International Journal of Geometric Methods in Modern Physics
Vol. 21, No. 11 (2024) 2450182 (18 pages)

DOI: 10.1142/S0219887824501822

Binary Darboux transformation of vector nonlocal reverse-space nonlinear Schrödinger equations

Wen-Xiu Ma*

Department of Mathematics, Zhejiang Normal University
Jinhua 321004, Zhejiang, P. R. China

Department of Mathematics, King Abdulaziz University
Jeddah 21598, Saudi Arabia

Department of Mathematics and Statistics
University of South Florida
Tampa, FL 33620-5700, USA

Material Science Innovation and Modelling
Department of Mathematical Sciences
North-West University, Mafikeng Campus
Mmabatho 2735, South Africa
mawx@cas.usf.edu

Yehui Huang

School of Mathematics and Physics
North China Electric Power University
Beijing 102206, P. R. China

Fudong Wang

School of Sciences, Great Bay University
Dongguan 523000, Guangdong, P. R. China

Yong Zhang

College of Mathematics and Systems Science
Shandong University of Science and Technology
Qingdao 266590, Shandong, P. R. China

Liyuan Ding

School of Science, Nanjing University of
Science and Technology
Nanjing 210094, Jiangsu, P. R. China

Received 31 October 2023
Accepted 22 March 2024
Published 10 April 2024

*Corresponding author.

For vector nonlocal reverse-space nonlinear Schrödinger equations, a binary Darboux transformation is formulated by using two sets of eigenfunctions and adjoint eigenfunctions. The resulting binary Darboux transformation has been decomposed into an N-fold product of single binary Darboux transformations. An application starting from zero seed potentials generates a class of soliton solutions.

Keywords: Nonlocal integrable equation; Darboux transformation; soliton solution.

Mathematics Subject Classification 2020: 35Q55, 37K15, 37K40

1. Introduction

The Darboux transformation stands among the most efficient approaches to soliton solutions of integrable equations in soliton theory. A Lax pair of matrix eigenvalue problems is the key to generating Darboux transformations. A binary Darboux transformation needs both a Lax pair of matrix eigenvalue problems and the other pair of adjoint matrix eigenvalue problems. We would like to analyze a general binary Darboux transformation associated with a multicomponent AKNS spatial eigenvalue problem.

Let μ stand for an eigenvalue parameter and $u = u(x,t)$, a potential vector, where x is the spatial variable and t is the temporal variable. We begin with a Lax pair of matrix eigenvalue problems:

$$-i\xi_x = U\xi, \quad -i\xi_t = V\xi, \tag{1.1}$$

where i is the unit imaginary number and ξ is an $m \times m$ matrix eigenfunction. The involved Lax pair, i.e. the pair of U and V, is defined by

$$\begin{cases} U = U(\mu) = U(u,\mu) = A(\mu) + P(u,\mu), \\ V = V(\mu) = V(u,\mu) = V(u, u_x, \ldots, u^{(n_0)}; \mu) = B(\mu) + Q(u,\mu). \end{cases} \tag{1.2}$$

It is often assumed that A and B are constant commuting $m \times m$ matrices, and P and Q are trace-less $m \times m$ matrices, which depend on the potential vector u. Usually, as the compatibility condition of the above Lax pair of matrix eigenvalue problems, the zero curvature equation

$$U_t - V_x + i[U,V] = 0, \tag{1.3}$$

where $[\cdot,\cdot]$ denotes the matrix commutator, yields an integrable equation, which are solvable by the inverse scattering transform [1–3]. Generally, we take reduced Lax pairs and zero curvature equations, in order to construct nonlocal integrable counterparts (see, for example [4]).

The pair of adjoint matrix eigenvalue problems is given by

$$i\tilde{\xi}_x = \tilde{\xi}U, \quad i\tilde{\xi}_t = \tilde{\xi}V. \tag{1.4}$$

The compatibility condition of this adjoint pair of eigenvalue problems engenders the same zero curvature equation as above. Obviously, the inverse ξ^{-1} of a matrix eigenfunction ξ satisfies the adjoint pair (1.4), and so, it gives rise to an adjoint matrix eigenfunction. This property has been used in establishing associated Riemann–Hilbert problems for classical local multicomponent integrable equations (see, for example [5]).

A binary Darboux transformation consists of

$$\xi' = S^{+}\xi, \quad \tilde{\xi}' = \tilde{\xi}S^{-}, \quad u' = f(u), \tag{1.5}$$

provided that

$$U' = -iS_x^{+}(S^{+})^{-1} + S^{+}U(S^{+})^{-1}, \quad V' = -iS_t^{+}(S^{+})^{-1} + S^{+}V(S^{+})^{-1}, \tag{1.6}$$

where S^{+} and S^{-} are inverse to each other, and $U' = U'(u', \mu) = U(u', \mu)$ and $V' = V'(u', \mu) = V(u', \mu)$. Therefore, ξ' and $\tilde{\xi}'$ satisfy

$$-i\xi'_x = U'\xi', \quad -i\xi'_t = V'\xi', \tag{1.7}$$

and

$$i\tilde{\xi}'_x = \tilde{\xi}'U', \quad i\tilde{\xi}'_t = \tilde{\xi}'V', \tag{1.8}$$

respectively. Either of these guarantees that the new Lax pair of U' and V' leads still to the same zero curvature equation, where u is replaced with u'. Consequently, the new potential u' provides another solution to the corresponding integrable equation. There exist abundant examples of applications of binary Darboux transformations to integrable nonlinear Schrödinger (NLS) equations in the literature (see, for example [6–11]).

In this paper, we would like to formulate a binary Darboux transformation and establish its link with the N-fold binary Darboux transformation. An application of the presenting binary Darboux transformation starting from the zero seed solution generates soliton solutions, and both local and nonlocal multicomponent integrable NLS equations will be illustrative examples. The conclusion, along with a few comments and remarks, is given in Sec. 5.

