqrt{3}\,xt$ and $\psi_{2,0} = x,\ \psi_{2,1} = \frac{1}{24}x^4 - \frac{\sqrt{3}}{2}x^2t + \frac{3}{2}t^2$, we can have the following Wronskian determinant and rational Wronskian solution of (1, 1)-order:

$$\begin{aligned} f &= W(\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}) = -\frac{1}{2}x^2 - \sqrt{3}\,t, \\ u &= 6\partial_x^2 \ln W(\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}) = -\frac{12(x^2 - 2\sqrt{3}\,t)}{(x^2 + 2\sqrt{3}\,t)^2}. \end{aligned}$$

Taking $\psi_{1,0} = 1,\ \psi_{1,1} = \frac{1}{6}x^3 - \sqrt{3}\,xt$ and $\psi_{2,0} = x^2 - 2\sqrt{3}\,t,\ \psi_{2,1} = \frac{1}{60}x^5 - \frac{\sqrt{3}}{3}x^3t + 3xt^2$, the corresponding Wronskian determinant and rational Wronskian solution of (1, 1)-order read

$$\begin{aligned} f &= W(\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}) = -\frac{1}{2}x^4 - 2\sqrt{3}\,x^2t + 6t^2, \\ u &= 6\partial_x^2 \ln W(\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}) = -\frac{24(x^6 + 2\sqrt{3}\,x^4t + 60x^2t^2 + 24\sqrt{3}\,t^3)}{(x^4 + 4\sqrt{3}\,x^2t - 12t^2)^2}. \end{aligned}$$

Taking $\psi_{1,0} = x,\ \psi_{1,1} = \frac{1}{24}x^4 - \frac{\sqrt{3}}{2}x^2t + \frac{3}{2}t^2$ and $\psi_{2,0} = x^2 - 2\sqrt{3}\,t,\ \psi_{2,1} = \frac{1}{60}x^5 - \frac{\sqrt{3}}{3}x^3t + 3xt^2$, the corresponding Wronskian determinant and rational Wronskian solution of (1, 1)-order are

$$\begin{aligned} f &= W(\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}) = -\frac{1}{20}\,x^6 - \frac{\sqrt{3}}{2}\,x^4t - 3\,x^2t^2 - 6\sqrt{3}\,t^3, \\ u &= 6\partial_x^2 \ln W(\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}) \\ &= -\frac{36(x^{10} + 10\sqrt{3}\,x^8t + 120\,x^6t^2 - 400\sqrt{3}\,x^4t^3 - 6000\,x^2t^4 - 2400\sqrt{3}\,t^5)}{(x^6 + 10\sqrt{3}\,x^4t + 60\,x^2t^2 + 120\sqrt{3}\,t^3)^2}. \end{aligned}$$

4.2. Wronskian solutions associated with nonzero real eigenvalues

For each real eigenvalue $\lambda_i \neq 0$, we start from the eigenfunction $\phi_i(\lambda_i)$, which is determined by

$$(\phi_i(\lambda_i))_{xxx} = \lambda_i\phi_i(\lambda_i),\ (\phi_i(\lambda_i))_t = -\sqrt{3}\,(\phi_i(\lambda_i))_{xx}. \tag{4.3}$$

The general solution to this system is

$$\begin{aligned} \phi_i(\lambda_i) &= c_{1i}\mathrm{e}^{-2a_ix - 4\sqrt{3}\,a_i^2t} + c_{2i}\mathrm{e}^{a_ix + \sqrt{3}\,(b_i^2 - a_i^2)t}\cos(b_i(x - 2\sqrt{3}\,a_it)) \\ &\quad + c_{3i}\mathrm{e}^{a_ix + \sqrt{3}\,(b_i^2 - a_i^2)t}\sin(b_i(x - 2\sqrt{3}\,a_it)), \end{aligned} \tag{4.4}$$

where c_{1i}, c_{2i} and c_{3i} are arbitrary real constants, and $-2a_i$, $a_i + b_i\mathrm{I}$ and $a_i - b_i\mathrm{I}$ are a set of solutions of $\mu^3 = \lambda_i$.

