Journal of Geometry and Physics 222 (2026) 105772

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Matrix integrable models associated with reduced AKNS Lax pairs

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A R T I C L E   I N F O

Article history:
Received 13 March 2025
Received in revised form 27 November 2025
Accepted 21 January 2026
Available online 23 January 2026

Keywords:
Matrix spectral problem
Lax pair
Group reduction or similarity transformation
Integrable model
Soliton hierarchy

A B S T R A C T

Pairs of group reductions or similarity transformations involving off-diagonal block matrices are proposed and analyzed for a specific type of Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problem. The corresponding reduced integrable hierarchies of AKNS matrix integrable models are presented, complementing the standard AKNS matrix integrable hierarchies. The Lax formulation plays a key role in generating these reduced matrix integrable models.

1. Introduction

The zero-curvature formulation provides a systematic framework for generating integrable models [1]. The key is to select a matrix spectral problem, from which an associated hierarchy of integrable models can be derived using zero-curvature equations. The inverse scattering transform leverages the matrix spectral problem to solve the Cauchy problems of integrable models, with the evolution of scattering data determined by the corresponding temporal matrix spectral problems [2].

Matrix spectral problems with free potentials are standard and natural, whereas reduced matrix spectral problems are more restrictive and challenging to apply. To address this, similarity transformations are used to formulate reduced matrix spectral problems, leading to integrable hierarchies (see, e.g., [3]). The goal of using similarity transformations is to maintain the invariance of the corresponding zero-curvature equations, thereby generating integrable models. Two typical examples of such integrable models are the nonlinear Schrödinger (NLS) equations and the modified Korteweg–de Vries (mKdV) equation. Both arise from Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems via a single similarity transformation. Moreover, applying a pair of similarity transformations can yield a broader class of integrable models. However, this approach introduces additional challenges, as the two reductions on potentials, corresponding to the pair of similarity transformations, impose new constraints on balancing the associated zero-curvature equations [4].

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https://doi.org/10.1016/j.geomphys.2026.105772

Recently, the use of group reductions or similarity transformations has also been applied to the construction of nonlocal integrable models [5]. Three types of reduced integrable nonlinear Schrödinger equations and two types of reduced integrable modified Korteweg–de Vries equations have been proposed and classified [6]. The inverse scattering transform has also been developed to solve nonlocal integrable models (see, e.g., [7–10]). Other efficient approaches have been explored for constructing nonlocal integrable models, particularly for deriving soliton solutions. The Hirota bilinear method, Darboux transformation, Bäcklund transformations, and the Riemann-Hilbert technique have proven to be powerful, leading to the development of numerous theoretical frameworks for various reduced integrable models, both local and nonlocal (see, e.g., [3,4], [11–15]).

In this paper, we propose a pair of local group reductions via similarity transformations for the AKNS matrix spectral problems to generate reduced integrable models. The rest of the paper is organized as follows. In Section 2, we review the AKNS matrix spectral problems and their associated hierarchies of matrix integrable models as a foundation for subsequent analyses. In Section 3, we introduce two local group reductions via similarity transformations for the AKNS matrix spectral problems and derive reduced local hierarchies of real matrix mKdV integrable models. In Section 4, we present illustrative examples that demonstrate the proposed formulation, showcasing a variety of reduced AKNS matrix spectral problems and corresponding matrix integrable models, including novel NLS-type and mKdV-type integrable models. Finally, in the conclusion, we summarize our findings and provide closing remarks.

2. The standard AKNS matrix integrable hierarchies

Let $m,n\geq 1$ be two arbitrarily given natural numbers. For each pair of $m,n\geq 1$, we state the AKNS matrix spectral problems and the associated AKNS hierarchies of matrix integrable models, to facilitate the subsequent analyses.

First, we denote the spectral parameter by λ, and assume that p and q are two submatrix potentials:

$$p=p(x,t)=(p_{jk})_{m\times n},\ q=q(x,t)=(q_{kj})_{n\times m}. \tag{2.1}$$

The standard AKNS matrix spectral problems read

$$-i\phi_x=U\phi,\ U=U(u,\lambda)=\lambda\Lambda+P, \tag{2.2}$$

and

$$-i\phi_t=V^{[r]}\phi,\ V^{[r]}=V^{[r]}(u,\lambda)=\lambda^r\Omega+Q^{[r]},\ r\geq 0, \tag{2.3}$$

where $u=u(p,q)$ is the potential consisting of the two submatrix potentials p and q. In the above Lax pair of matrix spectral problems, the $(m+n)$-th order square matrices, Λ and Ω, are defined by

$$\Lambda=\mathrm{diag}(\alpha_1 I_m,\alpha_2 I_n),\ \Omega=\mathrm{diag}(\beta_1 I_m,\beta_2 I_n), \tag{2.4}$$

where I_k is the identity matrix of size k, and α_1,α_2 and β_1,β_2 are two pairs of arbitrarily given distinct real constants, which will show the diversity of matrix spectral problems but do not have a serious effect on associated integrable models. The other two $(m+n)$-th order square matrices, P and $Q^{[r]}$, are given by

$$P=P(u)=\begin{bmatrix}0 & p\\ q & 0\end{bmatrix}, \tag{2.5}$$

which is called the potential matrix, and

$$Q^{[0]}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ Q^{[r]}=\sum_{s=0}^{r-1}\lambda^s\begin{bmatrix}a^{[r-s]} & b^{[r-s]}\\ c^{[r-s]} & d^{[r-s]}\end{bmatrix},\ r\geq 1, \tag{2.6}$$