2. Multicomponent Nonlinear Schrödinger Equations

Let $n \in \mathbb{N}$ be arbitrary, I_n denote the nth-order identity matrix, and $\{\delta_1, \delta_2\}$ and $\{\gamma_1, \gamma_2\}$ stand for two pairs of different constants. The matrix eigenvalue problems of the multicomponent NLS equations read (see, for example [12]):

$$-i\xi_x = U\xi = U(u, \mu)\xi, \quad -i\xi_t = V\xi = V(u, \mu)\xi, \tag{2.1}$$

with the Lax pair being defined by

$$U = \mu\Lambda + P, \quad V = \mu^2\Theta + Q. \tag{2.2}$$

Here, those involved four matrices are given by

$$\Lambda = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 I_n \end{bmatrix}, \quad P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.3}$$

$$\Theta = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 I_n \end{bmatrix}, \quad Q = Q(u, \mu) = \frac{\gamma}{\delta}\mu \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\gamma}{\delta^2} \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}, \tag{2.4}$$

where the potential vector is $u = (p, q^T)^T$ with $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)^T$, and two nonzero constants are $\delta = \delta_1 - \delta_2$ and $\gamma = \gamma_1 - \gamma_2$. Note that the matrix Q can be determined by the potential matrix P in the following way:

$$Q = \frac{\gamma}{\delta}\mu P - \frac{\gamma}{\delta^2}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix} P^2 + \begin{bmatrix} i & 0 \\ 0 & -iI_n \end{bmatrix} P_x\right\} = \frac{\gamma}{\delta}\mu P - \frac{\gamma}{\delta^2} I_{1,n}(P^2 + iP_x), \tag{2.5}$$

where $I_{1,n} = \mathrm{diag}(1, -I_n)$.

When only one pair of p_j and $q_j, 1 \leq j \leq n$, is nonzero, the spatial eigenvalue problem in (2.1) becomes the standard AKNS eigenvalue problem [13]. The multicomponent AKNS spatial matrix eigenvalue problem in (2.1) is degenerate because Λ has a multiple eigenvalue δ_2.

Clearly, the compatibility condition of the matrix eigenvalue problems above yields the classical local multicomponent NLS equations:

$$p_{j,t} = -\frac{\gamma}{\delta^2} i \left[p_{j,xx} + 2\left(\sum_{r=1}^{n} p_r q_r\right) p_j \right],$$

$$q_{j,t} = \frac{\gamma}{\delta^2} i \left[q_{j,xx} + 2\left(\sum_{r=1}^{n} p_r q_r\right) q_j \right], \quad 1 \leq j \leq n. \tag{2.6}$$

When $n = 2$, under a specific kind of symmetric reductions, the above multicomponent integrable NLS equations (2.6) can be reduced to the Manokov system [14], which has an N-fold decomposition into finite-dimensional integrable Hamiltonian systems [15].

Following the idea of conducting group reductions in [16], we can form a kind of particular nonlocal group reductions:

$$U^\dagger(-x, t, -\mu^*)C + CU(x, t, \mu) = 0, \quad C = \mathrm{diag}(1, \Sigma), \ \ \Sigma^\dagger = \Sigma, \tag{2.7}$$

for the spectral matrix U [17]. This, equivalently, demands a condition on P:

$$P^\dagger(-x, t)C + CP(x, t) = 0. \tag{2.8}$$

Henceforth, $\dagger$ stands for the Hermitian transpose, $*$ denotes the complex conjugate, and Σ is a constant invertible Hermitian matrix.

Based on the nonlocal group reductions determined by (2.8), we can arrive at the reductions between the two potential matrices:

$$\Sigma q(x, t) + p^\dagger(-x, t) = 0, \tag{2.9}$$

where Σ is an arbitrary invertible Hermitian matrix. It further follows that

$$V^\dagger(-x, t, -\mu^*) = CV(x, t, \mu)C^{-1}, \quad Q^\dagger(-x, t, -\mu^*) = CQ(x, t, \mu)C^{-1}, \tag{2.10}$$

where the two matrices V and Q are given by (2.2) and (2.4), respectively.

It is straightforward to see that there is no new condition for the spatial and temporal matrix eigenvalue problems in (2.1), besides the nonlocal reductions in (2.8) (or equivalently, (2.9)). The reductions in (2.8) put the multicomponent classical local integrable NLS equations (2.6) into the following multicomponent nonlocal reverse-space integrable NLS equations:

$$ip_t(x,t) = \frac{\gamma}{\delta^2}[p_{xx}(x,t) - 2p(x,t)\Sigma^{-1}p^\dagger(-x,t)p(x,t)], \tag{2.11}$$

where the Hermitian matrix Σ is arbitrary but invertible.

When $n = 1$, we can get a well-known example [18, 19]:

$$ip_t(x,t) = p_{xx}(x,t) + 2\sigma p^*(-x,t)p^2(x,t), \quad \sigma = \pm 1.$$

When $n = 2$, we can obtain a new system of nonlocal reverse-space integrable NLS equations [17]:

$$\begin{cases} ip_{1,t}(x,t) = p_{1,x,x}(x,t) + (\zeta_1 p_1^\dagger(-x,t)p_1(x,t) + \zeta_2 p_2^\dagger(-x,t)p_2(x,t))p_1(x,t), \\ ip_{2,t}(x,t) = p_{2,x,x}(x,t) + (\zeta_1 p_1^\dagger(-x,t)p_1(x,t) + \zeta_2 p_2^\dagger(-x,t)p_2(x,t))p_2(x,t). \end{cases}$$

where ζ_1 and ζ_2 are arbitrary nonzero constants.

3. Binary Darboux Transformation

3.1. *A general formulation*

Let us now propose a binary Darboux transformation, through applying a single pair of eigenfunction and adjoint eigenfunction:

$$-iv_{1,x} = U(u,\mu_1)v_1, \quad -iv_{1,t} = V(u,\mu_1)v_1, \tag{3.1}$$

and

$$i\hat{v}_{1,x} = \hat{v}_1 U(u,\hat{\mu}_1), \quad i\hat{v}_{1,t} = \hat{v}_1 V(u,\hat{\mu}_1), \tag{3.2}$$

where μ_1 and $\hat{\mu}_1$ are a pair of arbitrary complex eigenvalue and adjoint eigenvalue. The idea of using both kinds of eigenfunctions was also adopted in the study of symmetry constraints [20].

A single binary Darboux transformation can be stated below.

