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INTEGRABLE REDUCTIONS OF MATRIX AKNS SOLITON HIERARCHIES

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A type of group reduction or similarity transformation is proposed and studied for the matrix AKNS spectral problem with two square matrix potentials. The corresponding integrable hierarchies of the reduced matrix AKNS equations, including reduced matrix nonlinear Schrödinger equations, are presented, demonstrating the diversity of matrix AKNS soliton hierarchies. The zero curvature formulation plays a crucial role in the development of these integrable models.

Keywords: Matrix AKNS spectral problem, integrable hierarchy, zero-curvature equation.

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1. Introduction

The Lax pair formulation provides a general framework for constructing and classifying integrable models [1, 2]. The key step in this process is selecting a matrix spectral problem that enables the generation of an associated hierarchy of integrable models. These models exhibit many integrable properties, such as bi-Hamiltonian structures and hereditary recursion operators. The matrix spectral problem serves as the primary tool for solving their Cauchy problems via the inverse scattering transform, with the evolution of the scattering data governed by the associated temporal matrix spectral problems [1].

In this context, the potentials in matrix spectral problems are unconstrained. However, imposing certain restrictions makes the spectral problems significantly more constrained, making them harder to analyze and apply. A useful approach is to apply similarity transformations, which can be used to formulate reduced matrix spectral problems. Such transformations ensure the construction of integrable hierarchies, as they preserve the invariance of the associated zero-curvature equations (see, e.g. [3]).

Two typical examples of integrable models are the nonlinear Schrödinger equation and the modified Korteweg–de Vries equation. Both can be derived from the Ablowitz–Kaup–Newell–Segur (AKNS) matrix spectral problems by applying a single similarity transformation. By applying two similarity transformations, even more diverse integrable models can be formulated, although this may introduce additional complexity. Corresponding to these two similarity transformations, there are two constraints on the potentials, which impose more restrictions to balance the potentials in the associated Lax pairs [4].

Recently, the idea of using similarity transformations has also been employed to construct nonlocal integrable models [5]. Three types of reduced integrable nonlinear Schrödinger-type equations and two types of reduced integrable modified Korteweg–de Vries-type equations have been proposed and classified [6]. The inverse scattering transform has also been developed for constructing soliton solutions for nonlocal integrable models (see, e.g. [7–9]). In addition, other efficient methods have been explored for nonlocal integrable models, particularly for constructing soliton solutions. The Hirota bilinear method, Darboux transformation, Bäcklund transformations, and the Riemann–Hilbert technique have all proven to be powerful tools. Many theories have been proposed for various reduced integrable models, both local and nonlocal (see, for example, [3, 4, 10–13]).

In this paper, we propose a novel type of local group reduction or similarity transformation for the AKNS matrix spectral problems with two square matrix potentials, aiming to generate reduced integrable models. The rest of the paper is organized as follows. In Section 2, we review the general AKNS matrix spectral problems and their associated hierarchies of matrix integrable models to set the stage for the subsequent analysis. In Section 3, we introduce a method of local group reductions or similarity transformations for the AKNS matrix spectral problems with square matrix potentials, and we formulate reduced integrable hierarchies based on the reduced matrix spectral problems. In Section 4, we present several concrete examples using the proposed formulation, demonstrating the diversity of reduced AKNS matrix spectral problems and their corresponding integrable matrix nonlinear Schrödinger models. In the final section, we conclude with a summary of our results and offer some remarks.

2. The matrix AKNS soliton hierarchies

We assume that m and n are two arbitrary natural numbers. For a fixed pair of $m,n \geq 1$, we recall the AKNS matrix spectral problems and the associated AKNS hierarchies of matrix integrable models, in order to facilitate the subsequent analysis.