with $a^{[s]},b^{[s]},c^{[s]}$ and $d^{[s]}$ being determined recursively via

$$b^{[0]}=0,\ c^{[0]}=0,\ a^{[0]}=\beta_1 I_m,\ d^{[0]}=\beta_2 I_n, \tag{2.7}$$

and

$$\begin{cases} b^{[s+1]}=\dfrac{1}{\alpha}(-ib_x^{[s]}-pd^{[s]}+a^{[s]}p),\\ c^{[s+1]}=\dfrac{1}{\alpha}(ic_x^{[s]}+qa^{[s]}-d^{[s]}q), \\ a_x^{[s+1]}=i(pc^{[s+1]}-b^{[s+1]}q),\\ d_x^{[s+1]}=i(qb^{[s+1]}-c^{[s+1]}p), \end{cases}\qquad s\geq 0, \tag{2.8}$$

where $\alpha = \alpha_1 - \alpha_2$ and zero constants of integration are taken in computing $a^{[s]}$ and $d^{[s]}$. Obviously, we can work out

$$Q^{[1]} = \frac{\beta}{\alpha} P, \ Q^{[2]} = \frac{\beta}{\alpha} \lambda P - \frac{\beta}{\alpha^2} I_{m,n}(P^2 + iP_x),$$

and

$$Q^{[3]} = \frac{\beta}{\alpha} \lambda^2 P - \frac{\beta}{\alpha^2} \lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P, P_x] + P_{xx} + 2P^3),$$

where $\beta = \beta_1 - \beta_2$ and $I_{m,n} = \mathrm{diag}(I_m, -I_n)$. We can easily see from the recursive relations in (2.8) with the initial data in (2.7) that

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix} \tag{2.9}$$

provides a unique Laurent series solution to the stationary zero-curvature equation

$$W_x = i[U, W], \tag{2.10}$$

where U is the spectral matrix in (2.2). Such a formal series solution is a pivotal element for generating integrable hierarchies (see, e.g., [16–19] for examples).

Now, it directly follows that for each pair of $m, n \geq 1$, the compatibility conditions of the two matrix spectral problems in (2.2) and (2.3), which are the zero-curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0, \ r \geq 0, \tag{2.11}$$

determine an AKNS matrix integrable hierarchy:

$$p_t = i\alpha b^{[r+1]}, \ q_t = -i\alpha c^{[r+1]}, \ r \geq 0. \tag{2.12}$$

The case of $m = n = 1$ gives rise to the typical AKNS integrable hierarchy with two scalar potentials [20]. By applying the trace identity [21] as in [22], each system in this AKNS matrix integrable hierarchy can be showed to possess a bi-Hamiltonian structure and infinitely many symmetries and conserved quantities (see, e.g., [23–25] for more examples).

It is direct to see that the first and second nonlinear (when $r = 2, 3$) integrable models in (2.12) are the AKNS matrix NLS equations:

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \ q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq), \tag{2.13}$$

and the AKNS matrix mKdV equations:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_x qp), \ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_x pq + 3qpq_x), \tag{2.14}$$

where p and q are the two matrix potentials given by (2.1). Other examples of higher-order AKNS matrix integrable models could be found in [26].

3. Reduced AKNS matrix integrable hierarchies

In what follows, let us consider that

$$\alpha_1 = -\alpha_2 = 1, \ \beta_1 = -\beta_2 = 2, \ m = n. \tag{3.1}$$

Therefore, we will only consider integrable reductions of AKNS matrix hierarchies associated with a specific type of AKNS spectral problem involving two square matrix potentials. The condition $m = n$ is necessary when we impose group reductions or similarity transformations involving off-diagonal block square matrices, but it is not required when using diagonal block square matrices.

3.1. Reductions of the AKNS matrix spectral problems

We take four constant invertible square matrices of order n: $\Sigma_1, \Sigma_2, \Delta_1$ and Δ_2, and formulate the two off-diagonal block square matrices of order $2n$:

$$\Sigma = \begin{bmatrix} 0 & \Sigma_1 \\ \Sigma_2 & 0 \end{bmatrix}, \ \Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix}. \tag{3.2}$$

Obviously, we have the crucial properties for Λ and Δ:

$$\Sigma\Lambda\Sigma^{-1}=\Delta\Lambda\Delta^{-1}=-\Lambda,\ \Sigma\Omega\Sigma^{-1}=\Delta\Omega\Delta^{-1}=-\Omega, \tag{3.3}$$

which allows us to introduce two group reductions or similarity transformations that preserve the invariance of the original zero-curvature equations.