To construct Wronskian solutions corresponding to Jordan blocks of higher-order, we use the basic idea developed for the KdV equation [3]. Differentiating (4.3) with respect to λ_i, we can find that the vector function

$$\Phi_i = \Phi_i(\lambda_i) = \left(\phi_i(\lambda_i), \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i), \ldots, \frac{1}{(k_i - 1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i)\right)^{\mathrm{T}}, \tag{4.5}$$

satisfies

$$\Phi_{i,xxx} = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i\times k_i} \Phi_i,\quad \Phi_{i,t} = -\sqrt{3}\,\Phi_{i,xx},$$

where ∂_{λ_i} denotes the derivative with respect to λ_i and k_i is an arbitrary natural number. Therefore, through this set of eigenfunctions, we obtain a Wronskian solution:

$$u = 2\partial_x^2 \ln W\left(\phi_i(\lambda_i), \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i), \ldots, \frac{1}{(k_i - 1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i)\right). \tag{4.6}$$

When $c_{1i} = 0$, we can get positon solutions, i.e., solutions involving only one kind of transcendental functions — trigonometric functions. When $c_{1i}c_{2i} \neq 0$ or $c_{1i}c_{3i} \neq 0$, we can get complexiton solutions, i.e., solutions involving two kinds of transcendental functions — exponential functions and trigonometric functions.

A more general Wronskian solution of this type can be obtained by combining n sets of eigenfunctions associated with distinct real eigenvalues $\lambda_i \neq 0$:

$$u = 2\partial_x^2 \ln W\left(\phi_1(\lambda_1), \ldots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1)\right); \cdots; \phi_n(\lambda_n), \ldots, \frac{1}{(k_n-1)!}\partial_{\lambda_n}^{k_n-1}\phi_n(\lambda_n)). \tag{4.7}$$

This solution is an n-positon or n-complexiton of order $(k_1-1, k_2-1, \ldots, k_n-1)$, associated with real eigenvalues of the coefficient matrix $\Lambda = (\lambda_{ij})$.

(a) Positons: Two kinds of special positons of order k, associated with nonzero real eigenvalues of Λ, are

$$u = 6\partial_x^2 \ln W(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k-1}\phi), \quad \phi = e^{ax+\sqrt{3}\,(b^2-a^2)t}\cos(b(x-2\sqrt{3}\,at)+d), \tag{4.8}$$

$$u = 6\partial_x^2 \ln W(\psi, \partial_\lambda \psi, \ldots, \partial_\lambda^{k-1}\psi), \quad \psi = e^{ax+\sqrt{3}\,(b^2-a^2)t}\sin(b(x-2\sqrt{3}\,at)+d), \tag{4.9}$$

where $-2a$, $a+b\mathrm{I}$ and $a-b\mathrm{I}$ are three roots of $\mu^3 = \lambda\ (\lambda \neq 0)$ and d is an arbitrary constant.

Such two positons of zero-order are

$$u = 6\partial_x^2 \ln(\phi) = -\frac{9c^2}{2\cos^2\left(\frac{\sqrt{3}}{2}cx + \frac{3}{2}c^2t + d\right)}, \tag{4.10}$$

$$u = 6\partial_x^2 \ln(\psi) = -\frac{9c^2}{2\sin^2\left(\frac{\sqrt{3}}{2}cx + \frac{3}{2}c^2t + d\right)}, \tag{4.11}$$

where $c = \sqrt[3]{\lambda}$. Noting that

$$a_\lambda = -\frac{1}{24a^2}, \qquad b_\lambda = \frac{3\sqrt{3}}{24b}.$$

We can have the corresponding two Wronskian determinants and two positons of first-order:

$$\begin{aligned}
f &= W(\phi, \partial_\lambda\phi) = -\frac{1}{12c^2}\,e^{-cx+\sqrt{3}c^2t}(2\cos^2\xi + \sqrt{3}\,\sin 2\xi + 3\,cx + 6\,\sqrt{3}\,c^2t),\\
u &= 6\partial_x^2 \ln W(\phi, \partial_\lambda\phi)\\
&= -\frac{18c^2[(14+6cx+12\sqrt{3}\,c^2t)\cos^2\xi + (-\sqrt{3}\,+3\sqrt{3}\,cx+18c^2t)\sin 2\xi - 3cx - 6\sqrt{3}\,c^2t]}{(2\cos^2\xi + \sqrt{3}\,\sin 2\xi + 3cx + 6\sqrt{3}\,c^2t)^2},\\
f &= W(\psi, \partial_\lambda\psi) = -\frac{1}{12c^2}\,e^{-cx+\sqrt{3}c^2t}(2\sin^2\xi - \sqrt{3}\,\sin 2\xi + 3\,cx + 6\,\sqrt{3}\,c^2t),\\
u &= 6\partial_x^2 \ln W(\psi, \partial_\lambda\psi)\\
&= -\frac{18c^2[(14+6cx+12\sqrt{3}\,c^2t)\sin^2\xi - (-\sqrt{3}\,+3\sqrt{3}\,cx+18c^2t)\sin 2\xi - 3cx - 6\sqrt{3}\,c^2t]}{(2\sin^2\xi - \sqrt{3}\,\sin 2\xi + 3cx + 6\sqrt{3}\,c^2t)^2},
\end{aligned}$$