Theorem 3.1. *Assume that*

$$\begin{cases} S^+[1] = S^+[1](\mu) = I_{n+1} - \dfrac{\mu_1 - \hat{\mu}_1}{\mu - \hat{\mu}_1}\dfrac{v_1\hat{v}_1}{\hat{v}_1 v_1}, \\ S^-[1] = S^-[1](\mu) = I_{n+1} + \dfrac{\mu_1 - \hat{\mu}_1}{\mu - \mu_1}\dfrac{v_1\hat{v}_1}{\hat{v}_1 v_1}, \end{cases} \tag{3.3}$$

and

$$S_1^+[1] = \lim_{\mu\to\infty}[\mu(S^+[1] - I_{n+1})], \quad S_1^-[1] = \lim_{\mu\to\infty}[\mu(S^-[1] - I_{n+1})]. \tag{3.4}$$

Then we have

$$S^+[1](\mu_1)v_1 = 0, \quad \hat{v}_1 S^-[1](\hat{\mu}_1) = 0, \tag{3.5}$$

$$S^+[1](\mu)S^-[1](\mu) = I_{n+1}, \quad S_1^+[1] = -S_1^-[1]; \tag{3.6}$$

and

$$\xi' = S^+[1]\xi, \quad \tilde{\xi}' = \tilde{\xi}S^-[1], \quad P' = P + S_1^+[1]\Lambda + \Lambda S_1^-[1] = P + [S_1^+[1], \Lambda], \tag{3.7}$$

constitute a binary Darboux transformation for the multicomponent classical local integrable NLS equations (2.6).

Proof. It is easy to get

$$S_1^+[1] = -(\mu_1 - \hat{\mu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}, \quad S_1^-[1] = (\mu_1 - \hat{\mu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1 v_1}. \tag{3.8}$$

The properties (3.5) and (3.6) just follow some more direct computations.

What we need to check below are the following two conditions:

$$-i(S^+[1])_x = U'S^+[1] - S^+[1]U, \tag{3.9}$$

where $U' = \mu\Lambda + P'$ with P' defined in (3.7), and

$$-i(S^+[1])_t = V'S^+[1] - S^+[1]V, \tag{3.10}$$

where $V' = \mu\Theta + Q'$ with $Q' = Q|_{P=P'}$. Those two conditions guarantee that (3.7) presents a binary Darboux transformation.

The case of $\mu_1 = \hat{\mu}_1$ is obvious. Thus, we assume that $\mu_1 \neq \hat{\mu}_1$ below. We first consider the x-part of the above conditions. Note that

$$\begin{aligned}\left(\frac{\hat{v}_1 v_1}{\mu_1 - \hat{\mu}_1}\right)_x &= \frac{1}{\mu_1 - \hat{\mu}_1}(\hat{v}_{1,x}v_1 + \hat{v}_1 v_{1,x}) \\ &= \frac{1}{\mu_1 - \hat{\mu}_1}[-i\hat{v}_1 U(\hat{\mu}_1)v_1 + i\hat{v}_1 U(\mu_1)v_1] \\ &= i\hat{v}_1\frac{U(\mu_1) - U(\hat{\mu}_1)}{\mu_1 - \hat{\mu}_1}v_1 = i\hat{v}_1\Lambda v_1.\end{aligned} \tag{3.11}$$

Let us, on the one hand, compute that

$$\begin{aligned}-i(\mu - \hat{\mu}_1)(S^+[1])_x &= i\left(v_1\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\hat{v}_1\right)_x \\ &= iv_{1,x}\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\hat{v}_1 + iv_1\left(\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\right)_x\hat{v}_1 + iv_1\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\hat{v}_{1.x} \\ &= -U(\mu_1)v_1\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\hat{v}_1 + v_1\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}(\hat{v}_1\Lambda v_1)\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\hat{v}_1 \\ &\quad + v_1\frac{\mu_1 - \hat{\mu}_1}{\hat{v}_1 v_1}\hat{v}_1 U(\hat{\mu}_1) \\ &:= T_1 + T_2 + T_3,\end{aligned} \tag{3.12}$$

where we have used the result in (3.11) and the identity

$$(M^{-1})_x = -M^{-1}M_xM^{-1} \tag{3.13}$$

for a scalar or matrix function M.

On the other hand, we can have

$$U'S^+[1] - S^+[1]U = [U, S^+[1]] + [S_1^+[1], \Lambda]S^+[1].$$

Let us further compute the two terms in the above sum as follows:

$$\begin{aligned}
[U, S^+[1]] &= \left[U, -\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1}\right] \\
&= -(U(\mu_1)+(\mu-\mu_1)\Lambda)\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} \\
&\quad + \frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1}(U(\hat{\mu}_1)+(\mu-\hat{\mu}_1)\Lambda) \\
&= -U(\mu_1)\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} + \frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1}U(\hat{\mu}_1) \\
&\quad -(\mu-\mu_1)\Lambda\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} + \frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1}(\mu-\hat{\mu}_1)\Lambda \\
&= \frac{T_1}{\mu-\hat{\mu}_1} + \frac{T_3}{\mu-\hat{\mu}_1} - (\mu-\mu_1)\Lambda\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} - S_1^+[1]\Lambda,
\end{aligned}$$

and

$$\begin{aligned}
&[S_1^+[1], \Lambda]S^+[1] \\
&\quad = -[S_1^+[1], \Lambda]\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} + [S_1^+[1], \Lambda] \\
&\quad = (\mu_1-\hat{\mu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1v_1}\Lambda\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} - \Lambda(\mu_1-\hat{\mu}_1)\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} \\
&\qquad + S_1^+[1]\Lambda + \Lambda(\mu_1-\hat{\mu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1v_1} \\
&\quad = \frac{T_2}{\mu-\hat{\mu}_1} + S_1^+[1]\Lambda - \Lambda(\mu_1-\hat{\mu}_1)\frac{\mu_1-\hat{\mu}_1}{\mu-\hat{\mu}_1}\frac{v_1\hat{v}_1}{\hat{v}_1v_1} + \Lambda(\mu_1-\hat{\mu}_1)\frac{v_1\hat{v}_1}{\hat{v}_1v_1}.
\end{aligned}$$

Now, we can see that $(\mu-\hat{\mu}_1)(U'S^+[1]-S^+[1]U)$ does not depend on μ and it equals to $T_1+T_2+T_3$, which is $-i(\mu-\hat{\mu}_1)(S^+[1])_x$. Therefore, the condition (3.9) is satisfied.

