Commun Nonlinear Sci Numer Simulat 17 (2012) 3795–3801

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Invariant subspaces and exact solutions of a class of dispersive evolution equations

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ARTICLE INFO

Article history:
Received 4 January 2012
Accepted 18 February 2012
Available online 2 March 2012

Keywords:
Invariant subspace
Refined invariant subspace method
Soliton

ABSTRACT

The invariant subspace method is used to classify a class of systems of nonlinear dispersive evolution equations and determine their invariant subspaces and exact solutions. A crucial step is to take subspaces of solutions to linear ordinary differential equations as invariant subspaces that systems of evolution equations admit. A few examples of presenting exact solutions with generalized separated variables illustrate the effectiveness of the invariant subspace method in solving systems of nonlinear evolution equations.

1. Introduction

The invariant subspace method, recently proposed in [1,2] and refined in [3], is one of powerful methods to construct exact solutions to nonlinear evolution equations. Various invariant subspaces defined as subspaces of solutions to linear ordinary differential equations have been used to solve special nonlinear evolution equations (see, e.g. [4–6]), particularly nonlinear evolution equations in mechanics and physics (see, e.g. Galaktionov and Svirshchevskii's book [4]). Evolution equations that admit invariant subspaces define symmetries of given linear ordinary differential equations [5,7].

There are two important aspects on the invariant subspace method. One is that the linear superposition principle has a good effect on the formulation of exact solutions to nonlinear evolution equations. It is known that N-soliton solutions to soliton equations such as the KdV equation, the mKdV equation, the nonlinear Schrödinger equation and the sine–Gordon equation, derived by Hirota's bilinear method [8], are all in a linear space of exponential functions under change of variables, and the linear superposition principle plays an important role in presenting soliton, negaton and complexiton solutions [4], [8–14]. The other is the generalized separation of variables of either dependent variables [15,16] or independent variables [17]. Particularly, nonlinear differential equations in higher dimensions often possess variable separation solutions [17,18].

The basic solution procedure of the refined invariant subspace method [3] is as follows. Let us focus on a scalar evolution equation

$$u_t = F[u] = F(x, t, u, u_x, u_{xx}, \ldots), \tag{1.1}$$

where $u = u(x,t)$ is a function of $x,\ t \in \mathbb{R}$ and $u_x = \frac{\partial u}{\partial x},\ u_{xx} = \frac{\partial^2 u}{\partial x^2},\ \ldots$. Introduce a k-dimensional linear space

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1007-5704/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cnsns.2012.02.024

$$W_k = \mathcal{L}\{f_1(x), f_2(x), \ldots, f_k(x)\} = \left\{\sum_{i=1}^{k} C_i f_i(x) | C_i = \text{const.}, 1 \leqslant i \leqslant k\right\}, \tag{1.2}$$

by a subspace of solutions to an nth-order linear ordinary differential equation:

$$L[y] = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0, \quad y^{(i)} = D^i y, \quad D = \frac{d}{dx}, \ i \geqslant 0, \tag{1.3}$$

where $a_0, a_1, \ldots, a_{n-1}$ are given continuous functions. We assume that W_k is an invariant space of the evolution Eq. (1.1):

$$F[W_k] \subseteq W_k, \quad \text{i.e.,} \ F[u] \in W_k, \ \forall u \in W_k.$$

This invariance condition implies that there exist k functions $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_k$ such that

$$F\left[\sum_{i=1}^{k} C_i f_i(x)\right] = \sum_{i=1}^{k} \bar{F}_i(C_1, C_2, \ldots, C_k) f_i(x) \tag{1.4}$$

for whatever constants $C_1, C_2, \ldots, C_k$. Then, a system of ordinary differential equations

$$\frac{d\psi_i}{dt} = \bar{F}_i(\psi_1, \psi_2, \ldots, \psi_k), \quad 1 \leqslant i \leqslant k, \tag{1.5}$$

yields a set of exact solutions to the evolution Eq. (1.1):

$$u = \sum_{i=1}^{k} \psi_i(t) f_i(x) \tag{1.6}$$

with generalized separated variables. This refined approach was proposed and analyzed in [3]. We remark that one may not able to define a subspace W_k by a kth-order linear ordinary differential equation [3], and that when $k < n$, the invariance conditions $F[W_k] \subseteq W_k$ and $F[W_n] \subseteq W_n$ require different sets of conditions on the evolution Eq. (1.1) and its associated invariant subspaces.

