

A multiple exp-function method for nonlinear differential equations and its application

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Abstract

A multiple exp-function method for exact multiple wave solutions of nonlinear partial differential equations is proposed. The method is oriented towards the ease of use and capability of computer algebra systems and provides a direct and systematic solution procedure that generalizes Hirota's perturbation scheme. With the help of Maple, applying the approach to the (3 + 1)-dimensional potential-Yu-Toda-Sasa-Fukuyama equation yields exact explicit one-wave, two-wave and three-wave solutions, which include one-soliton, two-soliton and three-soliton type solutions. Two cases with specific values of the involved parameters are plotted for each of the two-wave and three-wave solutions.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Exact solutions to nonlinear partial differential equations (PDEs) help us understand the physical phenomena they describe in nature. Many solution methods have been proposed, which contain the tanh-function method [1–3], the sech-function method [4–6], the homogeneous balance method [7, 8], the extended tanh-function method [9–11], the sine-cosine method [12, 13], the tanh-coth method [14] and the exp-function method [15, 16]. The crucial idea of these methods is to search for rational solutions to variable coefficient ordinary differential equations (ODEs) transformed from given nonlinear PDEs. Following this observation, a unified approach to exact solutions to nonlinear equations was recently proposed, revealing relations between solvable ODEs and nonlinear PDEs [17]. Solitary waves, periodic waves and kink waves modeling various nonlinear motions have been presented for many nonlinear dispersive and dissipative equations.

However, those existing methods are only concerned with traveling wave solutions to nonlinear equations. It is known that there are multiple wave solutions to nonlinear

equations, for instance, multi-soliton solutions to many physically significant equations including the Korteweg-de Vries (KdV) equation and the Toda lattice equation [18] and multiple periodic wave solutions to the Hirota bilinear equations [19, 20]. Therefore, it naturally follows that there should be a similar direct approach for constructing multiple wave solutions to nonlinear equations. We would, in this paper, like to construct an answer by formulating a solution algorithm for computing multiple wave solutions to nonlinear equations. The approach will be illustrated step by step whilst being applied to an example, providing the general features of solving nonlinear equations by adopting linear ones.

The application example we will present is the (3 + 1)-dimensional potential-Yu-Toda-Sasa-Fukuyama equation (potential-YTSF equation):

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + 3u_{yy} = 0. \quad (1.1)$$

This equation is a potential-type counterpart of a (3 + 1)-dimensional nonlinear equation

$$[-4v_t + \Phi(v)v_z]_x + 3v_{yy} = 0, \quad \Phi = \partial^2 + 4v + 2v_x \partial^{-1}, \quad (1.2)$$

introduced by Yu *et al* in [21] while making a (3+1)-dimensional generalization from the (2+1)-dimensional Calogero-Bogoyavlenskii–Schiff equation (see e.g. [22] and references therein),

$$-4v_t + \Phi(v)v_z = 0, \quad \Phi = \partial^2 + 4v + 2v_x \partial^{-1}, \quad (1.3)$$

as they did for the Kadomtsev–Petviashvili (KP) equation from the KdV equation. Taking $v = u_x$ transforms equation (1.2) into the potential-YTTSF equation (1.1) [23]. We also remark that equation (1.1) itself becomes the potential KP equation if $z = x$ and reduces to the potential KdV equation if $u_y = 0$ is also taken. Therefore, various applications of the KP and KdV equations show great potential for application of (1.1) in the physical sciences.

Obviously, the potential-YTTSF equation (1.1) has solutions independent of two variables,

$$\begin{aligned} u &= f(z, t), & u &= f(x) + g(t), & u &= cx + f(z), \\ u &= cy + f(z), & u &= cy + f(t), \end{aligned} \quad (1.4)$$

and a particular variable separated solution,

$$u = (cy + d)x + yf(z, t) + g(z, t), \quad (1.5)$$

where c and d are arbitrary constants and f, g and h are arbitrary functions in the indicated variables. A known solution $u = u(x, y, z, t)$ will lead to a new one,

$$v = u(x, y, z, t) + cy + f(t), \quad (1.6)$$

where c is an arbitrary constant and f is an arbitrary function in t .

A Bäcklund transformation of the type $v = 2(\ln \phi)_x + u$ was constructed by Yan in [24] and a class of other variable separated solutions was constructed in [24–27]. It is worth noting that variable separated solutions exist ubiquitously for (2+1)-dimensional integrable equations (see, e.g. [28]). We will formulate a multiple exp-function solution method and present a few broad classes of exact wave solutions, including one-soliton, two-soliton and three-soliton type solutions, to the potential-YTTSF equation (1.1). In particular, our multiple exp-function method will yield two different classes of two-wave and three-wave solutions to the potential-YTTSF equation, and every class contains diverse soliton-type solutions, both analytic and singular.