Note that we have a formula (2.5) for Q, i.e., we can present Q in terms of P. A completely similar but lengthy argument can prove the t-part (3.10) of the conditions for the binary Darboux transformation. The proof is then finished. □

Let us now consider a general case. We begin with the two sets of given eigenfunctions and adjoint eigenfunctions, associated with arbitrary complex eigenvalues and adjoint eigenvalues, μ_k and $\hat{\mu}_k$, $1 \leq k \leq N$, and denote them by

$$-iv_{k,x} = U(u,\mu_k)v_k, \quad -iv_{k,t} = V(u,\mu_k)v_k, \;\; 1 \leq k \leq N, \tag{3.14}$$

and

$$i\hat{v}_{k,x} = \hat{v}_k U(u,\hat{\mu}_k), \quad i\hat{v}_{k,t} = \hat{v}_k V(u,\hat{\mu}_k), \;\; 1 \leq k \leq N. \tag{3.15}$$

A crucial step is to introduce a square matrix:

$$M = (m_{kl})_{N\times N}, \quad m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\mu_l - \hat{\mu}_k}, & \text{if } \mu_l \neq \hat{\mu}_k, \\ m^c_{kl}(x,t), & \text{if } \mu_l = \hat{\mu}_k, \end{cases} \quad \text{where } 1 \leq k,l \leq N. \tag{3.16}$$

The conditions on m^c_{kl} in the presentation of a new binary Darboux transformation will be determined later, but no condition on m^c_{kl} is required in the following analysis before Theorem 3.2.

Then, if M is invertible, we define

$$\begin{cases} S^+ = S^+(\mu) = I_{n+1} - \displaystyle\sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl}\hat{v}_l}{\mu - \hat{\mu}_l}, \\ S^- = S^-(\mu) = I_{n+1} + \displaystyle\sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl}\hat{v}_l}{\mu - \mu_k}, \end{cases} \tag{3.17}$$

and

$$S^\pm_1(\mu) = \lim_{\mu\to\infty} [\mu(S^\pm(\mu) - I_{n+1})], \tag{3.18}$$

where I_{n+1} is the $(n+1)$th-order identity matrix. We point out that the case of $\{\mu_k | 1 \leq k \leq N\} \cap \{\hat{\mu}_k | 1 \leq k \leq N\} = \emptyset$ and $\mu_k \neq \mu_l, \hat{\mu}_k \neq \hat{\mu}_l, 1 \leq k,l \leq N$, has been studied in the literature (see, for example [3, 21, 22]). However, the nonempty intersection is the case which produces soliton solutions to various nonlocal integrable equations, and we will see illustrative examples in the following section.

We can readily obtain the following results.

Proposition 3.1. *The following properties hold*:

$$S^+(\mu_k)v_k = 0, \quad \hat{v}_k S^-(\hat{\mu}_k) = 0, \;\; 1 \leq k \leq N, \tag{3.19}$$

if $S^+(\mu_k)$ and $S^-(\hat{\mu}_k)$ are well defined, respectively.

Taking advantage of the definition of the two Darboux matrices S^+ and S^- in (3.17), we can easily find their partial fraction decompositions in the following proposition.

Proposition 3.2. *Let the Darboux matrices S^+ and S^- be defined by* (3.17). *Then the partial fraction decompositions hold*:

$$S^+ = I_{n+1} - \sum_{k=1}^{N} \frac{v_k^M \hat{v}_k}{\mu - \hat{\mu}_k}, \quad S^- = I_{n+1} + \sum_{k=1}^{N} \frac{v_k \hat{v}_k^M}{\mu - \mu_k} \tag{3.20}$$

where

$$\begin{cases} (v_1^M, v_2^M, \ldots, v_N^M)M = (v_1, v_2, \ldots, v_N), \\ M((\hat{v}_1^M)^T, (\hat{v}_2^M)^T, \ldots, (\hat{v}_N^M)^T)^T = (\hat{v}_1^T, \hat{v}_2^T, \ldots, \hat{v}_N^T)^T. \end{cases} \tag{3.21}$$

Moreover, we can have the following property.

Proposition 3.3. *Under the orthogonal condition*

$$\hat{v}_k v_l = 0, \quad \text{if } \mu_l = \hat{\mu}_k, \text{ where } 1 \le k, l \le N, \tag{3.22}$$

we have

$$S^+(\mu) S^-(\mu) = I_{n+1}, \quad S_1^+ = -S_1^-, \tag{3.23}$$

which implies that $S^+(\mu)$ and $S^-(\mu)$ are inverse to each other.

Proof. For the sake of brevity, let us denote

$$v = (v_1, v_2, \ldots, v_N), \quad \hat{v} = (\hat{v}_1^T, \hat{v}_2^T, \ldots, \hat{v}_N^T)^T, \tag{3.24}$$

and

$$F = \begin{bmatrix} \frac{1}{\mu - \mu_1} & & & 0 \\ & \frac{1}{\mu - \mu_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\mu - \mu_N} \end{bmatrix}, \quad \hat{F} = \begin{bmatrix} \frac{1}{\mu - \hat{\mu}_1} & & & 0 \\ & \frac{1}{\mu - \hat{\mu}_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\mu - \hat{\mu}_N} \end{bmatrix}. \tag{3.25}$$

We can then show that

$$\hat{F} \hat{v} v F = MF - \hat{F} M. \tag{3.26}$$

Let us fix a pair of $1 \le k, l \le N$. If $\mu_l = \hat{\mu}_k$, then we first have $(\hat{F} \hat{v} v F)_{kl} = 0$, on account of (3.22). Second, we have

$$(MR - \hat{R}M)_{kl} = m_{kl} \frac{1}{\mu - \mu_l} - \frac{1}{\mu - \hat{\mu}_k} m_{kl} = 0,$$

by the definition of M in (3.16). Thus, $(\hat{F}\hat{v}vF)_{kl} = (MF - \hat{F}M)_{kl}$. On the other hand, if $\mu_l \neq \hat{\mu}_k$, we have

$$(\hat{F}\hat{v}vF)_{kl} = \frac{\hat{v}_k}{\mu - \hat{\mu}_k} \frac{v_l}{\mu - \hat{\mu}_l} = \left(\frac{1}{\mu - \mu_l} - \frac{1}{\mu - \hat{\mu}_k}\right) \frac{\hat{v}_k v_l}{\mu_l - \hat{\mu}_k} = (MF - \hat{F}M)_{kl}.$$

Therefore, the equality (3.26) holds.