where

$$\xi = \frac{1}{2}\sqrt{3}\,cx + \frac{3}{2}c^2t + d,\ c = \sqrt[3]{\lambda}.$$

(b) Complexitons: Let us take the following general eigenfunction

$$\begin{aligned}
\phi &= c_1 e^{-2ax-4\sqrt{3}\,a^2t} + c_2 e^{ax+\sqrt{3}\,(b^2-a^2)t}\cos(b(x-2\sqrt{3}\,a)t)\\
&\quad + c_3 e^{ax+\sqrt{3}\,(b^2-a^2)t}\sin(b(x-2\sqrt{3}\,a)t),
\end{aligned}$$

where $-2a$, $a+b\mathrm{I}$ and $a-b\mathrm{I}$ are three roots of $\mu^3 = \lambda\ (\lambda \neq 0)$. Assume that

$$c_1c_2 \neq 0 \quad \text{or } c_1c_3 \neq 0,$$

which keeps two kinds of transcendental functions – exponential functions and trigonometric functions – in play. Therefore, we obtain the following complexiton solution

$$u = 6\partial_x^2 \ln W(\phi) = \frac{f}{g},$$

$$f = \frac{9}{2}c^2\left[2c_1(c_2 - \sqrt{3}c_3)\cos\xi + 2c_1(\sqrt{3}c_2 + c_3)\sin\xi - (c_2^2 + c_3^2)e^{\frac{3}{2}c(-x+\sqrt{3}ct)}\right]$$

$$g = 2c_1c_2\cos\xi + 2c_1c_3\sin\xi + c_2^2e^{\frac{3}{2}c(-x+\sqrt{3}ct)}\cos^2\xi + c_3^2e^{\frac{3}{2}c(-x+\sqrt{3}ct)}\sin^2\xi$$
$$+c_2c_3e^{\frac{3}{2}c(-x+\sqrt{3}ct)}\sin 2\xi + c_1^2e^{-\frac{3}{2}c(-x+\sqrt{3}ct)},$$

where

$$\xi = \frac{1}{2}\sqrt{3}cx + \frac{3}{2}c^2t, \quad c = \sqrt[3]{\lambda}.$$

This solution is associated with a real eigenvalue of the coefficient matrix Λ, but it corresponds to a pair of conjugate complex roots of the characteristic equation of the first differential equation in the linear conditions.

4.3. Wronskian solutions associated with complex eigenvalues

This case only leads to complexiton solutions. For each complex eigenvalue $\lambda_i = \alpha_i + \beta_i \mathrm{I}$, we start from a pair of eigenfunctions $(\phi_{i,1}(\alpha_i, \beta_i), \phi_{i,2}(\alpha_i, \beta_i))$ determined by

$$\Phi_{i,xxx} = A_i\Phi_i, \qquad \Phi_{i,t} = -\sqrt{3}\,\Phi_{i,xx}, \qquad A_i = \begin{bmatrix}\alpha_i & -\beta_i\\ \beta_i & \alpha_i\end{bmatrix}, \qquad \Phi_i = \begin{bmatrix}\phi_{i,1}(\alpha_i, \beta_i)\\ \phi_{i,2}(\alpha_i, \beta_i)\end{bmatrix}. \tag{4.12}$$

As shown before, this system is equivalent to

$$\phi_{i,xxx} = \lambda_i\phi_i, \qquad \phi_{i,t} = -\sqrt{3}\,\phi_{i,xx} \quad \text{with } \phi_i = \phi_{i,1} + \phi_{i,2}I, \quad \lambda_i = \alpha_i + \beta_i I. \tag{4.13}$$

From the previous section, we know that the general solution to the above system is given by

$$\phi_i = \nu_{1,0}^{(i)}\,e^{\mu_1^{(i)}(x-\sqrt{3}\mu_1^{(i)}t)} + \nu_{2,0}^{(i)}\,e^{\mu_2^{(i)}(x-\sqrt{3}\mu_2^{(i)}t)} + \nu_{3,0}^{(i)}\,e^{\mu_3^{(i)}(x-\sqrt{3}\mu_3^{(i)}t)}, \tag{4.14}$$

where $\mu_1^{(i)}$, $\mu_2^{(i)}$ and $\mu_3^{(i)}$ are three distinct complex roots of $\mu^3 = \lambda_i$ and $\nu_{1,0}^{(i)},\ \nu_{2,0}^{(i)}$ and $\nu_{3,0}^{(i)}$ are arbitrary constants.