The paper is organized as follows. In section 2, a direct formulation of the procedure for generating multiple wave solutions to nonlinear equations is made, by seeking rational solutions in new variables defining individual waves. In section 3, the method is applied to construct multiple wave solutions to the (3+1)-dimensional potential-YTTSF equation. We conclude the paper in the final section, along with a discussion on polynomial solutions.

2. A multiple exp-function method

Let us formulate our solution procedure by focusing on a scalar (1+1)-dimensional PDE,

$$P(x, t, u_x, u_t, \dots) = 0, \quad (2.1)$$

which is assumed to be of differential polynomial type like the KdV equation. The solution method will also work for systems of nonlinear equations and high-dimensional ones.

Step 1. Defining solvable differential equations

We introduce a sequence of new variables $\eta_i = \eta_i(x, t)$, $1 \leq i \leq n$, by solvable PDEs, for instance, the linear ones,

$$\eta_{i,x} = k_i \eta_i, \quad \eta_{i,t} = -\omega_i \eta_i, \quad 1 \leq i \leq n, \quad (2.2)$$

where k_i , $1 \leq i \leq n$, are the angular wave numbers and ω_i , $1 \leq i \leq n$, are the wave frequencies. This is often a starting point for constructing exact solutions to nonlinear equations, since no method can solve nonlinear equations directly. Solving such linear equations leads to the exponential function solutions,

$$\eta_i = c_i e^{\xi_i}, \quad \xi_i = k_i x - \omega_i t, \quad 1 \leq i \leq n, \quad (2.3)$$

where c_i , $1 \leq i \leq n$, are any constants, positive or negative. The arbitrariness of the constants c_i , $1 \leq i \leq n$, brings more choices for solutions than was possible in [29]. Each of the functions η_i , $1 \leq i \leq n$, describing a single wave and a multiple wave solution will be a combination of all of those single waves.

We emphasize that the linear differential relations in (2.2) are extremely helpful while transforming differential equations to algebraic equations and carrying out related computations by computer algebra systems. The explicit solutions (2.3) offer reasons as to why the approach is called the multiple exp-function method. The idea of using linear differential conditions could also be applied to other systems, in which there might be diverse solutions [30]. Both the differential relations and the solution formulae are important in understanding and applying the approach.

The basic idea of using solvable differential equations was also successfully used to solve the (2+1)-dimensional KdV–Burgers equation through a second-order ODE $a\eta'' + b\eta' + c\eta^2 + d\eta = 0$ ($a, b, c, d = \text{const}$) in [31], and the Kolmogorov–Petrovskii–Piskunov equation through a first-order ODE $\eta' = 1 \pm \eta^2$ in [9]. It has been broadly adopted in the tanh-function type methods [10, 11, 14], the Jacobi elliptic function method [32, 33], the mapping method [34, 35], the F -expansion-type methods [36–38] and the G'/G -expansion method [39].

Step 2. Transforming nonlinear PDEs

Let us proceed by considering rational solutions in the new variables η_i , $1 \leq i \leq n$:

$$\begin{aligned} u(x, t) &= \frac{p(\eta_1, \eta_2, \dots, \eta_n)}{q(\eta_1, \eta_2, \dots, \eta_n)}, & p &= \sum_{r,s=1}^n \sum_{i,j=0}^M p_{rs,ij} \eta_r^i \eta_s^j, \\ q &= \sum_{r,s=1}^n \sum_{i,j=0}^N q_{rs,ij} \eta_r^i \eta_s^j, \end{aligned} \quad (2.4)$$

where $p_{kl,ij}$ and $q_{kl,ij}$ are all constants to be determined from the original equation (2.1). All Laurent polynomial and polynomial functions are only special examples of rational

functions and so we can similarly have a multiple tanh-coth method for obtaining multiple wave solutions to nonlinear equations.

By using the differential relations in (2.2), it is straightforward to express all partial derivatives of u with x and t in terms of η_i , $1 \leq i \leq n$. For example, we can have

$$u_t = \frac{q \sum_{i=1}^n p_{\eta_i} \eta_{i,t} - p \sum_{i=1}^n q_{\eta_i} \eta_{i,t}}{q^2} = \frac{-q \sum_{i=1}^n \omega_i p_{\eta_i} \eta_i + p \sum_{i=1}^n \omega_i q_{\eta_i} \eta_i}{q^2}, \quad (2.5)$$

and

$$u_x = \frac{q \sum_{i=1}^n p_{\eta_i} \eta_{i,x} - p \sum_{i=1}^n q_{\eta_i} \eta_{i,x}}{q^2} = \frac{q \sum_{i=1}^n k_i p_{\eta_i} \eta_i - p \sum_{i=1}^n k_i q_{\eta_i} \eta_i}{q^2}, \quad (2.6)$$

where p_{η_i} and q_{η_i} are partial derivatives of p and q with respect to η_i . This way, we can see that all partial derivatives, not only u_t and u_x , will still be rational functions in the new variables η_i , $1 \leq i \leq n$. Substituting those new expressions for partial derivatives into the original equation (2.1) generates a rational function equation in the new variables η_i , $1 \leq i \leq n$:

$$Q(x, t, \eta_1, \eta_2, \dots, \eta_n) = 0. \quad (2.7)$$

This is called the transformed equation of the original equation (2.1). The step here makes it possible to compute solutions to differential equations directly by computer algebra systems.