Now taking (3.26) into account, we can find that

$$S^+ S^- = I_{n+1} - vM^{-1}\hat{F}\hat{v} + vFM^{-1}\hat{v} - vM^{-1}\hat{F}\hat{v}vFM^{-1}\hat{v} = I_{n+1},$$

since S^+ and S^- can be rewritten as

$$S^+ = I_{n+1} - vM^{-1}\hat{F}\hat{v}, \quad S^- = I_{n+1} + vFM^{-1}\hat{v}. \tag{3.27}$$

The second part of (3.23) is a consequence of

$$S_1^+ = -\sum_{k,l=1}^{N} (M^{-1})_{kl} v_k \hat{v}_l = -vM^{-1}\hat{v}, \quad S_1^- = \sum_{k,l=1}^{N} (M^{-1})_{kl} v_k \hat{v}_l = vM^{-1}\hat{v}. \tag{3.28}$$

This completes the proof. □

Now, we are ready to state the required general binary Darboux transformation for the multicomponent classical local integrable NLS equations (2.6) in the following theorem.

Theorem 3.2. *If the condition* (3.22) *and the conditions in*

$$m^c_{kl,x} = i\hat{v}_k \Lambda v_l, \quad \text{if } \mu_l = \hat{\mu}_k, \text{ where } 1 \le k, l \le N, \tag{3.29}$$

and

$$m^c_{kl,t} = i\hat{v}_k \Theta_{[k,l]} v_l, \quad \text{if } \mu_l = \hat{\mu}_k, \text{ where } 1 \le k, l \le N, \tag{3.30}$$

where

$$\Theta_{[k,l]} = (\hat{\mu}_k^2 + \hat{\mu}_k \mu_l + \mu_l^2)\Theta + \frac{\gamma}{\delta}(\hat{\mu}_k + \mu_l)P - \frac{\gamma}{\delta^2} I_{1,n}(P^2 + iP_x), \tag{3.31}$$

are satisfied, then the multicomponent classical local integrable NLS equations (2.6) *possess the binary Darboux transformation*:

$$\xi' = S^+\xi, \quad \tilde{\xi}' = \tilde{\xi}S^-, \quad P' = P + S_1^+\Lambda + \Lambda S_1^- = P + [S_1^+, \Lambda]. \tag{3.32}$$

Proof. What we need to prove is to verify the conditions in (3.9) and (3.10), which ensure the binary Darboux transformation (3.32).

The proof is analogous to the one of Theorem 3.1. Below, let us verify the x-part (3.9) of the conditions. First note that we have

$$v_x = i(\Lambda v D + P v), \quad \hat{v}_x = -i(\hat{D}\hat{v}\Lambda + \hat{v}P), \tag{3.33}$$

where

$$D = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N), \quad \hat{D} = \mathrm{diag}(\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_N). \tag{3.34}$$

Moreover, like (3.11), we can have

$$M_x = i\hat{v}\Lambda v, \tag{3.35}$$

on account of (3.30).

Now taking advantage of the matrix expressions for S^+ and S_1^+ in (3.27) and (3.28), and applying the derivative formulas in (3.13), (3.33) and (3.35), we can have that

$$\begin{aligned} -iS_x^+ &= (-\Lambda v D M^{-1}\hat{F}\hat{v} - PvM^{-1}\hat{F}\hat{v}) + vM^{-1}\hat{v}\Lambda v M^{-1}\hat{F}\hat{v} \\ &\quad + (vM^{-1}\hat{F}\hat{D}\hat{v}\Lambda + vM^{-1}\hat{F}\hat{v}P), \\ U'S^+ &= (\mu\Lambda + P - vM^{-1}\hat{v}\Lambda + \Lambda v M^{-1}\hat{v}) - (\mu\Lambda v M^{-1}\hat{F}\hat{v} + PvM^{-1}\hat{F}\hat{v} \\ &\quad - vM^{-1}\hat{v}\Lambda v M^{-1}\hat{F}\hat{v} + \Lambda v M^{-1}\hat{v} v M^{-1}\hat{F}\hat{v}), \\ S^+U &= (\mu\Lambda + P) - (\mu v M^{-1}\hat{F}\hat{v}\Lambda + vM^{-1}\hat{F}\hat{v}P). \end{aligned}$$

Upon some simplification, we can compute that

$$\begin{aligned} &-iS_x^+ - U'S^+ + S^+U \\ &\quad = -\Lambda v A M^{-1}\hat{F}\hat{v} + vM^{-1}\hat{F}\hat{D}\hat{v}\Lambda - \mu v M^{-1}\hat{F}\hat{v}\Lambda + vM^{-1}\hat{v}\Lambda \\ &\qquad - \Lambda v M^{-1}\hat{v} + \mu\Lambda v M^{-1}\hat{F}\hat{v} + \Lambda v M^{-1}\hat{v} v M^{-1}\hat{F}\hat{v} \\ &\quad = (vM^{-1}\hat{F}\hat{D}\hat{v}\Lambda - \mu v M^{-1}\hat{F}\hat{v}\Lambda + vM^{-1}\hat{v}\Lambda) \\ &\qquad + (\mu\Lambda v M^{-1}\hat{F}\hat{v} - \Lambda v A M^{-1}\hat{F}\hat{v}) - (\Lambda v M^{-1}\hat{v} - \Lambda v M^{-1}\hat{v} v M^{-1}\hat{F}\hat{v}) \\ &\quad = (-vM^{-1}\hat{F}\hat{F}^{-1}\hat{v}\Lambda + vM^{-1}\hat{v}\Lambda) + \Lambda v F^{-1} M^{-1}\hat{F}\hat{v} \\ &\qquad - (\Lambda v M^{-1}\hat{v} - \Lambda v M^{-1}\hat{v} v M^{-1}\hat{F}\hat{v}) \\ &\quad = \Lambda v F^{-1} M^{-1}\hat{F}\hat{v} - (\Lambda v M^{-1}\hat{v} - \Lambda v M^{-1}\hat{v} v M^{-1}\hat{F}\hat{v}) = 0, \end{aligned}$$

the last step of which is a consequence of (3.26). This means that the x-part (3.9) of the conditions is satisfied.