Similarly, differentiating (4.12) with respect to α_i, we can see that

$$\begin{bmatrix}\Phi_i\\ \frac{1}{1!}\partial_{\alpha_i}\Phi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{l_i-1}\Phi_i\end{bmatrix}_{xxx} = \begin{bmatrix}A_i & & & 0\\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i\end{bmatrix}_{l_i\times l_i}\begin{bmatrix}\Phi_i\\ \frac{1}{1!}\partial_{\alpha_i}\Phi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{l_i-1}\Phi_i\end{bmatrix},$$

and

$$\left(\frac{1}{j!}\partial_{\alpha_i}^j\Phi_i\right)_t = -\sqrt{3}\left(\frac{1}{j!}\partial_{\alpha_i}^j\Phi_i\right)_{xx}, \quad 0 \le j \le l_i - 1.$$

Taking the derivative with respect to β_i, we can have

$$\begin{bmatrix}\Phi_i\\ \frac{1}{1!}\partial_{\beta_i}\Phi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\beta_i}^{l_i-1}\Phi_i\end{bmatrix}_{xxx} = \begin{bmatrix}A_i & & & 0\\ \Sigma_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & \Sigma_2 & A_i\end{bmatrix}_{l_i\times l_i}\begin{bmatrix}\Phi_i\\ \frac{1}{1!}\partial_{\beta_i}\Phi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\beta_i}^{l_i-1}\Phi_i\end{bmatrix},$$

and

$$\left(\frac{1}{j!}\partial_{\beta_i}^j\Phi_i\right)_t = -\sqrt{3}\left(\frac{1}{j!}\partial_{\beta_i}^j\Phi_i\right)_{xx}, \quad 0 \le j \le l_i - 1,$$

where

$$\Sigma_2 = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

Associated with those two sets of eigenfunctions, we have two Wronskian solutions to the Boussinesq equation I (1.2):

$$u = 6\partial_x^2 \ln W\left(\Phi_i^{\mathrm{T}}, \frac{1}{1!}\partial_{\alpha_i}\Phi_i^{\mathrm{T}}, \ldots, \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{l_i-1}\Phi_i^{\mathrm{T}}\right), \tag{4.15}$$

and

$$u = 6\partial_x^2 \ln W\left(\Phi_i^{\mathrm{T}}, \frac{1}{1!}\partial_{\beta_i}\Phi_i^{\mathrm{T}}, \ldots, \frac{1}{(l_i-1)!}\partial_{\beta_i}^{l_i-1}\Phi_i^{\mathrm{T}}\right). \tag{4.16}$$

A more general Wronskian solution is given by

$$u = 6\partial_x^2 \ln W\left(\Phi_1^{\mathrm{T}}, \ldots, \frac{1}{(l_1-1)!}\partial_{\zeta_1}^{l_1-1}\Phi_1^{\mathrm{T}}; \cdots; \Phi_n^{\mathrm{T}}, \ldots, \frac{1}{(l_n-1)!}\partial_{\zeta_n}^{l_n-1}\Phi_n^{\mathrm{T}}\right), \tag{4.17}$$

where ∂_{ζ_i} could be either of ∂_{α_i} or ∂_{β_i}. This solution is called an n-complexiton solution of order $(l_1-1, l_2-1, \ldots, l_n-1)$, to reflect the orders of derivatives of eigenfunctions with respect to eigenvalues. If $l_i = 1,\ 1 \le i \le n$, we simply say that it is an n-complexiton.

Taking into consideration that

$$W(\Re\phi, \Im\phi) = \begin{vmatrix} \Re\phi & \Re\phi_x \\ \Im\phi & \Im\phi_x \end{vmatrix} = \frac{1}{4}\begin{vmatrix} 1 & 1 \\ -\mathrm{I} & \mathrm{I} \end{vmatrix}\begin{vmatrix} \phi & \phi_x \\ \bar\phi & \bar\phi_x \end{vmatrix} = \frac{1}{2}\mathrm{I}\begin{vmatrix} \phi & \phi_x \\ \bar\phi & \bar\phi_x \end{vmatrix},$$

we can avoid computing real and imaginary parts of the complex eigenfunctions by choosing conjugate pairs of complex eigenvalues, while computing real solutions. Here are a few concrete examples of such complexitons of zero-order.