The invariant subspace method can be used to present exact solutions to systems of nonlinear evolution equations. On the basis of the existence of invariant subspaces that systems of linear ordinary differential equations define, Qu and Zhu [6] classified a particular class of systems of nonlinear parabolic equations. Zhu and Qu [19] presented an estimation of maximal dimensions of invariant subspaces for two-component systems of nonlinear evolution equations, and Shen et al. [20] generalized this estimation to multi-component systems of nonlinear evolution equations and presented certain classifications of systems of nonlinear parabolic equations and exact solutions, by observing invariant subspaces.

In this paper, we would like to apply the invariant subspace method to solve systems of dispersive evolution equations. A class of two-component nonlinear systems of dispersive equations is classified and a set of sufficient conditions is presented for the existence of invariant subspaces that the considered systems admit. A few concrete examples of the discussed systems illustrate the effectiveness of the invariant subspace theory in presenting exact and explicit solutions with generalized separated variables.

2. The refined invariant subspace method

We use the following notations

$$u_0^i = u^i(x,t), \quad u_j^i = \frac{\partial^j u^i(x,t)}{\partial x^j}, \quad 1 \leqslant i \leqslant q, \ j \geqslant 1, \tag{2.1}$$

which can be easily extended to cases of multiple spatial variables. Take a system of evolution equations of the form

$$u_t = F[u] = (F^1[u], F^2[u], \ldots, F^q[u])^T, \quad u = (u^1, u^2, \ldots, u^q)^T, \tag{2.2}$$

where all components of $F[u]$:

$$F^i[u] = F^i(x, t, u^1, \ldots, u^q, \ldots, u^1_{m_i}, \ldots, u^q_{m_i}), \quad 1 \leqslant i \leqslant q, \tag{2.3}$$

are given sufficient smooth functions in the indicated variables and can be considered as generalized differential operators.

Step 1: Determining invariant subspaces.

Let $W_{k_1,\ldots,k_q}$ denote a linear space $W^1_{k_1} \times \ldots \times W^q_{k_q}$, with $W^i_{k_i}$ being defined by

$$W^i_{k_i} = \mathcal{L}\{f^i_1(x), \ldots, f^i_{k_i}(x)\} = \left\{\sum_{j=1}^{k_i} C^i_j f^i_j(x) | C^i_j = \text{const.}, 1 \leqslant j \leqslant k_i\right\}, \quad 1 \leqslant i \leqslant q, \tag{2.4}$$

where for each $1 \leqslant i \leqslant q$, $f^i_1(x), \ldots, f^i_{k_i}(x)$ are linearly independent solutions to an n_i th-order linear ordinary differential equation with continuous function coefficients:

$$L_i[y_i] = y_i^{(n_i)} + a^i_{n_i-1}(x)y_i^{(n_i-1)} + \ldots + a^i_1(x)y_i' + a^i_0(x)y_i = 0, \tag{2.5}$$

where $n_i \geqslant k_i$. The invariance condition $F[W_{k_1,\ldots,k_q}] \subseteq W_{k_1,\ldots,k_q}$ for the subspace $W_{k_1,\ldots,k_q} = W^1_{k_1} \times \ldots \times W^q_{k_q}$ with respect to $F = (F^1, \ldots, F^q)^T$ reads

$$F^i[u] \in W^i_{k_i}, \quad \forall u \in W_{k_1,\ldots,k_q}, \quad 1 \leqslant i \leqslant q, \tag{2.6}$$

namely,

$$D^{n_i}F^i[u] + a^i_{n_i-1}(x)D^{n_i-1}F^i[u] + \ldots + a^i_0(x)F^i[u] = 0, \quad u \in W_{k_1,\ldots,k_q},\ 1 \leqslant i \leqslant q. \tag{2.7}$$

This set of equations provides a criterion for determining invariant subspaces that the system of evolution Eqs. (2.2) admits.