Another similar argument can show that the t-part (3.10) of the conditions is satisfied as well. The proof is then finished. □

3.2. *N-fold decomposition*

Moreover, by a careful computation, we would like to show that the above general binary Darboux transformation can be expressed as a product of N single binary Darboux transformations.

Let us introduce two sequences of basic matrices $S^+\{k\}$ and $S^-\{k\}$, $1 \le k \le N$, recursively as follows:

$$\begin{cases} S^+\{k\} = S^+\{k\}(\mu) = I_{n+1} - \dfrac{\mu_k - \hat{\mu}_k}{\mu - \hat{\mu}_k} \dfrac{v'_k \hat{v}'_k}{\hat{v}'_k v'_k}, \quad 1 \le k \le N, \\ S^-\{k\} = S^-\{k\}(\mu) = I_{n+1} + \dfrac{\mu_k - \hat{\mu}_k}{\mu - \mu_k} \dfrac{v'_k \hat{v}'_k}{\hat{v}'_k v'_k}, \quad 1 \le k \le N, \end{cases} \tag{3.36}$$

with

$$v'_k = S^+[\![k-1]\!](\mu_k) v_k, \quad \hat{v}'_k = \hat{v}_k S^-[\![k-1]\!](\hat{\mu}_k), \ \ 1 \le k \le N, \tag{3.37}$$

where

$$\begin{cases} S^+[\![0]\!] = S^-[\![0]\!] = I_{n+1}, \\ S^+[\![k]\!] = S^+\{k\} \cdots S^+\{2\} S^+\{1\}, \quad 1 \le k \le N-1, \\ S^-[\![k]\!] = S^-\{1\} S^-\{2\} \cdots S^-\{k\}, \quad 1 \le k \le N-1. \end{cases} \tag{3.38}$$

Note that $S^+\{1\}$ and $S^-\{1\}$ above are the same as $S^+[1]$ and $S^-[1]$ defined by (3.3).

Theorem 3.3. *If* $\{\mu_k | \, 1 \le k \le N\} \cap \{\hat{\mu}_k | \, 1 \le k \le N\} = \emptyset$, *then* S^+ *and* S^- *possess the following* N*-fold decompositions*:

$$S^+ = S^+\{N\} S^+\{N-1\} \cdots S^+\{1\}, \quad S^- = S^-\{1\} \cdots S^-\{N-1\} S^-\{N\}, \tag{3.39}$$

where $S^+\{k\}$ *and* $S^-\{k\}$, $1 \le k \le N$, *are recursively defined by* (3.36).

Proof. Based on the definition of $S^+[\![k]\!]$ and $S^-[\![k]\!]$ in (3.38), we see

$$S^+\{k\}(\mu_k) v'_k = 0, \quad \hat{v}'_k S^-\{k\}(\hat{\mu}_k) = 0, \ \ 1 \le k \le N. \tag{3.40}$$

It then follows that

$$\begin{cases} S^+[\![N]\!](\mu_k) v_k = S^+\{N\}(\mu_k) \cdots S^+\{k+1\}(\mu_k) S^+\{k\}(\mu_k) v'_k = 0, \quad 1 \le k \le N, \\ \hat{v}_k S^-[\![N]\!](\hat{\mu}_k) = \hat{v}'_k S^-\{k\}(\hat{\mu}_k) S^-\{k+1\}(\hat{\mu}_k) \cdots S^-\{N\}(\hat{\mu}_k) = 0, \quad 1 \le k \le N, \end{cases}$$

where

$$S^+[\![N]\!] = S^+\{N\} \cdots S^+\{2\} S^+\{1\}, \quad S^-[\![N]\!] = S^-\{1\} S^-\{2\} \cdots S^-\{N\}. \tag{3.41}$$

Now, because of the same property for S^+ and S^- in Proposition 3.1, this means that

$$S^+ = S^+[\![N]\!], \quad S^- = S^-[\![N]\!],$$

which are exactly the decompositions in (3.39). The proof is finished. □

3.3. *Nonlocal reduction*

To satisfy the nonlocal reduction condition for U' in (2.7), we choose

$$\hat{\mu}_k = \begin{cases} -\mu_k^*, & \text{if } \mu_k \notin i\mathbb{R}, \ 1 \le k \le N, \\ \text{any value } \in i\mathbb{R}, & \text{if } \mu_k \in i\mathbb{R}, \ 1 \le k \le N. \end{cases} \tag{3.42}$$

Then we know that

$$(S_1^+(-x,t))^\dagger C = C S_1^+(x,t), \tag{3.43}$$

ensures the nonlocal reduction condition (2.7) for the new spectral matrix U'. To ensure (3.43), we further take

$$\hat{v}_k(x,t,\hat{\mu}_k) = v_k^\dagger(-x,t,\mu_k)C, \quad 1 \le k \le N, \tag{3.44}$$

and require that

$$\begin{cases} v_k^\dagger(-x,t,\mu_k)Cv_l(x,t,\mu_l) = 0, \\ m_{kl,x}^c = iv_k^\dagger(-x,t,\mu_k)C\Lambda v_l(x,t,\mu_l), & \text{if } \mu_l = \hat{\mu}_k, \\ m_{kl,t}^c = iv_k^\dagger(-x,t,\mu_k)C\Theta_{[k,l]}v_l(x,t,\mu_l), \end{cases} \tag{3.45}$$

where $1 \le k,l \le N$. Finally, we know that the reduced binary Darboux transformation engenders a binary Darboux transformation for the multicomponent nonlocal reverse-space integrable NLS equations (2.11). We summarize the result below.

Theorem 3.4. *Let the set of adjoint eigenvalues,* $\{\hat{\mu}_k | 1 \le k \le N\}$, *be taken as in* (3.42). *Assume that the set of adjoint eigenfunctions,* $\{\hat{v}_k | 1 \le k \le N\}$, *is determined by* (3.44), *and the three basic conditions in* (3.45) *are satisfied. Then the reduced binary Darboux transformation* (3.32) *leads to a binary Darboux transformation for the multicomponent nonlocal reverse-space integrable NLS equations* (2.11).