Step 3. Solving algebraic systems

Now we set the numerator of the resulting rational function $Q(x, t, \eta_1, \eta_2, \dots, \eta_n)$ to zero. This yields a system of algebraic equations on all variables $k_i, \omega_i, p_{kl,ij}, q_{kl,ij}$. We solve this system to determine two polynomials p and q and the wave exponents ξ_i , $1 \leq i \leq n$. All computation can be done systematically by computer algebra systems such as Maple. We point out that the resulting algebraic systems may be complicated and so a computer program really helps. Now, the multiple wave solution u is computed and given by

$$u(x, t) = \frac{p(c_1 e^{k_1 x - \omega_1 t}, \dots, c_n e^{k_n x - \omega_n t})}{q(c_1 e^{k_1 x - \omega_1 t}, \dots, c_n e^{k_n x - \omega_n t})}. \quad (2.8)$$

Since we begin with the exponential function solutions to the initial linear equations, we call the above method a multiple exp-function method. If we choose some other linear equations we can, for instance, have a multiple sine-cosine method to get multiple periodic wave solutions to nonlinear equations. Clearly, our multiple exp-function method in the case of $n = 1$ becomes the so-called exp-function method proposed by He and Wu in [15].

The solution procedure described above provides a direct and systematic solution procedure for generating multiple wave solutions and allows us to carry out the involved

computation conveniently by powerful computer algebra systems such as Maple, Mathematica, MuPAD and Matlab. It also presents a generalization of Hirota's perturbation scheme to construct multi-soliton solutions [18]. We will analyze three cases of polynomials p and q for the (3+1)-dimensional potential-YTSF equation (1.1), to construct its multiple wave solutions.

3. One-wave, two-wave and three-wave solutions to the potential-YTSF equation

Let us apply our multiple exp-function method to the (3+1)-dimensional potential-YTSF equation (1.1). We will discuss three cases of two polynomial functions p and q to generate one-wave, two-wave and three-wave solutions as follows.

Case 1. One-wave solutions

We require the linear conditions,

$$\eta_{1,x} = k_1 \eta_1, \quad \eta_{1,y} = l_1 \eta_1, \quad \eta_{1,z} = m_1 \eta_1, \quad \eta_{1,t} = -\omega_1 \eta_1, \quad (3.1)$$

where k_1, l_1, m_1 and ω_1 are constants. We then try a pair of polynomials of degree one,

$$p(\eta_1) = a_0 + a_1 \eta_1, \quad q(\eta_1) = b_0 + b_1 \eta_1, \quad (3.2)$$

where a_0, a_1, b_0 and b_1 are constants to be determined. By the multiple exp-function method and using the differential relations in (3.1), we obtain the following solution to the resulting algebraic system with Maple,

$$a_1 = \frac{b_1(2k_1 b_0 + a_0)}{b_0}, \quad \omega_1 = -\frac{1}{4} k_1^2 m_1 - \frac{3l_1^2}{4k_1}, \quad (3.3)$$

and all other constants are arbitrary. Since we can have an exponential function solution to (3.1),

$$\eta_1 = e^{k_1 x + l_1 y + m_1 z - \omega_1 t}, \quad (3.4)$$

the corresponding one-wave solutions read

$$u = u(x, y, z, t) = \frac{p}{q} = \frac{a_0 + a_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}}{b_0 + b_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}}, \quad (3.5)$$

where a_1 and ω_1 are defined by (3.3) and all other involved constants are arbitrary. This is in agreement with the selection for the one-soliton solution in [40] and contains all exact solutions in [41]. Note that the wave frequency depends on all angular wave numbers in the one-wave solutions above, but we will see that it is not the case in the two-wave and three-wave solutions below.

Case 2. Two-wave solutions

Similarly, we require the linear conditions,

$$\eta_{i,x} = k_i \eta_i, \quad \eta_{i,y} = l_i \eta_i, \quad \eta_{i,z} = m_i \eta_i, \quad \eta_{i,t} = -\omega_i \eta_i, \quad 1 \leq i \leq 2, \quad (3.6)$$

where k_i, l_i, m_i, ω_i , $1 \leq i \leq 2$, are constants and thus the solutions η_1 and η_2 can be defined by

$$\eta_i = c_i e^{k_i x + l_i y + m_i z - \omega_i t}, \quad 1 \leq i \leq 2, \quad (3.7)$$

where c_1 and c_2 are arbitrary constants.