Step 2: Solving ODEs.

The invariance conditions in (2.7) mean that there exist functions $\bar{F}^i_j,\ 1 \leqslant j \leqslant k_i,\ 1 \leqslant i \leqslant q$, such that

$$F^i\left[\sum_{j=1}^{k_1} C^1_j f^1_j(x), \ldots, \sum_{j=1}^{k_q} C^q_j f^q_j(x)\right] = \sum_{j=1}^{k_i} \bar{F}^i_j(C^1_1, \ldots, C^1_{k_1}, \ldots, C^q_1, \ldots, C^q_{k_q}) f^i_j(x), \tag{2.8}$$

where $1 \leqslant i \leqslant q$. Now if a space $W_{k_1,\ldots,k_q}$ is invariant under the system of evolution Eqs. (2.2), then the system (2.2) possesses an exact solution of the form

$$u^i = \sum_{j=1}^{k_i} C^i_j(t) f^i_j(x), \quad 1 \leqslant i \leqslant q, \tag{2.9}$$

if and only if the $C^i_j(t)$'s satisfy a system of ordinary differential equations:

$$\frac{dC^i_j}{dt} = \bar{F}^i_j(C^1_1, \ldots, C^1_{k_1}, \ldots C^q_1, \ldots, C^q_{k_q}), \quad 1 \leqslant j \leqslant k_i, \quad 1 \leqslant i \leqslant q, \tag{2.10}$$

which is often much simpler than the system of evolution Eqs. (2.2). It then yields an exact solution (2.9) to the system (2.2) to solve this system of ordinary differential equations (2.10).

In concrete situations, we normally take linear ordinary differential equations with constant coefficients in (2.5) to begin with. The whole job of applying the refined invariant subspace method is to check the invariance conditions (2.7) and solve the resulting system of ordinary differential equations (2.10).

We point out that the orders of linear ordinary differential equations defining invariant subspaces are not arbitrary, and they are subject to the differential orders of the nonlinear operators $F^i,\ 1 \leqslant i \leqslant q$ (see, e.g. [4,19]). As soon as the maximal orders of the desired linear ordinary differential equations are determined, we can classify systems of evolution equations under consideration, and compute exact solutions from the corresponding invariant subspaces.

3. Applications

In this section, we analyze a (1 + 1)-dimensional nonlinear system of dispersive evolution equations to illustrate how to generate invariant subspaces and the corresponding exact solutions. We consider the following nonlinear system of dispersive evolution equations:

$$u_t = F = (u_{xxx} + \alpha_1 v v_x)_x + \alpha_2 v^2, \tag{3.1}$$

$$v_t = G = u_{xxx} + \beta_1 u + \beta_2 v, \tag{3.2}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants, α_1, α_2 are not simultaneously equal to zero to keep the nonlinearity, and we denote $u_x = \frac{\partial u}{\partial x},\ v_x = \frac{\partial v}{\partial x},\ u_{xx} = \frac{\partial^2 u}{\partial x^2},\ v_{xx} = \frac{\partial^2 v}{\partial x^2},\ \ldots$, as in soliton theory.

3.1. Classification

Let us take an invariant subspace $W_{2,2} = W^1_2 \times W^2_2$ defined by two second-order linear ordinary differential equations:

$$W^1_2 = \{y | L_1[y] = y'' + a_1 y' + a_0 y = 0\}, \quad W^2_2 = \{z | L_2[z] = z'' + b_1 z' + b_0 z = 0\}, \tag{3.3}$$

where a_0, a_1, b_0, b_1 are constants to be determined. The corresponding invariance conditions read

$$(D^2F + a_1 DF + a_0 F)|_{u \in W^1_2, v \in W^2_2} = 0, \tag{3.4}$$

$$(D^2G + b_1 DG + b_0 G)|_{u \in W^1_2, v \in W^2_2} = 0 \tag{3.5}$$

Substitute the expressions for F and G into the above equations, and replace u_{xx} by $-a_1 u_x - a_0 u$ and v_{xx} by $-b_1 v_x - b_0 v$ a few times to remove all higher-order partial derivatives of u and v with respect to x. Then, we collect the coefficients of the three terms $(v_x)^2$, vv_x and v^2 in the first simplified equation and the coefficients of the two terms u_x and u in the second simplified equation, and set all the resulting coefficients to be zero, to obtain the sufficient conditions:

$$(v_x)^2 : 7\alpha_1 b_1^2 + \alpha_1 a_0 + 2\alpha_2 - 4\alpha_1 b_0 - 3\alpha_1 a_1 b_1 = 0, \tag{3.6}$$

$$vv_x : 12\alpha_1 b_0 b_1 - \alpha_1 a_0 b_1 - 4\alpha_1 a_1 b_0 + 2\alpha_2 a_1 - \alpha_1 b_1^3 - 2\alpha_2 b_1 + \alpha_1 a_1 b_1^2 = 0, \tag{3.7}$$

$$v^2 : 4\alpha_1 b_0^2 + \alpha_2 a_0 - \alpha_1 b_0 b_1^2 + \alpha_1 a_1 b_0 b_1 - \alpha_1 a_0 b_0 - 2\alpha_2 b_0 = 0, \tag{3.8}$$

$$u_x : a_1^4 - 3a_0 a_1^2 + a_0^2 - \beta_1 a_1 - a_1^3 b_1 + 2a_0 a_1 b_1 + \beta_1 b_1 + a_1^2 b_0 - a_0 b_0 = 0, \tag{3.9}$$

$$u : a_0 a_1^3 - 2a_0^2 a_1 - \beta_1 a_0 - a_0 a_1^2 b_1 + a_0^2 b_1 + a_0 a_1 b_0 + \beta_1 b_0 = 0. \tag{3.10}$$

This guarantees the invariance conditions (3.4) and (3.5). We began with two second-order ordinary differential equations with constant coefficients, and so there exist linearly dependent terms in $(v_x)^2$, vv_x and v^2 for whatever solution v, but u and u_x could be linearly independent (see a theorem in [3]). Therefore, the conditions ((3.6)–(3.8)) are sufficient but not necessary to guarantee the first invariance condition (3.4), but the conditions (3.9) and (3.10) are both sufficient and necessary to guarantee the second invariance condition (3.5).

Solving the above system of algebraic equations by Maple, we obtain the following list of 11 examples, each of which consists of a system of evolution equations and a system of linear ordinary differential equations defining an invariant subspace:

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{1}{8}\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + \frac{8}{27} a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' + \frac{2}{9} a_1^2 y = 0, \\ L_2(z) = z'' + \frac{1}{2} a_1 z' + \frac{1}{18} a_1^2 z = 0; \end{cases} \tag{3.11}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{2}{25}\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + \frac{216}{125} a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' - \frac{6}{25} a_1^2 y = 0, \\ L_2(z) = z'' + \frac{2}{5} a_1 z' - \frac{3}{25} a_1^2 z = 0; \end{cases} \tag{3.12}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{2}{81}\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + \frac{8}{27} a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' + \frac{2}{9} a_1^2 y = 0, \\ L_2(z) = z'' + \frac{2}{9} a_1 z' - \frac{1}{27} a_1^2 z = 0; \end{cases} \tag{3.13}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{8}{25}\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + \frac{27}{125} a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' + \frac{6}{25} a_1^2 y = 0, \\ L_2(z) = z'' + \frac{3}{5} a_1 z' + \frac{2}{25} a_1^2 z = 0; \end{cases} \tag{3.14}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{1}{8}\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} - a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' + a_1^2 y = 0, \\ L_2(z) = z'' + \frac{1}{2} a_1 z' + \frac{1}{4} a_1^2 z = 0; \end{cases} \tag{3.15}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - 2\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + \beta_1 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' = 0, \\ L_2(z) = z'' + a_1 z' = 0; \end{cases} \tag{3.16}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{1}{8}\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' = 0, \\ L_2(z) = z'' + \frac{1}{2} a_1 z' = 0; \end{cases} \tag{3.17}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x - 2\alpha_1 a_1^2 v^2, \\ v_t = u_{xxx} + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' = 0, \\ L_2(z) = z'' - a_1^2 z = 0; \end{cases} \tag{3.18}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x, \\ v_t = u_{xxx} + \frac{8}{27} a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' + \frac{2}{9} a_1^2 y = 0, \\ L_2(z) = z'' + \frac{1}{3} a_1 z' = 0; \end{cases} \tag{3.19}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x, \\ v_t = u_{xxx} + a_1^3 u + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' + a_1 y' = 0, \\ L_2(z) = z'' = 0; \end{cases} \tag{3.20}$$