For a given AKNS spectral matrix U in (2.2), we now consider a pair of group reductions or similarity transformations:

$$\Sigma U(\lambda)\Sigma^{-1}=-U^*(\lambda^*)=-(U(\lambda^*))^*, \tag{3.4}$$

and

$$\Delta U(\lambda)\Delta^{-1}=-U^T(\lambda)=-(U(\lambda))^T. \tag{3.5}$$

In (3.4), A^* stands for the complex of a matrix A, and in (3.5), A^T denotes the matrix transpose of a matrix A. These two reductions exhibit both invariance properties (see also [27]). Noting the specific form of the spectral matrix U, we can see that these two group reductions equivalently generate

$$\Sigma P\Sigma^{-1}=-P^*, \tag{3.6}$$

and

$$\Delta P\Delta^{-1}=-P^T, \tag{3.7}$$

respectively. Evidently, these require the following corresponding constraints for the two submatrix potentials p and q:

$$p^*=-\Sigma_1 q\Sigma_2^{-1},\ q^*=-\Sigma_2 p\Sigma_1^{-1}, \tag{3.8}$$

and

$$p^T=-\Delta_2 p\Delta_1^{-1},\ q^T=-\Delta_1 q\Delta_2^{-1}, \tag{3.9}$$

respectively.

To ensure the two reductions in (3.8) are compatible, we impose the following condition:

$$\Sigma_1^*\Sigma_2=\gamma I_n, \tag{3.10}$$

where $\gamma\in\mathbb{R}$ and $\gamma\neq 0$. Under this condition (3.10), to ensure the compatibility of the two constraints in (3.9), a sufficient condition could be

$$\Sigma_1^T\Delta_1^*\Sigma_2=\eta\Delta_2, \tag{3.11}$$

where $\eta\in\mathbb{C}$ and $|\eta|^2=\gamma^2$. It is direct to see that (3.10) and (3.11) guarantee that the two constraints in (3.8) or (3.9) are equivalent to each other.

To summarize, under the conditions (3.10) and (3.11) on the two matrices Σ and Δ, the two group reductions in (3.4) and (3.5) generate a class of reduced AKNS matrix spectral problems:

$$-i\phi_x=U\phi,\ U=\begin{bmatrix}\lambda I_n & p\\ -\Sigma_1^{-1}p^*\Sigma_2 & -\lambda I_n\end{bmatrix}, \tag{3.12}$$

where the matrix potential p needs to satisfy the first constraint in (3.9), or equivalently,

$$-i\phi_x=U\phi,\ U=\begin{bmatrix}\lambda I_n & -\Sigma_2^{-1}q^*\Sigma_1\\ q & -\lambda I_n\end{bmatrix}, \tag{3.13}$$

where the matrix potential q needs to satisfy the second constraint in (3.9).

3.2. Reduced AKNS matrix integrable hierarchies

Let us consider the solution W determined by (2.9), with the initial data in (2.7). Under the two group reductions in (3.4) and (3.5), we can obtain the equalities at the initial value, $\lambda=\infty$:

$$\begin{cases}\Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty}=-W^*(\lambda^*)|_{\lambda=\infty},\\ \Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty}=-W^T(\lambda)|_{\lambda=\infty}.\end{cases} \tag{3.14}$$

By the uniqueness of solutions to the stationary zero-curvature equations, these guarantee that

$$\begin{cases} \Sigma W(\lambda)\Sigma^{-1} = -W^*(\lambda^*) = -(W(\lambda^*))^*, \\ \Delta W(\lambda)\Delta^{-1} = -W^T(\lambda) = -(W(\lambda))^T, \end{cases} \tag{3.15}$$

respectively. It follows from these relations that for each $r \geq 0$, we have the following two pairs of invariance properties:

$$\begin{cases} \Sigma V^{[r]}(\lambda)\Sigma^{-1} = -V^{[r]*}(\lambda^*) = -(V^{[r]}(\lambda^*))^*, \\ \Delta V^{[r]}(\lambda)\Delta^{-1} = -V^{[r]T}(\lambda) = -(V^{[r]}(\lambda))^T, \end{cases} \tag{3.16}$$

which are equivalent to

$$\begin{cases} \Sigma Q^{[r]}(\lambda)\Sigma^{-1} = -Q^{[r]*}(\lambda^*) = -(Q^{[r]}(\lambda^*))^*, \\ \Delta Q^{r]}(\lambda)\Delta^{-1} = -Q^{[r]T}(\lambda) = -(Q^{[r]}(\lambda))^T, \end{cases} \tag{3.17}$$

where $r \geq 0$ and $V^{[r]}$ and $Q^{[r]}$ are defined in (2.3) and (2.6), respectively. It then follows that under the potential constraints (3.8) and (3.9), we have

$$\begin{cases} \Sigma(U_t - V_x^{[r]} + i[U, V^{[r]}])(\lambda)\Sigma^{-1} = ((-U^*)_t - (-V^{[r]*})_x + i[-U^*, -V^{[r]*}])(\lambda^*) \\ \qquad = -(U_t^* - V_x^{[r]*} - i[U^*, V^{[r]*}])(\lambda^*) \\ \qquad = -(U_t - V_x^{[r]} + i[U, V^{[r]}])^*(\lambda^*), \\ \Delta(U_t - V_x^{[r]} + i[U, V^{[r]}])(\lambda)\Delta^{-1} = ((-U^T)_t - (-V^{[r]T})_x + i[-U^T, -V^{[r]T}])(\lambda) \\ \qquad = -(U_t^T - V_x^{[r]T} - i[U^T, V^{[r]T}])(\lambda) \\ \qquad = -(U_t - V_x^{[r]} + i[U, V^{[r]}])^T(\lambda), \end{cases} \tag{3.18}$$

where $r \geq 0$, and thus, the AKNS matrix integrable models in (2.12) with $r \geq 0$ become a hierarchy of reduced AKNS matrix integrable models:

$$p_t = 2ib^{[r+1]}|_{q=-\Sigma_2^* p^* (\Sigma_1^*)^{-1}},\ r \geq 0, \tag{3.19}$$

where the matrix potential p is a reduced $n \times n$ matrix potential being subject to the first constraint in (3.9), or equivalently,

$$q_t = -2ic^{[r+1]}|_{p=-\Sigma_1^* q^* (\Sigma_2^*)^{-1}},\ r \geq 0, \tag{3.20}$$

where the matrix potential q is a reduced $n \times n$ matrix potential being subject to the second constraint in (3.9).