First, we use λ to denote the spectral parameter, and assume that p and q are two matrix potentials:

$$p = p(x,t) = (p_{jk})_{m\times n}, \quad q = q(x,t) = (q_{kj})_{n\times m}. \tag{1}$$

The standard matrix AKNS spectral problems are defined by

$$-i\phi_x = U\phi, \quad U = U(u,\lambda) = \lambda\Lambda + P, \tag{2}$$

and

$$-i\phi_t = V^{[r]}\phi, \quad V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]}, \quad r \geq 0, \tag{3}$$

where $u = u(p,q)$ is the potential consisting of the two matrix potentials p and q. In these Lax pairs of matrix spectral problems, two bigger square matrices, Λ and Ω, are given by

$$\Lambda = \operatorname{diag}(\alpha_1 I_m, \alpha_2 I_n), \quad \Omega = \operatorname{diag}(\beta_1 I_m, \beta_2 I_n), \tag{4}$$

where I_k denotes the identity matrix of size k, and α_1, α_2 and β_1, β_2 are two pairs of distinct arbitrary constants, illustrating the diversity of matrix spectral problems but having minimal impact on the associated integrable models. The other two bigger square matrices, P and $Q^{[r]}$, are defined as follows:

$$P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{5}$$

which is called the potential matrix, and

$$Q^{[0]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q^{[r]} = Q^{[r]}(u,\lambda) = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}, \quad r \geq 1. \tag{6}$$

Here $a^{[s]}, b^{[s]}, c^{[s]}$ and $d^{[s]}$ are determined recursively through

$$\begin{cases} b^{[s+1]} = \dfrac{1}{\alpha}\left(-ib_x^{[s]} - pd^{[s]} + a^{[s]}p\right), \quad c^{[s+1]} = \dfrac{1}{\alpha}\left(ic_x^{[s]} + qa^{[s]} - d^{[s]}q\right), \\ a_x^{[s+1]} = i\left(pc^{[s+1]} - b^{[s+1]}q\right), \quad d_x^{[s+1]} = i\left(qb^{[s+1]} - c^{[s+1]}p\right), \end{cases} \quad s \geq 0, \tag{7}$$

where $\alpha = \alpha_1 - \alpha_2$ and zero constants of integration are taken in computing $a^{[s]}$ and $d^{[s]}$, with the initial data:

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a^{[0]} = \beta_1 I_m, \quad d^{[0]} = \beta_2 I_n. \tag{8}$$

Obviously, we can determine

$$Q^{[1]} = \frac{\beta}{\alpha}P, \quad Q^{[2]} = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + iP_x), \tag{9}$$

and

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}\left(i[P,P_x] + P_{xx} + 2P^3\right), \tag{10}$$

where $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \operatorname{diag}(I_m, -I_n)$. The above recursive relations in (7) with the initial data in (8) form the unique Laurent series solution

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{11}$$

to the stationary zero-curvature equation

$$W_x = i[U, W], \tag{12}$$

where U is the spectral matrix in (2). This formal series solution is a crucial element in generating integrable hierarchies (see, e.g. [14, 15]).

Now, it directly follows that for a fixed pair of $m, n \geq 1$, the compatibility conditions of the two matrix spectral problems in (2) and (3) are the zero-curvature equations

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \quad r \geq 0, \tag{13}$$

and they present a matrix AKNS soliton hierarchy

$$p_t = i\alpha b^{[r+1]}, \quad q_t = -i\alpha c^{[r+1]}, \quad r \geq 0. \tag{14}$$

The case of $m = n = 1$ corresponds exactly to the typical AKNS hierarchy with two scalar potentials [16]. The trace identity [17], as used in [18], provides a generating energy functional for the soliton hierarchy, which yields a bi-Hamiltonian structure along with infinitely many symmetries and conserved quantities (see, e.g. [19] for more examples).

The first and second nonlinear models in (14), i.e. the cases of $r = 2$ and $r = 3$, give rise to the AKNS matrix nonlinear Schrödinger (NLS) equations:

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \quad q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq), \tag{15}$$

and the AKNS matrix modified Korteweg–de Vries equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \quad q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \tag{16}$$

where p and q are the two matrix potentials defined by (1). More examples of higher-order matrix AKNS integrable models could be worked out as well (see, e.g. [20]).