We choose a complex eigenvalue $\lambda = -b^3\mathrm{I}\,(b \neq 0)$, then the eigenfunction ϕ defined by (3.21) with $h = 0$ reads

$$\phi = \nu_{1,0}\,\mathrm{e}^{b\,(x\mathrm{I}+\sqrt{3}\,bt)} + \nu_{2,0}\,\mathrm{e}^{-\frac{1}{2}\,b\,(x\mathrm{I}+\sqrt{3}\,x+\sqrt{3}\,bt+3\,bt\mathrm{I})} + \nu_{3,0}\,\mathrm{e}^{-\frac{1}{2}\,b\,(x\mathrm{I}-\sqrt{3}\,x+\sqrt{3}\,bt-3\,bt\mathrm{I})}, \tag{4.18}$$

where $\nu_{1,0}$, $\nu_{2,0}$ and $\nu_{3,0}$ are three arbitrary constants.

First take $\nu_{1,0} = c,\ \nu_{2,0} = d$ and $\nu_{3,0} = 0$. Then, the corresponding Wronskian determinant and complexiton solution of zero-order read

$$\begin{aligned}
f &= W(\Re\phi, \Im\phi) = b\,p,\\
p &= \frac{1}{2}\,c\,d\,\mathrm{e}^{\frac{1}{2}\sqrt{3}\,b\,(b\,t-x)}\cos\left(\frac{3}{2}\,b\,(x+b\,t)\right) + c^2\,\mathrm{e}^{2\sqrt{3}\,b^2\,t}\\
&\quad + \frac{1}{2}\sqrt{3}\;c\,d\,\mathrm{e}^{\frac{\sqrt{3}}{2}\,b\,(b\,t-x)}\sin\left(\frac{3}{2}\,b\,(x+b\,t)\right) - \frac{1}{2}\,d^2\,\mathrm{e}^{-\sqrt{3}\,b\,(x+b\,t)},\\
u &= 6\partial_x^2 \ln W(\Re\phi, \Im\phi) = \frac{9}{2}b^2cd\frac{q}{p^2},\\
q &= -d^2\,\mathrm{e}^{-\frac{\sqrt{3}}{2}\,b\,(b\,t+3\,x)}\cos\left(\frac{3}{2}\,b\,(x+b\,t)\right) - 4\,c^2\,\mathrm{e}^{\frac{\sqrt{3}}{2}\,b\,(5\,b\,t-x)}\cos\left(\frac{3}{2}\,b\,(x+b\,t)\right)\\
&\quad - 5\,c\,d\,\mathrm{e}^{\sqrt{3}\,b\,(b\,t-x)} + \sqrt{3}\,d^2\,\mathrm{e}^{-\frac{\sqrt{3}}{2}\,b\,(b\,t+3\,x)}\sin\left(\frac{3}{2}\,b\,(x+b\,t)\right),
\end{aligned}$$

where b, c and d are arbitrary constants.

Second take $\nu_{1,0} = c,\ \nu_{3,0} = d$ and $\nu_{2,0} = 0$. Then the corresponding Wronskian determinant and complexiton solution of zero-order are

$$\begin{aligned}
f &= W(\Re\phi, \Im\phi) = b\,p,\\
p &= -\frac{1}{2}\,d^2\,\mathrm{e}^{-\sqrt{3}\,b\,(b\,t-x)} + \frac{1}{2}\,c\,d\,\mathrm{e}^{\frac{\sqrt{3}}{2}\,b\,(x+b\,t)}\cos\left(\frac{3}{2}\,b\,(b\,t-x)\right)\\
&\quad + c^2\,\mathrm{e}^{2\sqrt{3}\,b^2\,t} + \frac{\sqrt{3}}{2}\,c\,d\,\mathrm{e}^{\frac{\sqrt{3}}{2}\,b\,(x+b\,t)}\sin\left(\frac{3}{2}\,b\,(b\,t-x)\right),\\
u &= 6\partial_x^2 \ln W(\Re\phi, \Im\phi) = \frac{9}{2}b^2cd\frac{q}{p^2},\\
q &= -d^2\,\mathrm{e}^{-\frac{\sqrt{3}}{2}\,b\,(b\,t-3\,x)}\cos\left(\frac{3}{2}\,b\,(b\,t-x)\right) - 5\,c\,d\,\mathrm{e}^{\sqrt{3}\,b\,(x+b\,t)}\\
&\quad - 4\,c^2\,\mathrm{e}^{\frac{\sqrt{3}}{2}\,b\,(5\,b\,t+x)}\cos\left(\frac{3}{2}\,b\,(b\,t-x)\right) + \sqrt{3}\,d^2\,\mathrm{e}^{-\frac{\sqrt{3}}{2}\,b\,(b\,t-3\,x)}\sin\left(\frac{3}{2}\,b\,(b\,t-x)\right),
\end{aligned}$$

where b, c and d are arbitrary constants.