Moreover, every member in the reduced hierarchy (3.19) or (3.20) possesses a Lax pair consisting of the reduced matrix spectral problems in (2.2) and (2.3) with $r \geq 0$, and has a hierarchy of commuting symmetries and conserved densities, reduced from those for the matrix integrable AKNS equations in (2.12) with $r \geq 0$. The matrix spectral problems (3.12) and

$$-i\phi_t = V^{[r]}|_{q=-\Sigma_2^* p^* (\Sigma_1^*)^{-1}}\phi,\ r \geq 0, \tag{3.21}$$

constitute a Lax pair for every member in the reduced hierarchy (3.19), or equivalently, the matrix spectral problems (3.13) and

$$-i\phi_t = V^{[r]}|_{p=-\Sigma_1^* q^* (\Sigma_2^*)^{-1}}\phi,\ r \geq 0, \tag{3.22}$$

constitute a Lax pair for each member in the reduced hierarchy (3.20).

Noting that the pairs of Σ_1, Σ_2 and Δ_1, Δ_2 are generally arbitrary invertible square matrices of order n, subject to minor conditions, we can present various reduced hierarchies of AKNS matrix integrable models.

4. Applications

In this section, we aim to apply the general theory to several specific cases and present illustrative examples of reduced AKNS matrix spectral problems and reduced AKNS matrix integrable NLS and mKdV equations.

4.1. Symmetric and skew-symmetric cases

If we take

$$\Sigma = \begin{bmatrix} 0 & I_n \\ -\sigma I_n & 0 \end{bmatrix}, \tag{4.1}$$

where $\sigma = \pm 1$, then the group reduction (3.4), i.e.,

$$\Sigma U \Sigma^{-1} = -U^* \tag{4.2}$$

yields

$$p^* = \sigma q \ \text{ or } \ q^* = \sigma p. \tag{4.3}$$

This gives rise to a relation between the two potentials p and q.

If we take

$$\Delta = \begin{bmatrix} 0 & I_n \\ -\delta I_n & 0 \end{bmatrix}, \tag{4.4}$$

where $\delta = \pm 1$, then the group reduction (3.5), i.e.,

$$\Delta U \Delta^{-1} = -U^T \tag{4.5}$$

engenders

$$p^T = \delta p, \ q^T = \delta q. \tag{4.6}$$

This requires that p and q are either symmetric or skew-symmetric.

These two group reductions are in agreement, since both the conditions in (3.10) and (3.11) are satisfied, where $\gamma = -\sigma$ and $\eta = \sigma\delta$. Together, the two reductions lead to

$$q = \sigma p^*, \ p^T = \delta p, \tag{4.7}$$

and thus, the reduced spectral matrix reads

$$U = \begin{bmatrix} \lambda I_n & p \\ \sigma p^* & -\lambda I_n \end{bmatrix}, \tag{4.8}$$

where $p^T = \delta p$. The case with $\sigma = \pm 1$, $\delta = 1$ and $n = 2$ was analyzed via the inverse scattering transform [28,29], and a Darboux transformation was also provided for the subcase of $\sigma = 1$ in [30].

Examples in symmetric cases:

First, let us fix $n = 2$. Then, we have

$$p = \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix}. \tag{4.9}$$

Consequently, the corresponding reduced AKNS matrix integrable NLS and mKdV equations are expressed as

$$\begin{cases} ip_{1,t} = p_{1,xx} + 2\sigma[(|p_1|^2 + 2|p_3|^2)p_1 + p_2^* p_3^2], \\ ip_{2,t} = p_{2,xx} + 2\sigma[(|p_2|^2 + 2|p_3|^2)p_2 + p_1^* p_3^2], \\ ip_{3,t} = p_{3,xx} + 2\sigma[(|p_1|^2 + |p_2|^2 + |p_3|^2)p_3 + p_1 p_2 p_3^*], \end{cases} \tag{4.10}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2} p_{1,xxx} - 3\sigma[(|p_1|^2 + |p_3|^2)p_{1,x} + (p_1 p_3^* + p_3 p_2^*)p_{3,x}], \\ p_{2,t} = -\frac{1}{2} p_{2,xxx} - 3\sigma[(|p_2|^2 + |p_3|^2)p_{2,x} + (p_2 p_3^* + p_3 p_1^*)p_{3,x}], \\ p_{3,t} = -\frac{1}{2} p_{3,xxx} - \frac{3}{2}\sigma[(p_2 p_3^* + p_1^* p_3)p_{1,x} + (p_1 p_3^* + p_3 p_2^*)p_{2,x} \\ \qquad + (|p_1|^2 + |p_2|^2 + 2|p_3|^2)p_{3,x}], \end{cases} \tag{4.11}$$

respectively, where $\sigma = \pm 1$. For the integrable NLS equations in (4.10), dark solitons and the inverse scattering transformation were presented in [28,29], respectively. The case $\sigma = -1$ of (4.10) gives an example formulated through using Hermitian symmetric spaces in [32].