Let us try a particular pair of polynomials of degree two,

$$\begin{cases} p(\eta_1, \eta_2) = 2[k_1\eta_1 + k_2\eta_2 + a_{12}(k_1 + k_2)\eta_1\eta_2], \\ q(\eta_1, \eta_2) = 1 + \eta_1 + \eta_2 + a_{12}\eta_1\eta_2, \end{cases} \quad (3.8)$$

where a_{12} is a constant to be determined. By the multiple exp-function method and using the differential relations in (3.6), we can have two solutions to the resulting algebraic system with Maple,

$$\omega_i = -\frac{3}{4}k_i - \frac{1}{4}k_i^2 m_i, \quad 1 \leq i \leq 2, \quad (3.9)$$

and

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad (3.10)$$

when $l_i = k_i, 1 \leq i \leq 2$, and

$$\omega_i = -\frac{1}{4}k_i^3 - \frac{3l_i^2}{4k_i}, \quad 1 \leq i \leq 2 \quad (3.11)$$

and

$$a_{12} = \frac{(k_1k_2^2 - k_1^2k_2 + k_1l_2 - l_1k_2)(k_1k_2^2 - k_1^2k_2 - k_1l_2 + l_1k_2)}{(k_1k_2^2 + k_1^2k_2 + k_1l_2 - l_1k_2)(k_1k_2^2 + k_1^2k_2 - k_1l_2 + l_1k_2)}, \quad (3.12)$$

when $m_i = k_i, 1 \leq i \leq 2$.

Then, the two corresponding two-wave solutions are determined by

$$u = u(x, y, z, t) = \frac{p(\eta_1, \eta_2, \eta_3)}{q(\eta_1, \eta_2, \eta_3)} = \frac{2[k_1\eta_1 + k_2\eta_2 + a_{12}(k_1 + k_2)\eta_1\eta_2]}{1 + \eta_1 + \eta_2 + a_{12}\eta_1\eta_2}, \quad (3.13)$$

where η_1 and η_2 are defined by (3.7), either with the frequencies ω_1 and ω_2 being given by (3.9) and a_{12} by (3.10) when $l_i = k_i, 1 \leq i \leq 2$, or with the frequencies ω_1 and ω_2 being given by (3.11) and a_{12} by (3.12) when $m_i = k_i, 1 \leq i \leq 2$. All unspecified constants involved in the solutions are arbitrary. There is a different selection of frequencies in [42] but it does not lead to exact non-constant solutions. Two specific solutions of the above two-wave solutions are plotted in figures 1 and 2. In each figure, the first plot is three dimensional, and the other plots exploit the x -, y - and z -curves or the contour plots with $z = 0$ at different times.

Case 3. Three-wave solutions

Again similarly, we require the linear conditions,

$$\begin{aligned} \eta_{i,x} &= k_i\eta_i, & \eta_{i,y} &= l_i\eta_i, & \eta_{i,z} &= m_i\eta_i, \\ \eta_{i,t} &= -\omega_i\eta_i, & 1 \leq i &\leq 3, \end{aligned} \quad (3.14)$$

where $k_i, l_i, m_i, \omega_i, 1 \leq i \leq 3$, are constants and thus the solutions η_1, η_2 and η_3 can be defined by

$$\eta_i = c_i e^{k_i x + l_i y + m_i z - \omega_i t}, \quad 1 \leq i \leq 3, \quad (3.15)$$

where c_1, c_2 and c_3 are arbitrary constants.

Let us now try a particular pair of polynomials of degree three,

$$\begin{cases} p(\eta_1, \eta_2, \eta_3) = 2[k_1\eta_1 + k_2\eta_2 + k_3\eta_3 + a_{12}(k_1 + k_2)\eta_1\eta_2 \\ \quad + a_{13}(k_1 + k_3)\eta_1\eta_3 + a_{23}(k_2 + k_3)\eta_2\eta_3 \\ \quad + a_{12}a_{13}a_{23}(k_1 + k_2 + k_3)\eta_1\eta_2\eta_3], \\ q(\eta_1, \eta_2, \eta_3) = 1 + \eta_1 + \eta_2 + \eta_3 + a_{12}\eta_1\eta_2 + a_{13}\eta_1\eta_3 \\ \quad + a_{23}\eta_2\eta_3 + a_{12}a_{13}a_{23}\eta_1\eta_2\eta_3, \end{cases} \quad (3.16)$$

where a_{12}, a_{13} and a_{23} are constants to be determined. By the multiple exp-function method and using the differential relations in (3.14), we can have two solutions to the resulting algebraic system with Maple,