3. Reduced matrix AKNS soliton hierarchies

In what follows, let us consider the values $\alpha_1 = \alpha_2 = 1$, $\beta_1 = -\beta_2 = 2$, and $m = n$. This gives rise to the AKNS spectral problems with two square matrix potentials p and q.

3.1. Reductions of the matrix AKNS spectral problems

We would like to formulate integrable reductions of the matrix AKNS spectral problems in the specific cases mentioned above (see also, e.g. [21–25] for other examples of similar reductions).

Let us take two constant invertible square matrices of order n, denoted by Σ_1 and Σ_2, and formulate the following $2n$th-order invertible constant square matrix of off-diagonal type

$$\Sigma = \begin{bmatrix} 0 & \Sigma_1 \\ \Sigma_2 & 0 \end{bmatrix}. \tag{17}$$

Obviously, we have the two important properties for Λ and Ω:

$$\Sigma\Lambda\Sigma^{-1} = -\Lambda, \quad \Sigma\Omega\Sigma^{-1} = -\Omega. \tag{18}$$

Based on these properties, we can consider a type of group reduction or similarity transformation for a given matrix AKNS spectral matrix U in (2),

$$\Sigma U(\lambda)\Sigma^{-1} = -U^*(\lambda^*) = -\big(U(\lambda^*)\big)^*, \tag{19}$$

where A^* stands for the complex of a matrix A. This reduction will demonstrate an invariance property for the zero-curvature equations (see also [26]). Noting the specific form of the spectral matrix U, we can see that the group reduction (19) equivalently generates

$$\Sigma P\Sigma^{-1} = -P^*. \tag{20}$$

Clearly, this requires the corresponding constraints for the two square matrix potentials p and q,

$$p^* = -\Sigma_1 q\Sigma_2^{-1}, \quad q^* = -\Sigma_2 p\Sigma_1^{-1}. \tag{21}$$

To ensure the two constraints in (21) are compatible, we impose the following requirement,

$$\Sigma_1^*\Sigma_2 = \gamma I_n, \tag{22}$$

where $\gamma \in \mathbb{R}$ and $\gamma \neq 0$. It is straightforward to see that (22) guarantees that one constraint in (21) implies the other.

To summarize, under the condition (22) on the square matrix Σ, the group reduction in (19) generates a class of reduced matrix AKNS spectral problems:

$$-i\phi_x = U\phi, \quad U = U(u,\lambda) = \begin{bmatrix} \lambda I_n & p \\ -\Sigma_1^{-1}p^*\Sigma_2 & -\lambda I_n \end{bmatrix}, \tag{23}$$

where $u = u(p)$ involves only the square matrix potential p, and Σ_1 and Σ_2 are assumed to satisfy (22), or equivalently,

$$-i\phi_x = U\phi, \quad U = \begin{bmatrix} \lambda I_n & -\Sigma_2^{-1}q^*\Sigma_1 \\ q & -\lambda I_n \end{bmatrix}, \tag{24}$$

where $u = u(q)$ involves only the square matrix potential q, and Σ_1 and Σ_2 are again assumed to satisfy (22).

3.2. Reduced matrix AKNS soliton hierarchies

Let us consider the solution W determined by (11), with the initial data in (8). Under the group reduction in (19), we can prove the equality at the initial value, $\lambda = \infty$,

$$\Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty} = -W^*(\lambda^*)|_{\lambda=\infty} = -\big(W(\lambda^*)\big)^*|_{\lambda=\infty}. \tag{25}$$