Second, let us fix $n=3$. Then, we have

$$p=\begin{bmatrix} p_1 & p_4 & p_6 \\ p_4 & p_2 & p_5 \\ p_6 & p_5 & p_3 \end{bmatrix}. \tag{4.12}$$

Accordingly, the corresponding reduced AKNS matrix integrable NLS and mKdV equations are novel integrable models and given by

$$\begin{cases} ip_{1,t}=p_{1,xx}+2\sigma[(|p_1|^2+2|p_2|^2+2|p_4|^2)p_1+p_2^2p_3^*+p_4^2p_5^*+2p_2p_4p_6^*], \\ ip_{2,t}=p_{2,xx}+2\sigma[(|p_1|^2+|p_2|^2+|p_3|^2+|p_4|^2+|p_6|^2)p_2 \\ \qquad +p_1(p_3p_2^*+p_6p_4^*)+p_4(p_6p_5^*+p_3p_6^*)], \\ ip_{3,t}=p_{3,xx}+2\sigma[(2|p_2|^2+|p_3|^2+2|p_6|^2)p_3+p_2^2p_1^*+p_6^2p_5^*+2p_2p_6p_4^*], \\ ip_{4,t}=p_{4,xx}+2\sigma[(|p_1|^2+|p_2|^2+|p_4|^2+|p_5|^2+|p_6|^2)p_4 \\ \qquad +p_1(p_6p_2^*+p_5p_4^*)+p_2(p_6p_3^*+p_5p_6^*)], \\ ip_{5,t}=p_{5,xx}+2\sigma[(2|p_4|^2+|p_5|^2+2|p_6|^2)p_5+p_6^2p_3^*+p_4^2p_1^*+2p_4p_6p_2^*], \\ ip_{6,t}=p_{6,xx}+2\sigma[(|p_2|^2+|p_3|^2+|p_4|^2+|p_5|^2+|p_6|^2)p_6 \\ \qquad +p_2(p_4p_1^*+p_5p_4^*)+p_3(p_4p_2^*+p_5p_6^*)], \end{cases} \tag{4.13}$$

and

$$\begin{cases} p_{1,t}=-\frac{1}{2}p_{1,xxx}-3\sigma[(|p_1|^2+|p_4|^2+|p_6|^2)p_{1,x}+(p_1p_4^*+p_4p_2^*+p_6p_5^*)p_{4,x} \\ \qquad +(p_1p_6^*+p_4p_5^*+p_6p_3^*)p_{6,x}], \\ p_{2,t}=-\frac{1}{2}p_{2,xxx}-3\sigma[(|p_2|^2+|p_4|^2+|p_5|^2)p_{2,x}+(p_2p_4^*+p_4p_1^*+p_5p_6^*)p_{4,x} \\ \qquad +(p_2p_5^*+p_4p_6^*+p_5p_3^*)p_{5,x}], \\ p_{3,t}=-\frac{1}{2}p_{3,xxx}-3\sigma[(|p_3|^2+|p_5|^2+|p_6|^2)p_{3,x}+(p_3p_5^*+p_5p_2^*+p_6p_4^*)p_{5,x} \\ \qquad +(p_3p_6^*+p_5p_4^*+p_6p_1^*)p_{6,x}], \\ p_{4,t}=-\frac{1}{2}p_{4,xxx}-\frac{3}{2}\sigma[(p_2p_4^*+p_4p_1^*+p_5p_6^*)p_{1,x}+(p_1p_4^*+p_4p_2^*+p_6p_5^*)p_{2,x} \\ \qquad +(|p_1|^2+|p_2|^2+2|p_4|^2+|p_5|^2+|p_6|^2)p_{4,x}+(p_1p_6^*+p_4p_5^*+p_6p_3^*)p_{5,x} \\ \qquad +(p_2p_5^*+p_4p_6^*+p_5p_3^*)p_{6,x}], \\ p_{5,t}=-\frac{1}{2}p_{5,xxx}-\frac{3}{2}\sigma[(p_3p_5^*+p_5p_2^*+p_6p_4^*)p_{2,x}+(p_2p_5^*+p_4p_6^*+p_5p_3^*)p_{3,x} \\ \qquad +(p_3p_6^*+p_5p_4^*+p_6p_1^*)p_{4,x}+(|p_2|^2+|p_3|^2+|p_4|^2+2|p_5|^2+|p_6|^2)p_{5,x} \\ \qquad +(p_2p_4^*+p_4p_1^*+p_5p_6^*)p_{6,x}], \\ p_{6,t}=-\frac{1}{2}p_{6,xxx}-\frac{3}{2}\sigma[(p_3p_6^*+p_5p_4^*+p_6p_1^*)p_{1,x}+(p_1p_6^*+p_4p_5^*+p_6p_3^*)p_{3,x} \\ \qquad +(p_3p_5^*+p_5p_2^*+p_6p_4^*)p_{4,x}+(p_1p_4^*+p_4p_2^*+p_6p_5^*)p_{5,x} \\ \qquad +(|p_2|^2+|p_3|^2+|p_4|^2+|p_5|^2+2|p_6|^2)p_{6,x}], \end{cases} \tag{4.14}$$

respectively, where $\sigma=\pm 1$.