$$\begin{cases} u_t = (u_{xxx} + \alpha_1 v v_x)_x + 2\alpha_1 b_0 v^2, \\ v_t = u_{xxx} + \beta_2 v, \end{cases} \quad \begin{cases} L_1(y) = y'' = 0, \\ L_2(z) = z'' + b_0 z = 0. \end{cases} \tag{3.21}$$

3.2. Illustrative examples

In what follows, we discuss three examples of getting exact solutions with generalized separated variables.

Example 3.1. Let us first consider the system in (3.11):

$$u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{1}{8}\alpha_1 a_1^2 v^2, \tag{3.22}$$

$$v_t = u_{xxx} + \frac{8}{27} a_1^3 u + \beta_2 v, \tag{3.23}$$

which admits an invariant subspace $W_{2,2}$ defined through

$$L_1[y] = y'' + a_1 y' + \frac{2}{9} a_1^2 y = 0, \tag{3.24}$$

$$L_2[z] = z'' + \frac{1}{2} a_1 z' + \frac{1}{18} a_1^2 z = 0, \tag{3.25}$$

where a_1, α_1, β_2 are given constants.

From the two equations $L_1[y] = 0$ and $L_2[z] = 0$, we obtain an invariant subspace

$$W_2^1 \times W_2^2 = \mathcal{L}\left\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{2}{3}a_1 x}\right\} \times \mathcal{L}\left\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{1}{6}a_1 x}\right\} \tag{3.26}$$

that the system of (3.22) and (3.23) admits. It then follows that an exact solution can take the form

$$u = C_1(t)e^{-\frac{1}{3}a_1 x} + C_2(t)e^{-\frac{2}{3}a_1 x}, \quad v = D_1(t)e^{-\frac{1}{3}a_1 x} + D_2(t)e^{-\frac{1}{6}a_1 x}, \tag{3.27}$$

where the coefficients are functions of t to be determined. Substituting this solution into the system of (3.22) and (3.23), we find the following system of ordinary differential equations for computing the coefficients:

$$C_1' = \frac{16}{81} a_1^4 C_1 + \frac{7}{72}\alpha_1 a_1^2 D_2^2, \quad C_2' = \frac{1}{81} a_1^4 C_2 + \frac{5}{72}\alpha_1 a_1^2 D_1^2, \tag{3.28}$$

$$D_1' = \beta_2 D_1, \quad D_2' = \frac{7}{27} a_1^3 C_2 + \beta_2 D_2. \tag{3.29}$$

This system can be solved one by one, and its solution reads

$$D_1(t) = d_1 e^{\beta_2 t}, \tag{3.30}$$

$$C_2(t) = \left[-\frac{5}{72}\alpha_1 a_1^2 \int_0^t D_1^2(s) e^{-\frac{1}{81}a_1^4 s}\, ds + c_2\right] e^{\frac{1}{81}a_1^4 t}, \tag{3.31}$$

$$D_2(t) = \left[\frac{7}{27} a_1^3 \int_0^t C_2(s) e^{-\beta_2 s}\, ds + d_2\right] e^{\beta_2 t}, \tag{3.32}$$

$$C_1(t) = \left[\frac{7}{27}\alpha_1 a_1^2 \int_0^t D_2^2(s) e^{-\frac{16}{81}a_1^4 s}\, ds + c_1\right] e^{\frac{16}{81}a_1^4 t}, \tag{3.33}$$

where $c_i, d_i,\ 1 \leqslant i \leqslant 2$, are arbitrary constants. Now, (3.27) presents an exact solution to the system of (3.22) and (3.23).