By the uniqueness of solutions to the stationary zero-curvature equation (12), this implies that

$$\Sigma W(\lambda)\Sigma^{-1} = -W^*(\lambda^*) = -\big(W(\lambda^*)\big)^*, \tag{26}$$

since both $\Sigma W(\lambda)\Sigma^{-1}$ and $-W^*(\lambda^*)$ solve Eq. (12). It then follows that for each $r \geq 0$, we have the invariance property,

$$\Sigma V^{[r]}(\lambda)\Sigma^{-1} = -V^{[r]*}(\lambda^*) = -\big(V^{[r]}(\lambda^*)\big)^*, \tag{27}$$

which is equivalent to

$$\Sigma Q^{[r]}(\lambda)\Sigma^{-1} = -Q^{[r]*}(\lambda^*) = -\big(Q^{[r]}(\lambda^*)\big)^*, \tag{28}$$

where $r \geq 0$ and $V^{[r]}$ and $Q^{[r]}$ are defined in (3) and (6), respectively. Therefore, under the potential constraints in (21), we can compute

$$\begin{aligned}\Sigma\big(U_t - V_x^{[r]} + i[U, V^{[r]}]\big)(\lambda)\Sigma^{-1} &= \big((-U^*)_t - (-V^{[r]*})_x + i[-U^*, -V^{[r]*}]\big)(\lambda^*)\\ &= -\big(U_t^* - V_x^{[r]*} - i[U^*, V^{[r]*}]\big)(\lambda^*)\\ &= -\big(U_t - V_x^{[r]} + i[U, V^{[r]}]\big)\big)^*(\lambda^*),\end{aligned} \tag{29}$$

where $r \geq 0$, and further, the matrix AKNS integrable models in (14) with $r \geq 0$ become a hierarchy of reduced matrix AKNS integrable models,

$$p_t = 2ib^{[r+1]}|_{q=-\Sigma_2^* p^* (\Sigma_1^*)^{-1}}, \quad r \geq 0, \tag{30}$$

where the square matrix potential p produces the potential vector u, or equivalently,

$$q_t = -2ic^{[r+1]}|_{p=-\Sigma_1^* q^* (\Sigma_2^*)^{-1}}, \quad r \geq 0, \tag{31}$$

where the square matrix potential q produces the potential vector u.

Moreover, every member in the reduced soliton hierarchy (30) or (31) possesses a Lax pair consisting of the reduced matrix spectral problems in (2) and (3) with $r \geq 0$, and has a hierarchy of commuting symmetries and conserved densities, reduced from those for the matrix AKNS integrable equations in (14) with $r \geq 0$. The matrix spectral problems (23) and

$$-i\phi_t = V^{[r]}|_{q=-\Sigma_2^* p^* (\Sigma_1^*)^{-1}}\phi, \quad r \geq 0, \tag{32}$$

constitute a Lax pair for each member in the reduced soliton hierarchy (30), or equivalently, the matrix spectral problems (24) and

$$-i\phi_t = V^{[r]}|_{p=-\Sigma_1^* q^* (\Sigma_2^*)^{-1}}\phi, \quad r \geq 0, \tag{33}$$

constitute a Lax pair for every member in the reduced soliton hierarchy (31).

Noting that Σ_1 and Σ_2 are arbitrary invertible square matrices of order n, we can formulate diverse reduced hierarchies of matrix AKNS integrable models via the group reduction (19), including many examples in nonlinear physics (see, e.g. [22, 23, 27, 28]).

4. Illustrative examples

In this section, we apply the presented general theory to a few specific cases. We take $n = 2$ and specify the matrix potential p as follows,

$$p = \begin{bmatrix} p_1 & p_3 \\ p_4 & p_2 \end{bmatrix}. \tag{34}$$

Case 1: Let us first take

$$\Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}, \qquad \Sigma_2 = -\sigma \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}, \tag{35}$$

where $\sigma = \pm 1$ and $\delta = \pm 1$, indicating four possible combinations. Then, the group reduction (19) yields

$$q = -\Sigma_2^* p^* (\Sigma_1^*)^{-1} = \sigma \begin{bmatrix} p_1^* & \delta p_3^* \\ \delta p_4^* & p_2^* \end{bmatrix},$$

and thus the reduced spectral matrix is given as

$$U = \left[\begin{array}{cc|cc} \lambda & 0 & p_1 & p_3 \\ 0 & \lambda & p_4 & p_2 \\ \hline \sigma p_1^* & \sigma\delta p_3^* & -\lambda & 0 \\ \sigma\delta p_4^* & \sigma p_2^* & 0 & -\lambda \end{array}\right]. \tag{36}$$