$$\omega_i = -\frac{3}{4}k_i - \frac{1}{4}k_i^2 m_i, \quad 1 \leq i \leq 3 \quad (3.17)$$

and

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i, j \leq 3, \quad (3.18)$$

when $l_i = k_i, 1 \leq i \leq 3$, and

$$\omega_i = -\frac{1}{4}k_i^3 - \frac{3l_i^2}{4k_i}, \quad 1 \leq i \leq 3 \quad (3.19)$$

and

$$a_{ij} = \frac{(k_i k_j^2 - k_i^2 k_j + k_i l_j - l_i k_j)(k_i k_j^2 - k_i^2 k_j - k_i l_j + l_i k_j)}{(k_i k_j^2 + k_i^2 k_j + k_i l_j - l_i k_j)(k_i k_j^2 + k_i^2 k_j - k_i l_j + l_i k_j)}, \quad 1 \leq i, j \leq 3, \quad (3.20)$$

when $m_i = k_i, 1 \leq i \leq 3$.

Then the two corresponding three-wave solutions are given by

$$u = u(x, y, z, t) = \frac{p(\eta_1, \eta_2, \eta_3)}{q(\eta_1, \eta_2, \eta_3)}, \quad (3.21)$$

where p and q are defined by (3.16) and η_1, η_2 and η_3 are defined by (3.15), either with the frequencies ω_1, ω_2 and ω_3 being given by (3.17) and a_{12}, a_{13} and a_{23} by (3.18) when $l_i = k_i, 1 \leq i \leq 3$, or with the frequencies ω_1, ω_2 and ω_3 being given by (3.19) and a_{12}, a_{13} and a_{23} by (3.20) when $m_i = k_i, 1 \leq i \leq 3$. All unspecified constants involved in the solutions are arbitrary. Two specific solutions of those three-wave solutions are plotted in figures 3 and 4. In each figure, the first plot is three dimensional, and the other plots exploit the x -curves with $y = 1$ and different z -values at different times or the contour plots with $z = 0$ at different times.

We emphasize that through the proposed multiple exp-function algorithm, two kinds of two-wave solutions and three-wave solutions are easily obtained for the potential-YTSF equation (1.1). If, for two-wave and three-wave solutions, we take the general wave frequencies as in (3.3), where m_1, k_1 and l_1 have no relation, we will meet contradictions in the resulting algebraic systems. On the other hand, if the involved constants in (3.5) satisfy $b_0 b_1 < 0$ and some of the constants $c_i, 1 \leq i \leq n$, in (3.13) and (3.21) are negative, the corresponding exact solutions become singular. Moreover, for the second case (i.e. $m_i = k_i$), even if the constants c_i are positive in (3.13) and (3.21), the constants a_{ij}