4. Soliton Solutions

4.1. *Unreduced classical case*

We take two arbitrary sets of complex numbers $\{\mu_k | 1 \le k \le N\}$ and $\{\hat{\mu}_k | 1 \le k \le N\}$ as eigenvalues and adjoint eigenvalues. Starting from $P = 0$, namely, the zero seed solution, we can arrive at the corresponding eigenfunctions and adjoint eigenfunctions

$$v_k(x,t) = \mathrm{e}^{i\mu_k\Lambda x + i\mu_k^2\Theta t} w_k, \quad 1 \le k \le N, \tag{4.1}$$

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\mu}_k\Lambda x - i\hat{\mu}_k^2\Theta t}, \quad 1 \le k \le N, \tag{4.2}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively. Moreover, we choose $m_{kl}^c = 0$, and so we need to satisfy

$$\hat{w}_k w_l = \hat{w}_k \Lambda w_l = \hat{w}_k \Theta w_l = 0, \quad \text{if } \mu_l = \hat{\mu}_k, \text{ where } 1 \le k, l \le N. \tag{4.3}$$

Now from the binary Darboux transformation in (3.32), we immediately obtain a novel potential matrix:

$$P' = [S_1^+, \Lambda], \quad S_1^+ = -\sum_{k,l=1}^{N} (M^{-1})_{kl} v_k \hat{v}_l. \tag{4.4}$$

Further, this leads to a class of N-soliton solutions to the multicomponent local integrable NLS equations (2.6):

$$p_j = \delta \sum_{k,l=1}^{N} (M^{-1})_{kl} v_k^{(1)} \hat{v}_l^{(j+1)}, \quad q_j = -\delta \sum_{k,l=1}^{N} (M^{-1})_{kl} v_k^{(j+1)} \hat{v}_l^{(1)}, \ 1 \le j \le n, \tag{4.5}$$

where

$$v_k = (v_k^{(1)}, v_k^{(2)}, \ldots, v_k^{(n+1)})^T \quad \text{and} \quad \hat{v}_k = (\hat{v}_k^{(1)}, \hat{v}_k^{(2)}, \ldots, \hat{v}_k^{(n+1)}), 1 \le k \le N.$$

4.2. *Reduced nonlocal case*

We need to check the involution condition

$$(S_1^+(-x,t))^\dagger C = C S_1^+(x,t), \tag{4.6}$$

so that we can generate soliton solutions to the multicomponent nonlocal reverse-space integrable NLS equations (2.11). This condition, equivalently, demands that the novel potential matrix P' defined by the resulting binary Darboux transformation needs to satisfy the group reduction condition (2.8). If the above condition (4.6) is satisfied, then the obtained soliton solution to the multicomponent local integrable NLS equations (2.6) generates a class of soliton solutions:

$$p_j = \delta \sum_{k,l=1}^{N} (M^{-1})_{kl} v_k^{(1)} \hat{v}_l^{(j+1)}, \quad 1 \le j \le n, \tag{4.7}$$

for the multicomponent nonlocal reverse-space integrable NLS equations (2.11).

To satisfy the crucial property (4.6), we define the adjoint eigenvalues $\{\hat{\mu}_k|\, 1 \le k \le N\}$ by (3.42), upon fixing a set of N eigenvalues

$$\mu_k \in \mathbb{C}, \quad 1 \le k \le N. \tag{4.8}$$

Further, we can have the associated eigenfunctions v_k, $1 \le k \le N$, as follows:

$$v_k(x,t) = v_k(x,t,\mu_k) = \mathrm{e}^{i\mu_k \Lambda x + i\mu_k^2 \Theta t} w_k, \quad 1 \le k \le N, \tag{4.9}$$

respectively, where $w_k, 1 \le k \le N$, are arbitrary column vectors. Moreover, thanks to the previous analysis on the nonlocal reductions, the corresponding adjoint eigenfunctions $\hat{v}_k$, $1 \le k \le N$, can be taken as

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\mu}_k) = v_k^\dagger(-x,t,\mu_k)C = w_k^\dagger \mathrm{e}^{-i\hat{\mu}_k \Lambda x - i\hat{\mu}_k^2 \Theta t} C, \quad 1 \le k \le N, \tag{4.10}$$

respectively. In this way, under $P = 0$, the properties in (3.45) become the following orthogonal conditions:

$$w_k^\dagger C w_l = w_k^\dagger C \Lambda w_l = w_k^\dagger C \Theta w_l = 0, \quad \text{if } \mu_l = \hat{\mu}_k, \ \text{ where } 1 \le k, l \le N, \tag{4.11}$$

for $\{w_k|\, 1 \le k \le N\}$, if we choose $m_{kl}^c = 0$. We point out that the case of $\mu_k = \hat{\mu}_k$ happens only when taking $\mu_k \in i\mathbb{R}$ and $\hat{\mu}_k = -\mu_k^*$. It is direct to see that the three conditions in (4.11) lead equivalently to

$$\begin{cases} w_k^{(1)*} w_l^{(1)} = 0, \\ (w_k^{(2)*}, \ldots, w_k^{(n+1)*}) \Sigma (w_l^{(2)}, \ldots, w_l^{(n+1)})^T = 0, \end{cases} \quad \text{if } \mu_l = \hat{\mu}_k, \ \text{ where } 1 \le k, l \le N, \tag{4.12}$$

with $w_k^{(j)}$ denoting the jth component of w_k, where $1 \le j \le n+1$, for each $1 \le k \le N$.

To conclude, the formula (4.7) provides us with a class of soliton solutions to the multicomponent nonlocal reverse-space integrable NLS equations (2.11), along with (3.16) and (3.45). When we take (4.9), (4.10) and $m_{kl}^c = 0$, the conditions in (3.45) are reduced to (4.12).