Moreover, the corresponding reduced AKNS matrix integrable NLS equations are given by

$$\begin{cases} ip_{1,t} = p_{1,xx} + 2\sigma\big[\delta(p_3^* p_4 + p_4^* p_3)p_1 + |p_1|^2 p_1 + p_2^* p_3 p_4\big], \\ ip_{2,t} = p_{2,xx} + 2\sigma\big[\delta(p_3^* p_4 + p_4^* p_3)p_2 + |p_2|^2 p_2 + p_1^* p_3 p_4\big], \\ ip_{3,t} = p_{3,xx} + 2\sigma\big[\delta(p_4^* p_3^2 + p_3^* p_1 p_2) + (|p_1|^2 + |p_2|^2)p_3\big], \\ ip_{4,t} = p_{4,xx} + 2\sigma\big[\delta(p_4^* p_1 p_2 + p_3^* p_4^2) + (|p_1|^2 + |p_2|^2)p_4\big], \end{cases} \tag{37}$$

where the two real constants satisfy $\sigma^2 = 1$ and $\delta^2 = 1$.

Case 2: Let us second take

$$\Sigma_1 = \begin{bmatrix} 0 & 1 \\ \delta & 0 \end{bmatrix}, \qquad \Sigma_2 = -\sigma \begin{bmatrix} 0 & \delta \\ 1 & 0 \end{bmatrix}, \tag{38}$$

where $\sigma = \pm 1$ and $\delta = \pm 1$, again representing four possible combinations. Then, the group reduction (19) generates

$$q = -\Sigma_2^* p^* (\Sigma_1^*)^{-1} = \sigma \begin{bmatrix} \delta p_2^* & p_4^* \\ p_3^* & \delta p_1^* \end{bmatrix},$$

and therefore, the reduced spectral matrix is

$$U = \left[\begin{array}{cc|cc} \lambda & 0 & p_1 & p_3 \\ 0 & \lambda & p_4 & p_2 \\ \hline \sigma\delta p_2^* & \sigma p_4^* & -\lambda & 0 \\ \sigma p_3^* & \sigma\delta p_1^* & 0 & -\lambda \end{array}\right]. \tag{39}$$

Accordingly, the corresponding reduced AKNS matrix integrable NLS equations read

$$\begin{cases} ip_{1,t} = p_{1,xx} + 2\sigma\big[\delta(p_2^* p_1^2 + p_1^* p_3 p_4) + (|p_3|^2 + |p_4|^2) p_1\big], \\ ip_{2,t} = p_{2,xx} + 2\sigma\big[\delta(p_2^* p_3 p_4 + p_1^* p_2^2) + (|p_3|^2 + |p_4|^2) p_2\big], \\ ip_{3,t} = p_{3,xx} + 2\sigma\big[\delta(p_2^* p_1 + p_1^* p_2) p_3 + |p_3|^2 p_3 + p_4^* p_1 p_2\big], \\ ip_{4,t} = p_{4,xx} + 2\sigma\big[\delta(p_2^* p_1 + p_1^* p_2) p_4 + p_3^* p_1 p_2 + |p_4|^2 p_4\big], \end{cases} \tag{40}$$

where the two real constants satisfy $\sigma^2 = 1$ and $\delta^2 = 1$.

Case 3: Finally, let us take

$$\Sigma_1 = \begin{bmatrix} 1 & 1 \\ \delta & 0 \end{bmatrix}, \qquad \Sigma_2 = -\sigma \begin{bmatrix} 0 & \delta \\ 1 & -\delta \end{bmatrix}, \tag{41}$$

where $\sigma = \pm 1$ and $\delta = \pm 1$, covering four possible combinations once more. Then, the group reduction (19) engenders

$$q = -\Sigma_2^* p^* (\Sigma_1^*)^{-1} = \sigma \begin{bmatrix} \delta p_2^* & p_4^* - p_2^* \\ p_3^* - \delta p_2^* & \delta(p_1^* - p_3^*) + (p_2^* - p_4^*) \end{bmatrix},$$