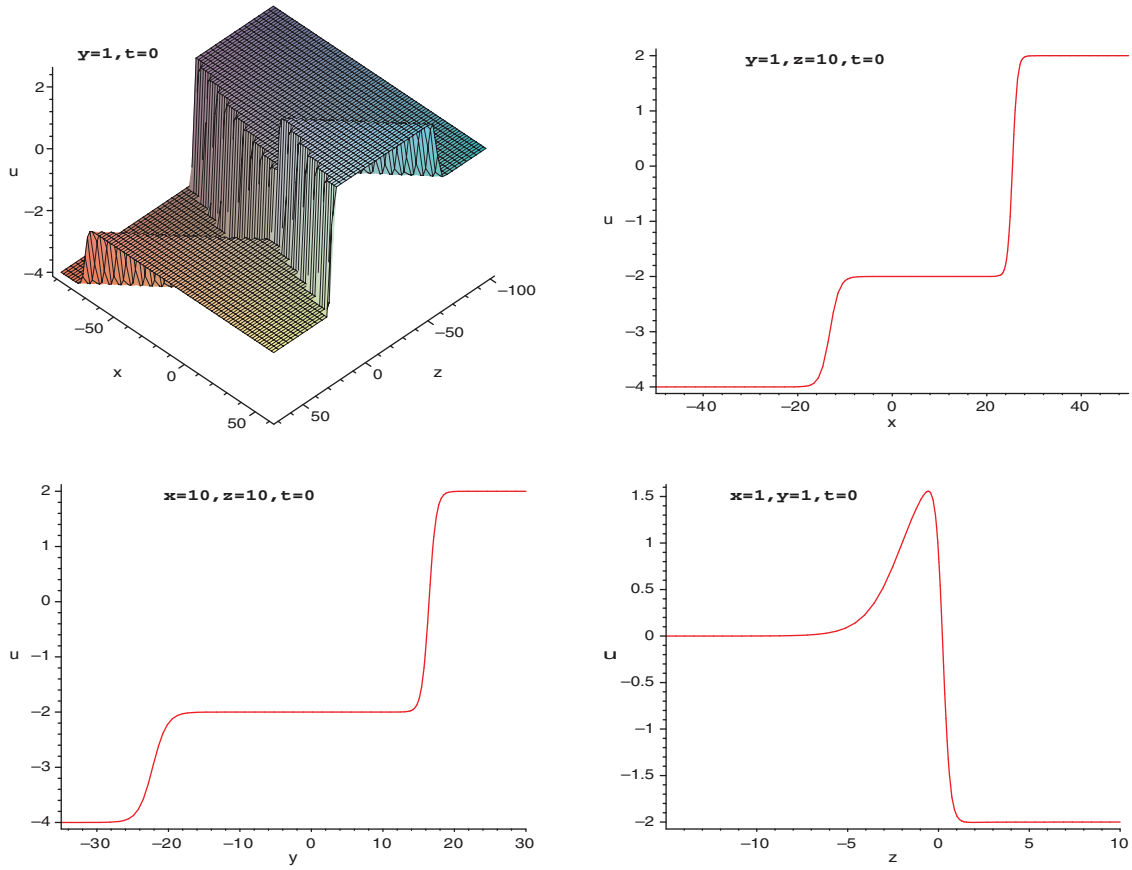


Figure 1. The first two-wave solution with $k_1 = 1, k_2 = -2, m_1 = 1, m_2 = 5, c_1 = 1$ and $c_2 = 2$.

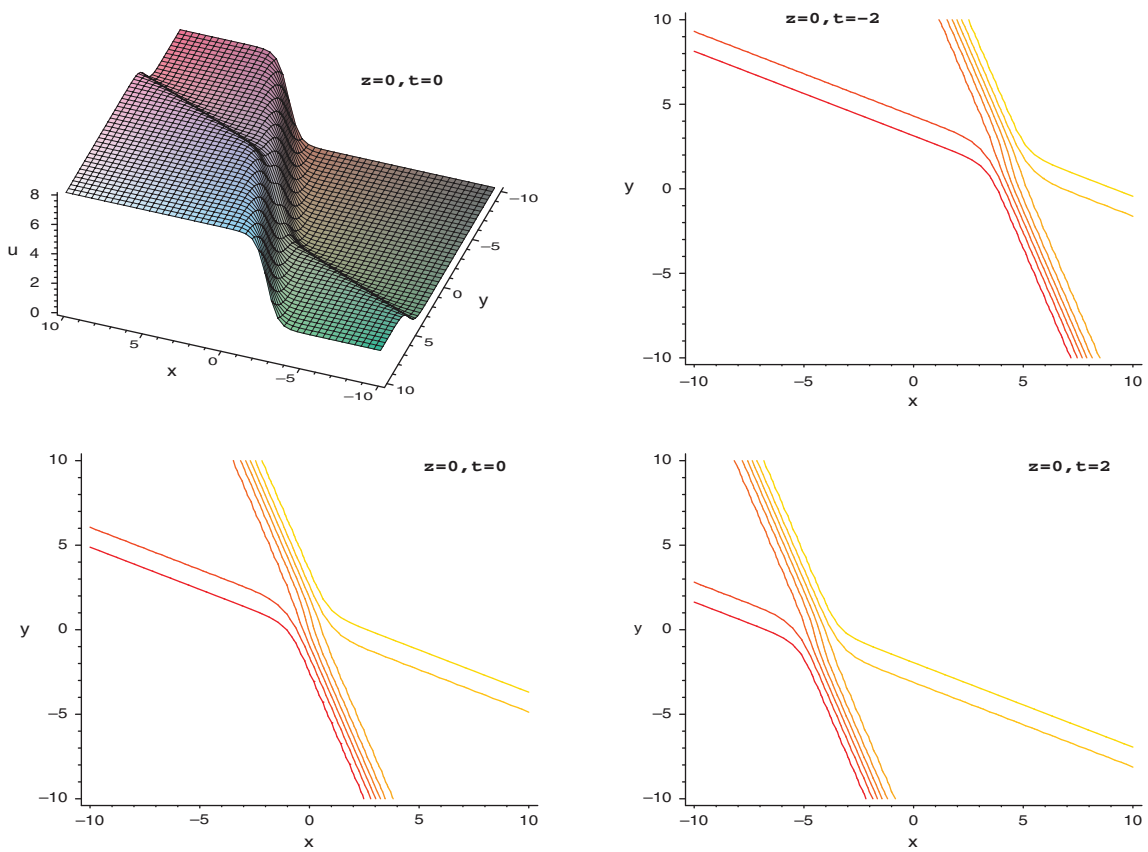


Figure 2. The second two-wave solution with $k_1 = 1, k_2 = 3, l_1 = 2, l_2 = 1, c_1 = 1$ and $c_2 = 2$.