Third take $\nu_{1,0} = 0,\ \nu_{2,0} = c$ and $\nu_{3,0} = d$. Then we have the following Wronskian determinant and complexiton solution of zero-order:

$$f = W(\Re\phi, \Im\phi) = \frac{1}{2}\,b\,p,$$

$$p=-2c\ d\,\mathrm{e}^{-\sqrt{3}\,b^2\,t}\cos(3\,b^2\,t)+2\,\sqrt{3}\,c\,d\,\mathrm{e}^{-\sqrt{3}\,b^2\,t}\sin(3\,b^2\,t)-c^2\,\mathrm{e}^{-\sqrt{3}\,b\,(x+b\,t)}-d^2\,\mathrm{e}^{-\sqrt{3}\,b\,(-x+b\,t)}$$

$$u=6\partial_x^2\ln W(\Re\phi,\Im\phi)=36b^2cd\frac{q}{p^2},$$

$$q\ =\ c^2\mathrm{e}^{-\sqrt{3}\,b\,(2\,b\,t+x)}\cos(3\,b^2\,t)+d^2\,\mathrm{e}^{-\sqrt{3}\,b\,(2\,b\,t-x)}\cos(3\,b^2\,t)-\sqrt{3}\,c^2\,\mathrm{e}^{-\sqrt{3}\,b\,(2\,b\,t+x)}\sin(3\,b^2\,t)$$
$$-\sqrt{3}\,d^2\,\mathrm{e}^{-\sqrt{3}\,b\,(2\,b\,t-x)}\sin(3\,b^2\,t)+2\,c\,d\,\mathrm{e}^{-2\,\sqrt{3}\,b^2\,t},$$

where b, c and d are arbitrary constants.

5. Concluding remarks

A new Wronskian formulation leading to rational solutions to the Boussinesq equation I has been presented by means of its bilinear form. By solving the representative systems of the linear conditions in the Wronskian formulation, rational solutions, positons and complexitons to the Boussinesq equation I (1.2) are computed explicitly. The resulting theory provides us with an effective way to construct exact solutions, which enriches the solution structure of the Boussinesq equation I (1.2).

Our Wronskian formulation yields various solutions through different Jordan canonical forms of the coefficient matrix in the linear conditions (2.3). However, the question of whether the eigenvalues of the coefficient matrix could describe the stability of the resulting solutions remains to be answered. Any study on this would help us understand the nonlinear effects of the Boussinesq equation more deeply.

It can also be directly checked that any polynomial solution to the Boussinesq equation I (1.2) is of the following form

$$u=c_1+c_2x+c_3t+c_4xt-{c_2}^2t^2-\frac{2}{3}c_2c_4t^3-\frac{1}{6}{c_4}^2t^4,$$

where $c_i,\ 1\le i\le 4$, are arbitrary constants. Similar polynomial solutions to the Boussinesq equation II (1.4) were presented in [33]. There also exits many other solutions such as soliton solutions [34], rational solutions [30,35,36] and breezer-type solutions [37,33]. Blow-up of its solutions was shown for the Eq. (1.2) in the case of special initial conditions [38]. Dynamics of nonlinear waves determined by the equation was studied both numerically and analytically [15–17]. But unfortunately, a similar Wronskian formulation does not work for real solutions of the Boussinesq equation II. A special formulation [31] only presents complex solutions, but no real solutions.

In addition to rational solutions, positons and complexitons to the Boussinesq equation I, one can also construct interaction solutions between any two different kinds of solutions within the established Wronskian formulation (see [32,4] for the cases of the KdV equation and the Toda lattice equation). There exists algebro-geometric solutions to the Boussinesq equation I (1.2) on the circle [25]. All this shows the richness of the solution space of the Boussinesq equation and the resulting solutions are expected to help understand wave dynamics in weakly nonlinear and dispersive media.

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