Examples in skew-symmetric cases:

In this case, let us first consider $n=3$, and thus, we have

$$p=\begin{bmatrix} 0 & p_1 & p_3 \\ -p_1 & 0 & p_2 \\ -p_3 & -p_2 & 0 \end{bmatrix}. \tag{4.15}$$

Now, the corresponding reduced coupled integrable NLS and mKdV equations give rise to the standard AKNS system of three-component integrable NLS and mKdV equations:

$$\begin{cases} ip_{1,t} = p_{1,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + |p_3|^2)p_1, \\ ip_{2,t} = p_{2,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + |p_3|^2)p_2, \\ ip_{3,t} = p_{3,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + |p_3|^2)p_3, \end{cases} \tag{4.16}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2}p_{1,xxx} + \frac{3}{2}\sigma[(2|p_1|^2 + |p_2|^2 + |p_3|^2)p_{1,x} + p_1p_2^*p_{2,x} + p_1p_3^*p_{3,x}], \\ p_{2,t} = -\frac{1}{2}p_{2,xxx} + \frac{3}{2}\sigma[p_2p_1^*p_{1,x} + (|p_1|^2 + 2|p_2|^2 + |p_3|^2)p_{2,x} + p_2p_3^*p_{3,x}], \\ p_{3,t} = -\frac{1}{2}p_{3,xxx} + \frac{3}{2}\sigma[p_3p_1^*p_{1,x} + p_3p_2^*p_{2,x} + (|p_1|^2 + |p_2|^2 + 3|p_3|^2)p_{3,x}], \end{cases} \tag{4.17}$$

where $\sigma = \pm 1$. For the integrable NLS equations in (4.16), the long-time asymptotics of solutions was analyzed in [31]. The integrable mKdV equations in (4.17) are different from those reduced via diagonal block matrix group reductions (see, e.g., [3]).

The second case is $n = 4$. At this point, we have

$$p = \begin{bmatrix} 0 & p_1 & p_4 & p_6 \\ -p_1 & 0 & p_2 & p_5 \\ -p_4 & -p_2 & 0 & p_3 \\ -p_6 & -p_5 & -p_3 & 0 \end{bmatrix}. \tag{4.18}$$

Accordingly, the corresponding reduced AKNS matrix integrable NLS and mKdV equations provide novel integrable models, which are given as follows:

$$\begin{cases} ip_{1,t} = p_{1,xx} + 2\sigma[(|p_1|^2 + |p_2|^2 + |p_4|^2 + |p_5|^2 + |p_6|^2)p_1 - (p_2p_6 - p_4p_5)p_3^*], \\ ip_{2,t} = p_{2,xx} + 2\sigma[(|p_1|^2 + |p_2|^2 + |p_3|^2 + |p_4|^2 + |p_5|^2)p_2 - (p_1p_3 - p_4p_5)p_6^*], \\ ip_{3,t} = p_{3,xx} + 2\sigma[(|p_2|^2 + |p_3|^2 + |p_4|^2 + |p_5|^2 + |p_6|^2)p_3 - (p_2p_6 - p_4p_5)p_1^*], \\ ip_{4,t} = p_{4,xx} + 2\sigma[(|p_1|^2 + |p_2|^2 + |p_3|^2 + |p_4|^2 + |p_6|^2)p_4 + (p_1p_3 + p_2p_6)p_5^*], \\ ip_{5,t} = p_{5,xx} + 2\sigma[(|p_1|^2 + |p_2|^2 + |p_3|^2 + |p_5|^2 + |p_6|^2)p_5 + (p_1p_3 + p_2p_6)p_4^*], \\ ip_{6,t} = p_{6,xx} + 2\sigma[(|p_1|^2 + |p_3|^2 + |p_4|^2 + |p_5|^2 + |p_6|^2)p_6 - (p_1p_3 - p_4p_5)]p_2^*, \end{cases} \tag{4.19}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2}p_{1,xxx} + \frac{3}{2}\sigma[(2|p_1|^2 + |p_2|^2 + |p_4|^2 + |p_5|^2 + |p_6|^2)p_{1,x} + (p_1p_2^* - p_6p_3^*)p_{2,x} \\ \qquad +(p_1p_4^* + p_5p_3^*)p_{4,x} + (p_1p_5^* + p_4p_3^*)p_{5,x} + (p_1p_6^* - p_2p_3^*)p_{6,x}], \\ p_{2,t} = -\frac{1}{2}p_{2,xxx} + \frac{3}{2}\sigma[(p_2p_1^* - p_3p_6^*)p_{1,x} + (|p_1|^2 + 2|p_2|^2 + |p_3|^2 + |p_4|^2 + |p_5|^2)p_{2,x} \\ \qquad -(p_1p_6^* - p_2p_3^*)p_{3,x} + (p_2p_4^* + p_5p_6^*)p_{4,x} + (p_2p_5^* + p_4p_6^*)p_{5,x}], \\ p_{3,t} = -\frac{1}{2}p_{3,xxx} + \frac{3}{2}\sigma[(p_3p_2^* - p_6p_1^*)p_{2,x} + (|p_2|^2 + 2|p_3|^2 + |p_4|^2 + |p_5|^2 + |p_6|^2)p_{3,x} \\ \qquad +(p_3p_4^* + p_5p_1^*)p_{4,x} + (p_3p_5^* + p_4p_1^*)p_{5,x} - (p_2p_1^* - p_3p_6^*)p_{6,x}], \\ p_{4,t} = -\frac{1}{2}p_{4,xxx} + \frac{3}{2}\sigma[(p_3p_5^* + p_4p_1^*)p_{1,x} + (p_4p_2^* + p_6p_5^*)p_{2,x} + (p_1p_5^* + p_4p_3^*)p_{3,x} \\ \qquad (|p_1|^2 + |p_2|^2 + |p_3|^2 + 2|p_4|^2 + |p_6|^2)p_{4,x} + (p_2p_5^* + p_4p_6^*)p_{6,x}], \\ p_{5,t} = -\frac{1}{2}p_{5,xxx} + \frac{3}{2}\sigma[(p_3p_4^* + p_5p_1^*)p_{1,x} + (p_5p_2^* + p_6p_4^*)p_{2,x} + (p_1p_4^* + p_5p_3^*)p_{3,x} \\ \qquad (|p_1|^2 + |p_2|^2 + |p_3|^2 + 2|p_5|^2 + |p_6|^2)p_{5,x} + (p_2p_4^* + p_5p_6^*)p_{6,x}], \\ p_{6,t} = -\frac{1}{2}p_{6,xxx} + \frac{3}{2}\sigma[-(p_3p_2^* - p_6p_1^*)p_{1,x} - (p_1p_2^* - p_6p_3^*)p_{3,x} + (p_5p_2^* + p_6p_4^*)p_{4,x} \\ \qquad +(p_4p_2^* + p_6p_5^*)p_{5,x} + (|p_1|^2 + |p_3|^2 + |p_4|^2 + |p_5|^2 + 2|p_6|^2)p_{6,x}], \end{cases} \tag{4.20}$$