Example 3.2. Let us second consider the system in (3.16):

$$u_t = (u_{xxx} + \alpha_1 v v_x)_x - 2\alpha_1 a_1^2 v^2, \tag{3.34}$$

$$v_t = u_{xxx} + \beta_1 u + \beta_2 v, \tag{3.35}$$

which admits an invariant subspace $W_{2,2}$ defined by

$$L_1[y] = y'' + a_1 y' = 0, \tag{3.36}$$

$$L_2[z] = z'' + a_1 z' = 0, \tag{3.37}$$

where $a_1, \alpha_1, \beta_1, \beta_2$ are given constants.

Similarly, from the two equations $L_1[y] = 0$ and $L_2[z] = 0$, we get an invariant subspace

$$W_2^1 \times W_2^2 = \mathcal{L}\{1, e^{-a_1 x}\} \times \mathcal{L}\{1, e^{-a_1 x}\} \tag{3.38}$$

that the system of (3.34) and (3.35) admits. It then follows that an exact solution can take the form with generalized separated variables:

$$u = C_1(t) + C_2(t)e^{-a_1 x}, \quad v = D_1(t) + D_2(t)e^{-a_1 x}. \tag{3.39}$$

Substituting this solution into the system of (3.34) and (3.35), we find the following system of ordinary differential equations:

$$C_1' = -2\alpha_1 a_1^2 D_1^2, \quad C_2' = a_1^4 C_2 - 3\alpha_1 a_1^2 D_1 D_2, \tag{3.40}$$

$$D_1' = \beta_1 C_1 + \beta_2 D_1, \quad D_2' = -a_1^3 C_2 + \beta_1 C_2 + \beta_2 D_2. \tag{3.41}$$

This is a really nonlinear system, and it is difficult to solve explicitly. To present exact solutions, let us only focus on a smaller invariant subspace

$$W_{1,1} = W_1^1 \times W_1^2 = \mathcal{L}\{e^{-a_1 x}\} \times \mathcal{L}\{e^{-a_1 x}\}. \tag{3.42}$$

Solving the system of (3.40) and (3.41) with $C_1 = D_1 = 0$, we arrive at

$$C_2 = ce^{a_1^4 t}, \quad D_2 = de^{\beta_2 t} - \frac{c(a_1^3 - \beta_1)}{a_1^4 - \beta_2} e^{a_1^4 t}, \tag{3.43}$$

and according to (3.39), we eventually obtain an exact solution to the system of (3.34) and (3.35):

$$u = ce^{-a_1 x + a_1^4 t}, \quad v = de^{-a_1 x + \beta_2 t} - \frac{c(a_1^3 - \beta_1)}{a_1^4 - \beta_2} e^{-a_1 x + a_1^4 t}, \tag{3.44}$$

where c and d are arbitrary constants. This gives an application of the refined invariant subspace method to exact solutions to nonlinear systems of evolutions equations.

Example 3.3. Let us third consider the system in (3.19):

$$u_t = (u_{xxx} + \alpha_1 v v_x)_x, \tag{3.45}$$

$$v_t = u_{xxx} + \frac{8}{27} a_1^3 u + \beta_2 v, \tag{3.46}$$

which admits an invariant subspace $W_{2,2}$ defined by

$$L_1[y] = y'' + a_1 y' + \frac{2}{9} a_1^2 y = 0, \tag{3.47}$$

$$L_2[z] = z'' + \frac{1}{3} a_1 z' = 0, \tag{3.48}$$

where a_1, α_1, β_2 are given constants.