and thus, the reduced spectral matrix becomes

$$U = \left[\begin{array}{cc|cc} \lambda & 0 & p_1 & p_3 \\ 0 & \lambda & p_4 & p_2 \\ \hline \sigma\delta p_2^* & \sigma(p_4^* - p_2^*) & -\lambda & 0 \\ \sigma(p_3^* - \delta p_2^*) & \sigma\big[\delta(p_1^* - p_3^*) + (p_2^* - p_4^*)\big] & 0 & -\lambda \end{array}\right]. \tag{42}$$

Consequently, the corresponding reduced AKNS matrix integrable NLS equations are given by

$$\begin{cases} ip_{1,t} = p_{1,xx}+2\sigma\{\delta[p_2^*(p_1-p_3)p_1+(p_1^*-p_3^*)p_3p_4]+(p_2^*-p_4^*)(p_3-p_1)p_4+|p_3|^2p_1\}, \\ ip_{2,t} = p_{2,xx}+2\sigma\{\delta[(p_1^*-p_3^*)p_2^2+p_2^*(p_4-p_2)p_3]+(p_2^*-p_4^*)(p_2-p_4)p_2+|p_3|^2p_2\}, \\ ip_{3,t} = p_{3,xx}+2\sigma\{\delta[p_2^*(p_1-p_3)p_3+(p_1^*-p_3^*)p_2p_3]+(p_2^*-p_4^*)(p_3-p_1)p_2+|p_3|^3p_3\}, \\ ip_{4,t} = p_{4,xx}+2\sigma\{\delta[p_2^*(p_4-p_2)p_1+(p_1^*-p_3^*)p_2p_4]+(p_2^*-p_4^*)(p_2-p_4)p_4+p_3^*p_1p_2\}, \end{cases} \tag{43}$$

where the two real constants satisfy $\sigma^2 = 1$ and $\delta^2 = 1$.

The above reduced spectral matrices, U, are quite special and involve potential restrictions. A remarkable aspect is that soliton hierarchies can still be generated from the corresponding reduced matrix spectral problems.

5. Conclusion and remarks

A novel type of local group reduction, or similarity transformation, has been introduced and analyzed for a specific class of AKNS spectral problem with two square matrix potentials. The group reductions impose a constraint on the two square matrix potentials in the AKNS potential matrix. The reductions yield reduced matrix AKNS integrable hierarchies, each of which possesses infinitely many symmetries and conservation laws. A few concrete examples of reduced AKNS matrix spectral problems and reduced matrix integrable NLS models have been computed. The introduced group reductions are distinct from those discussed in the literature [4, 29], where the reduction matrices, Σ, are of diagonal block matrix type.

Soliton solutions are known to be constructible using various analytical techniques, such as the Darboux transformation, the Hirota bilinear method, Bäcklund transforms, and the Wronskian determinant technique (see, e.g. [30, 31]). Among the most interesting solutions are rational solutions (see, e.g. [32]), lump wave solutions (see, e.g. [33–35]), invariant solutions (see, e.g. [36, 37]), breather and rogue wave solutions (see, e.g. [38–41]) and their interaction solutions (see, e.g. [42]). Furthermore, the Riemann–Hilbert technique offers a powerful method for constructing soliton solutions for integrable models that involve multiple poles in the scattering coefficients [43]. It is important to note that reduced integrable models involve balancing different potentials in the original equations, and as such, they must satisfy specific constraints. Consequently, these models and their solutions tend to be more challenging to determine.

It is hoped that the analysis presented here will contribute to the development of integrable models and further enrich the field by offering insights into specific matrix spectral problems (see, e.g. [44, 45]). Standard integrable NLS models provide excellent perturbative frameworks with wave numbers for studying water waves and optical waves [1]. Accordingly, we anticipate that our novel integrable NLS models could be valuable for exploring the mathematical properties of nonlinear dispersive waves in physical and engineering sciences, such as fluid dynamics and nonlinear optics.

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