respectively, where $\sigma=\pm 1$. The case $\sigma=1$ of (4.19) gives an example presented through Hermitian symmetric spaces in [32]. There are some other interesting examples in the skew-symmetric case (see, e.g., [33,34]).

4.2. Non-symmetric and skew-symmetric cases

Let us first consider $n=2$ and choose

$$\Sigma_1=I_2,\ \Sigma_2=-\sigma I_2,\ \Delta_1=\begin{bmatrix} 0 & 1\\ \delta & 0\end{bmatrix},\ \Delta_2=-\delta\Delta_1, \tag{4.21}$$

where $\sigma=\pm 1$ and $\delta=\pm 1$, indicating four possible combinations. Then, the first group reduction (3.4) yields

$$q=\sigma p^*, \tag{4.22}$$

and the second group reduction (3.5) determines

$$p=\begin{bmatrix} p_2 & p_1\\ p_3 & \delta p_2\end{bmatrix}. \tag{4.23}$$

Consequently, the associated reduced AKNS integrable NLS and mKdV equations are given by

$$\begin{cases} ip_{1,t}=p_{1,xx}+2\sigma[(p_1p_3^*+2|p_2|^2)p_1+\delta p_2^2p_1^*],\\ ip_{2,t}=p_{2,xx}+2\sigma[(p_1p_3^*+p_3p_1^*+|p_2|^2)p_2+\delta p_1p_3p_2^*]\\ ip_{3,t}=p_{3,xx}+2\sigma[(p_3p_1^*+2|p_2|^2)p_3+\delta p_2^2p_3^*],\end{cases} \tag{4.24}$$

and

$$\begin{cases} p_{1,t}=-\frac{1}{2}p_{1,xxx}-3\sigma[(p_1p_3^*+|p_2|^2)p_{1,x}+(p_1p_2^*+\delta p_2p_1^*)p_{2,x}],\\ p_{2,t}=-\frac{1}{2}p_{2,xxx}-\frac{3}{2}\sigma[(p_2p_3^*+\delta p_3p_2^*)p_{1,x}+(p_1p_3^*+2|p_2|^2+p_3p_1^*)p_{2,x}+(\delta p_1p_2^*+p_2p_1^*)p_{3,x}],\\ p_{3,t}=-\frac{1}{2}p_{3,xxx}-3\sigma[(\delta p_2p_3^*+p_3p_2^*)p_{2,x}+(|p_2|^2+p_3p_1^*)p_{3,x}],\end{cases} \tag{4.25}$$

respectively, where $\sigma=\pm 1$ and $\delta=\pm 1$.

Next, let us consider $n=3$ and take

$$\Sigma_1=I_3,\ \Sigma_2=-\sigma I_3,\ \Delta_1=\Delta_2=\begin{bmatrix} 0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}, \tag{4.26}$$

where $\sigma=\pm 1$. Then, by the two group reductions in (3.4) and (3.5), we obtain

$$q=\sigma p^*,\ p=\begin{bmatrix} p_2 & p_1 & 0\\ p_3 & 0 & p_1\\ 0 & p_3 & -p_2\end{bmatrix}. \tag{4.27}$$