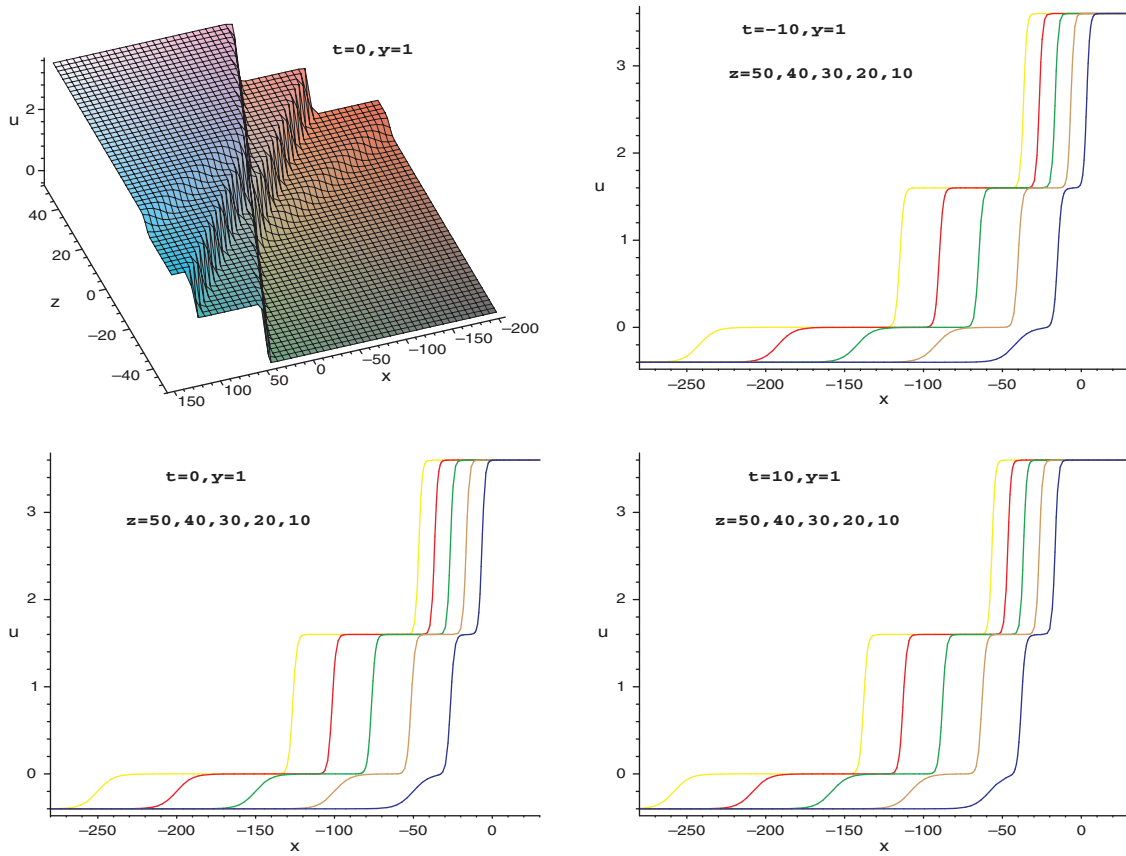


Figure 3. The first three-wave solution with $k_1 = 0.8, k_2 = 1.6, k_3 = -0.6, l_1 = -2, l_2 = 3, l_3 = -1.5, c_1 = 0.9, c_2 = 0.8$ and $c_3 = 1.2$.