Starting from the two equations $L_1[y] = 0$ and $L_2[z] = 0$, we get an invariant subspace

$$W_2^1 \times W_2^2 = \mathcal{L}\left\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{2}{3}a_1 x}\right\} \times \mathcal{L}\left\{1, e^{-\frac{1}{3}a_1 x}\right\} \tag{3.49}$$

that the system of (3.34) and (3.35) admits. Then, we can form an exact solution with generalized separated variables:

$$u = C_1(t)e^{-\frac{1}{3}a_1 x} + C_2(t)e^{-\frac{2}{3}a_1 x}, \quad v = D_1(t) + D_2(t)e^{-\frac{1}{3}a_1 x}, \tag{3.50}$$

where the coefficients are functions of t to be determined. Substituting this solution into the system of (3.45) and (3.46), we find the following system of ordinary differential equations for computing the coefficients:

$$C_1' = \frac{1}{81} a_1^4 C_1 + \frac{1}{9} \alpha_1 a_1^2 D_1 D_2, \quad C_2' = \frac{16}{81} a_1^4 C_2 + \frac{2}{9} \alpha_1 a_1^2 D_2^2, \tag{3.51}$$

$$D_1' = \beta_2 D_1, \quad D_2' = \frac{7}{27} a_1^3 C_1 + \beta_2 D_2. \tag{3.52}$$

This is a nonlinear system, too. To present exact solutions, let us only focus on two smaller invariant subspaces

$$W_{1,1} = W_1^1 \times W_1^2 = \mathcal{L}\{e^{-\frac{2}{3}a_1 x}\} \times \mathcal{L}\{1\}, \tag{3.53}$$

and

$$W_{2,1} = W_2^1 \times W_1^2 = \mathcal{L}\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{2}{3}a_1 x}\} \times \mathcal{L}\{e^{-\frac{1}{3}a_1 x}\}. \tag{3.54}$$

Solving the system of (3.51) and (3.52) with $C_1 = D_2 = 0$ tells

$$C_2 = ce^{\frac{16}{81}a_1^4 t}, \quad D_1 = de^{\beta_2 t}, \tag{3.55}$$

and according to (3.50), leads eventually to an exact solution to the system of (3.45) and (3.46):

$$u = ce^{-\frac{2}{3}a_1 x + \frac{16}{81}a_1^4 t}, \quad v = de^{\beta_2 t}, \tag{3.56}$$

where c and d are arbitrary constants.

The system of (3.51) and (3.52) with $D_1 = 0$ can be solved one by one from C_1, D_2 to C_2, and it solution formula is given by

$$\begin{cases} C_1(t) = c_1 e^{\frac{1}{81}a_1^4 t}, \quad D_2(t) = \left[\frac{7}{27} a_1^3 \int_0^t C_1(s) e^{-\beta_2 s}\, ds + d_1\right] e^{\beta_2 t}, \\ C_2(t) = \left[\frac{2}{9} \alpha_1 a_1^2 \int_0^t D_2^2(s) e^{-\frac{16}{81}a_1^4 s}\, ds + c_2\right] e^{\frac{16}{81}a_1^4 t}, \end{cases} \tag{3.57}$$

where c_1, c_2 and d_1 are arbitrary constants. Further, the formula (3.50), together with (3.57), presents an exact solution to the system of (3.45) and (3.46).

These are two application examples of the refined invariant subspace method to construct exact solutions to nonlinear systems of evolutions equations.

4. Concluding remarks

The invariant subspace method was used to classify a class of nonlinear systems of dispersive evolution equations and determine their invariant subspaces and exact solutions. A few examples illustrated the effectiveness of the invariant subspace theory for exploring solution structures of systems of nonlinear evolution equations.

The invariant subspace method can be considered as a generalized separation of variables for nonlinear differential equations. It generates a kind of complexiton-like solutions [17,10,21,22] and exhibits an integrability characteristic [23] that integrable differential equations possess, complementing existing approaches such as the transformed rational function method [24] and the multiple exp-function method [25].

It is also interesting to see that the linear superposition principle takes on key role in computing exact solutions to both evolution equations [4] and Hirota bilinear equations [26]. All related theories furnish linear combination solutions of trigonometric and exponential functions with generalized separated variables.

Acknowledgements

The work was supported in part by the Natural Science Foundation of Shanghai and the Shanghai Leading Academic Discipline Project (No. J50101), Zhejiang Innovation Project (Grant No. T200905), the State Administration of Foreign Experts Affairs of China, the National Natural Science Foundation of China (Nos. 11071274, 10971136, 10831003, 61072147 and 11071159), and Chunhui Plan of the Ministry of Education of China.

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