Accordingly, the associated reduced AKNS integrable NLS and mKdV equations are given as follows:

$$\begin{cases} ip_{1,t}=p_{1,xx}+2\sigma(p_1p_3^*+|p_2|^2+p_3p_1^*)p_1,\\ ip_{2,t}=p_{2,xx}+2\sigma(p_1p_3^*+|p_2|^2+p_3p_1^*)p_2,\\ ip_{3,t}=p_{3,xx}+2\sigma(p_1p_3^*+|p_2|^2+p_3p_1^*)p_3,\end{cases} \tag{4.28}$$

and

$$\begin{cases} p_{1,t}=-\frac{1}{2}p_{1,xxx}-\frac{3}{2}\sigma[(2p_1p_3^*+|p_2|^2+p_3p_1^*)p_{1,x}+p_1p_2^*p_{2,x}+|p_1|^2p_{3,x}],\\ p_{2,t}=-\frac{1}{2}p_{2,xxx}-\frac{3}{2}\sigma[p_2p_3^*p_{1,x}+(p_1p_3^*+2|p_2|^2+p_3p_1^*)p_{2,x}+p_2p_1^*p_{3,x}],\\ p_{3,t}=-\frac{1}{2}p_{3,xxx}-\frac{3}{2}\sigma[|p_3|^2p_{1,x}+p_3p_2^*p_{2,x}+(p_1p_3^*+|p_2|^2+2p_3p_1^*)p_{3,x}],\end{cases} \tag{4.29}$$

respectively, where $\sigma=\pm 1$.

Now, let us take up the third case:

$$\Sigma_1=I_3,\ \Sigma_2=-\sigma I_3,\ \Delta_1=\Delta_2=\begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}, \tag{4.30}$$

where $\sigma = \pm 1$. Then, the corresponding two group reductions yield

$$q = \sigma p^*, \ p = \begin{bmatrix} p_1 & p_2 & 0 \\ p_3 & 0 & -p_2 \\ -p_1 & -p_3 - p_2 & -p_1 \end{bmatrix}. \tag{4.31}$$

Therefore, the associated reduced AKNS integrable NLS and mKdV equations become

$$\begin{cases} ip_{1,t} = p_{1,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + p_2 p_3^* + p_3 p_2^*)p_1, \\ ip_{2,t} = p_{2,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + p_2 p_3^* + p_3 p_2^*)p_2, \\ ip_{3,t} = p_{3,xx} + 2\sigma(|p_1|^2 + |p_2|^2 + p_2 p_3^* + p_3 p_2^*)p_3, \end{cases} \tag{4.32}$$

and

$$\begin{cases} p_{1,t} = -\frac{1}{2} p_{1,xxx} - \frac{3}{2}\sigma[(2|p_1|^2 + |p_2|^2 + p_2 p_3^* + p_3 p_2^*)p_{1,x} + p_1(p_2^* + p_3^*)p_{2,x} + p_1 p_2^* p_{3,x}], \\ p_{2,t} = -\frac{1}{2} p_{2,xxx} - \frac{3}{2}\sigma[p_2 p_1^* p_{1,x} + (|p_1|^2 + 2|p_2|^2 + 2p_2 p_3^* + p_3 p_2^*)p_{2,x} + |p_2|^2 p_{3,x}], \\ p_{3,t} = -\frac{1}{2} p_{3,xxx} - \frac{3}{2}\sigma[p_3 p_1^* p_{1,x} + p_3(p_2^* + p_3^*)p_{2,x} + (|p_1|^2 + |p_2|^2 + p_2 p_3^* + 2p_3 p_2^*)p_{3,x}], \end{cases} \tag{4.33}$$

respectively, where $\sigma = \pm 1$.

The three examples in these specific cases represent novel models of integrable NLS and mKdV equations. We note that more specific integrable reductions can be achieved by choosing different group reductions or similarity transformations for the zero-curvature equations (see, e.g., [35–39]).

5. Concluding remarks

A pair of local group reductions, or similarity transformations, has been proposed and analyzed for a specific type of AKNS spectral problem. The reductions yield reduced AKNS matrix integrable hierarchies, whose members commute with each other. A few concrete examples of reduced AKNS matrix spectral problems and reduced AKNS matrix integrable NLS and mKdV models have been presented. One of the two group reductions imposes a constraint on the two square matrix potentials in the AKNS matrix potential, while the other imposes a constraint on the square matrix potentials. The novelty of this work lies in introducing group reductions involving off-diagonal block matrices. Traditionally, group reductions are formulated using diagonal block matrices, but not off-diagonal ones (see, e.g., [4,40]).

It is known that soliton solutions can be constructed using analytical techniques, including the Darboux transformation, the Hirota bilinear method, Bäcklund transforms, and the Wronskian determinant technique. Rational solutions (see, e.g., [41]), lump wave solutions (see, e.g., [42–44]), breather wave and rogue wave solutions (see, e.g., [45–49]), and their interaction solutions (see, e.g., [50,51]) are among the most significantly interesting solutions. Moreover, the Riemann-Hilbert technique provides a powerful way to construct soliton solutions for integrable models with multiple poles in the scattering coefficients [52,53]. We note that reduced integrable models balance different potentials in the original equations and, therefore, must satisfy certain constraints (see, e.g., [54–56]). These models and their solutions are generally more difficult to solve and obtain.

It is expected that the analysis presented here will contribute to the refinement and classification of integrable models, further enriching the field based on specific matrix spectral problems.

Acknowledgements

The work was supported in part by the Ministry of Science and Technology of China (G2021016032L and G2023016011L) and the National Natural Science Foundation of China (12271488 and 11975145).

Data availability

No data was used for the research described in the article.

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