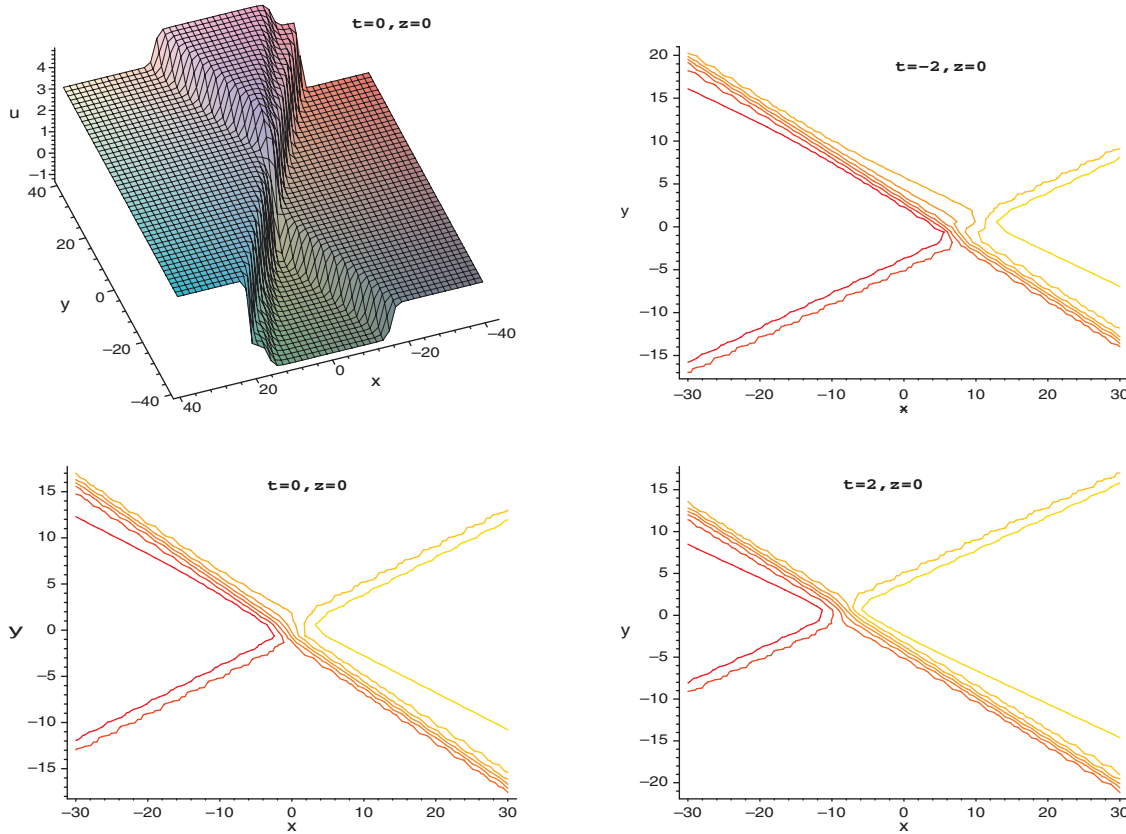


Figure 4. The second three-wave solution with $k_1 = 0.8, k_2 = 1.6, k_3 = -0.6, l_1 = -2, l_2 = 3, l_3 = -1.5, c_1 = 0.9, c_2 = 0.8$ and $c_3 = 1.2$.

can be negative and thus the solutions (3.13) and (3.21) can be singular. Taking special constants in our one-wave, two-wave and three-wave solutions and considering equal angular wave numbers $l_i = m_i = k_i$ yields all special soliton solutions to the potential-YTSF equation (1.1), as presented by Wazwaz in [43].

4. Concluding remarks

A direct and systematic solution procedure for constructing multiple wave solutions to nonlinear partial differential equations is proposed. The presented method is oriented towards the ease of use and capability of computer algebra systems, allowing us to carry out the involved computations conveniently through powerful computer algebra systems. It is by the use of computer algebra systems that in each of the cases of two-wave and three-wave solutions, we are able to present two classes of concrete exact explicit solutions to the (3+1)-dimensional potential-YTSF equation, only in the form of $u = f(t, x + y, z)$ or $u = f(t, x + z, y)$ (but not in a general form or in the form $u = f(t, x, y + z)$). The key point of our approach is to seek rational solutions in a set of new variables defining individual waves. The application of our method yields specific one-wave, two-wave and three-wave solutions to the (3+1)-dimensional potential-YTSF equation. The method can also be easily applied to other nonlinear evolution and wave equations in mathematical physics.

It is easy to check that the (3+1)-dimensional potential-YTSF equation (1.1) has a class of polynomial solutions,

$$u_1 = u_1(x, y, z, t) = a_1 + a_2x + a_3y + a_4z + a_5t + a_6xy + a_7yz + a_8yt + a_9zt + a_{10}yzt, \quad (4.1)$$

where a_i , $1 \leq i \leq 10$, are arbitrary constants. These are all polynomial solutions among a class of polynomial functions with $\deg(u_1, x) = \deg(u_1, y) = \deg(u_1, z) = \deg(u_1, t) = 1$. On the other hand, there are two other solutions,

$$u_2 = u_2(x, y, z, t) = a_1 + a_2x + a_3y + a_4z + a_5t + f(z, t) \quad (4.2)$$

and

$$u_3 = u_3(x, y, z, t) = a_1 + a_2x + a_3z + a_4t + g(x) + h(t), \quad (4.3)$$

where a_i , $1 \leq i \leq 5$, are arbitrary constants and f , g and h are arbitrary functions in the indicated variables. Taking f , g and h as polynomials engenders other polynomial solutions to the potential-YTSF equation (1.1), which can be of high degree. But the third one reduces to solutions to the (2+1)-dimensional potential Calogero–Bogoyavlenskii–Schiff equation, independent of the variable y .

It is our guess that higher-wave solutions to the (3+1)-dimensional potential-YTSF equation (1.1) could be presented in a parallel manner. But the required computation is complicated, even in the case of four-wave solutions. We hope that they could be presented and verified in some analytic way. Any general form of two-waves and three-waves, which does not involve any relation among the angular wave numbers k_i , l_i and m_i , would be more interesting and important.

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