5. Conclusion

The paper aims to explore a binary Darboux transformation for a class of vector integrable NLS equations and their nonlocal reverse-space integrable counterparts. The key is to apply a pair of eigenfunction and adjoint eigenfunction. A general formulation was proposed, together with an N-fold decomposition. A reduction of the M-matrix was also analyzed, which leads to the orthogonal conditions on eigenfunctions and adjoint eigenfunctions. The resultant binary Darboux transformation was applied to constructing soliton solutions to both local and nonlocal vector integrable NLS equations.

The primary result in our theory is a general framework of Darboux transformations for both local and nonlocal cases. The crucial step is to form a generalized M matrix, and the motivation to achieve the goal comes from recent studies on nonlocal integrable equations, such as nonlocal integrable NLS and mKdV equations (see, for example [4, 25–27]). The basic procedure adopted in our formulation has also been applied to the study of Riemann–Hilbert problems (see, for example [5]). It will be of much interest to search for other kinds of exact solutions, e.g. lump solutions [28] and breather waves [29–31], to nonlocal integrable equations, and generalize the resultant binary Darboux transformations to local and nonlocal integrable models with self-consistent sources [7, 32].

Acknowledgments

The work was supported in part by the NSFC under the grants 12271488, 11975145 and 11972291. The Ministry of Science and Technology of China (G2021016032L and G2023016011L), and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020). The authors would also like to thank Alle Adjiri, Yushan Bai, Qingxian Chen, Yan Jiang, Wenting Li and Morgan McAnally for their helpful discussions.

ORCID

Wen-Xiu Ma https://orcid.org/0000-0001-5309-1493
Yehui Huang https://orcid.org/0000-0003-1269-724X
Fudong Wang https://orcid.org/0000-0001-6183-017X
Yong Zhang https://orcid.org/0009-0002-2257-6169
Liyuan Ding https://orcid.org/0000-0002-4370-5188

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[2] F. Calogero and A. Degasperis, *Solitons and Spectral Transform I* (North-Holland, Amsterdam, 1982).
[3] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method* (Consultants Bureau, New York, 1984).
[4] W. X. Ma, Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations, *Appl. Math. Lett.* **102** (2020) 106161.
[5] W. X. Ma, Riemann–Hilbert problems and N-soliton solutions for a coupled mKdV system, *J. Geom. Phys.* **132** (2018) 45–54.
[6] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, Heidelberg, 1991).
[7] Y. B. Zeng, W. X. Ma and Y. J. Shao, Two binary Darboux transformations for the KdV hierarchy with self-consistent sources, *J. Math. Phys.* **42** (2001) 2113–2128.
[8] E. V. Doktorov and S. B. Leble, *A Dressing Method in Mathematical Physics* (Springer, Dordrecht, 2007).

[9] H. Q. Zhang, Y. Wang and W. X. Ma, Binary Darboux transformation for the coupled Sasa-Satsuma equations, *Chaos* **27** (2017) 073102.

[10] W. X. Ma and S. Batwa, A binary Darboux transformation for multicomponent NLS equations and their reductions, *Anal. Math. Phys.* **11** (2021) 44.

[11] W. X. Ma, A novel kind of reduced integrable matrix mKdV equations and their binary Darboux transformations, *Mod. Phys. Lett. B* **36** (2022) 2250094.

[12] W. X. Ma and R. G. Zhou, Adjoint symmetry constraints of multicomponent AKNS equations, *Chin. Ann. Math. Ser. B* **23** (2002) 373–384.

[13] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249–315.

[14] S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Sov. Phys. JETP* **38** (1974) 248–253.

[15] S. T. Chen and R. G. Zhou, An integrable decomposition of the Manakov equation, *Comput. Appl. Math.* **31** (2012) 1–18.

[16] W. X. Ma, Riemann–Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction, *Math. Meth. Appl. Sci.* **42** (2019) 1099–1113.

[17] W. X. Ma, Y. H. Huang and F. D. Wang, Inverse scattering transforms for non-local reverse-space matrix non-linear Schrödinger equations, *Eur. J. Appl. Math.* **33** (2022) 1062–1082.

[18] M. J. Ablowitz and Z. H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, *Phys. Rev. Lett.* **110** (2013) 064105.

[19] M. J. Ablowitz and Z. H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, *Nonlinearity* **29** (2016) 915–946.

[20] W. X. Ma and W. Strampp, An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, *Phys. Lett. A* **185** (1994) 277–286.

[21] T. Kawata, Riemann spectral method for the nonlinear evolution equation, in *Advances in Nonlinear Waves,* Vol. I, ed. L. Debnath (Pitman, Boston, MA, 1984), pp. 210–225.

[22] J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (SIAM, Philadelphia, 2010).

[23] C. H. Gu, H. S. Hu and Z. X. Zhou, *Darboux Transformations in Integrable Systems* (Springer, Netherlands, 2005).

[24] W. X. Ma, Darboux transformations for a Lax integrable system in $2n$-dimensions, *Lett. Math. Phys.* **39** (1997) 33–49.

[25] M. Gürses and A. Pekcan, Nonlocal nonlinear Schrödinger equations and their soliton solutions, *J. Math. Phys.* **59** (2018) 051501.

[26] A. S. Fokas, Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation, *Nonlinearity* **29** (2016) 319–324.

[27] C. Q. Song, D. M. Xiao and Z. N. Zhu, Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation, *Commun. Nonlinear Sci. Numer. Simul.* **45** (2017) 13–28.

[28] W. X. Ma and Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, *J. Differ. Equ.* **264** (2018) 2633–2659.

[29] X. Wang and C. Z. Li, Solitons, breathers and rogue waves in the coupled nonlocal reverse-time nonlinear Schrödinger equations, *J. Geom. Phys.* **180** (2022) 104619.

[30] M. Wang, B. Tian and T. Y. Zhou, Darboux transformation, generalized Darboux transformation and vector breathers for a matrix Lakshmanan–Porsezian–Daniel equation in a Heisenberg ferromagnetic spin chain, *Chaos Solitons Fractals* **152** (2021) 111411.

[31] H. Yilmaz, Binary Darboux transformation for the Gerdjikov–Ivanov equation, *Wave Motion* **113** (2022) 102991.
[32] Y. H. Huang, H. Q. Jing, R. L. Lin and Y. Q. Yao, The rogue wave of the nonlinear Schrödinger equation with self-consistent sources, *Mod. Phys. Lett. B* **32** (2018) 1850367.