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Inverse scattering transforms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies

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Abstract

The aim of the paper is to construct nonlocal reverse-space nonlinear Schrödinger (NLS) hierarchies through nonlocal group reductions of eigenvalue problems and generate their inverse scattering transforms and soliton solutions. The inverse scattering problems are formulated by Riemann-Hilbert problems which determine generalized matrix Jost eigenfunctions. The Sokhotski-Plemelj formula is used to transform the Riemann-Hilbert problems into Gelfand-Levitan-Marchenko type integral equations. A solution formulation to special Riemann-Hilbert problems with the identity jump matrix, corresponding to the reflectionless transforms, is presented and applied to N-soliton solutions of the nonlocal NLS hierarchies.

KEYWORDS

integrable hierarchy, inverse scattering, matrix eigenvalue problem, nonlocal reduction, Riemann-Hilbert problem, soliton solution

JEL CLASSIFICATION

35Q55, 37K15, 37K40

1 INTRODUCTION

Nonlocal integrable equations have become one of the most popular topics in soliton theory. Three types of nonlocal nonlinear Schrödinger (NLS) equation arises while taking group reductions.[1] The corresponding inverse scattering transforms have been recently established for the scalar case[2–6] and the multicomponent case,[7,8] and soliton solutions have been constructed from the Riemann-Hilbert problems whose jump is the identity,[8,9] through Darboux transformations,[10–12] and by the Hirota bilinear method.[13] Some other multicomponent generalizations[1,14,15] and nonlocal integrable equations[16] were also presented. We would like to present a class of NLS hierarchies of nonlocal reverse-space integrable equations associated with the multicomponent Ablowitz-Kaup-Newell-Segur (AKNS) eigenvalue problem, and construct their inverse scattering transforms first and then soliton solutions through formulating and solving the associated Riemann-Hilbert problems with matrix eigenvalue problems.

Like Darboux transformations, the Riemann-Hilbert problems are successfully used to study integrable equations and further construct their soliton solutions.[17] Many integrable equations, such as the multiple wave interaction equations,[17] the general coupled NLS equations,[18] the generalized Sasa-Satsuma equation,[19] the Harry Dym equation,[20] and multicomponent modified Korteweg-de Vires (mKdV) equations,[21] have been studied through exploring the Riemann-Hilbert problems associated with their matrix eigenvalue problems.

A general procedure for formulating Riemann-Hilbert problems on $\mathbb{R}$ is stated as follows. Assume that there is a pair of matrix eigenvalue problems:

$$\left\{ \begin{array}{l} -i\varphi_x = U\varphi,\ U = U(u,\lambda) = A(\lambda) + P(u,\lambda), \\ -i\varphi_t = V\varphi,\ V = V(u,\lambda) = B(\lambda) + Q(u,\lambda), \end{array} \right) \tag{1}$$

in which i is the unit imaginary number, ϕ is a square matrix eigenfunction, λ is an eigenvalue, u is a potential, and we usually assume that A and B are commuting constant diagonal square matrices, and P and Q are square matrices being traceless. The above two matrix eigenvalue problems need to satisfy the compatibility condition, namely, the zero curvature equation,

$$U_t - V_x + i[U,V] = 0, \tag{2}$$

where $[\cdot,\cdot]$ is the matrix commutator, and this generates an integrable equation. To furnish this integrable equation with a Riemann-Hilbert problem, we utilize the following equivalent pair of matrix eigenvalue problems:

$$\left\{ \begin{array}{l} \psi_x = i[A(\lambda),\psi] + \check{P}(u,\lambda)\psi, \\ \psi_t = i[B(\lambda),\psi] + \check{Q}(u,\lambda)\psi, \end{array} \right. \tag{3}$$

where $\check{P} = iP$ and $\check{Q} = iQ$, and ψ is a square matrix eigenfunction. Obviously, the equivalence between (1) and (3) is guaranteed by the commutativity of A and B. The connection between ϕ and ψ reads

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{iA(\lambda)x+iB(\lambda)t}. \tag{4}$$

The uniqueness of matrix eigenfunctions needs the standard boundary conditions

$$\psi^{\pm} \to I,\ \text{as}\ x,t \to \pm\infty, \tag{5}$$

in which I represents the identity matrix. From those eigenfunctions $\psi^{\pm}$, we need to pick the entries to define two generalized matrix Jost eigenfunctions $T^{\pm}(x,t,\lambda)$, being analytical with respect to λ in $\mathbb{C}^+$ and $\mathbb{C}^-$ (the upper and lower half-planes), respectively, and continuous with respect to λ in $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$ (the closed upper and lower half-planes), respectively. Then, based on those two generalized matrix Jost eigenfunctions $T^{\pm}$, we can formulate a so-called Riemann-Hilbert problem on $\mathbb{R}$:

$$G^+(x,t,\lambda) = G^-(x,t,\lambda)G_0(x,t,\lambda),\ \lambda \in \mathbb{R}, \tag{6}$$

in which $G^{\pm}$ are two unimodular generalized matrix Jost eigenfunctions and G_0 is the jump matrix.

Note that the two eigenfunctions, ψ^- and ψ^+, are linearly dependent. It therefore follows that

$$\psi^- E_g = \psi^+ E_g S(\lambda), \tag{7}$$

in which $S(\lambda)$ is called the scattering matrix of the associated matrix eigenvalue problems. The jump matrix G_0 contains basic scattering data inherited from $S(\lambda)$. Solutions of the Riemann-Hilbert problems could be presented by using the Sokhotski-Plemelj formula and used to construct the required generalized matrix Jost eigenfunctions for recovering the potentials in the matrix eigenvalue problems, which solve the corresponding integrable equation. Such a procedure formulates an inverse scattering transform. Soliton solutions are presented from solutions to special Riemann-Hilbert problems whose jumps G_0 are the identity matrix (or equivalently, the reflectionless inverse scattering transforms).

In this paper, first by making a special kind of nonlocal group reductions, we propose a class of nonlocal reverse-space NLS hierarchies, and then from a perspective of Riemann-Hilbert problems, we construct their inverse scattering transforms and soliton solutions. We organize the other sections of the paper as follows. In Section 2, we make a kind of nonlocal group reductions and construct nonlocal reverse-space NLS hierarchies from the AKNS integrable hierarchy possessing multiple potentials. In Section 3, we establish the associated Riemann-Hilbert problems, based on the analytic Fredholm theory on integral equations. In Section 4, we transform, by means of the Sokhotski-Plemelj formula, the resulting Riemann-Hilbert problems into Gelfand-Levitan-Marchenko type integral equations, to formulate the inverse scattering transforms. In Section 5, we first propose a solution formulation to special Riemann-Hilbert problems whose jumps are $G_0 = I$, and then construct soliton solutions of the nonlocal reverse-space NLS hierarchies. In the final section, we summarize our results and give several concluding remarks.

2 | NONLOCAL REDUCTIONS AND NONLOCAL NLS HIERARCHIES

2.1 | AKNS hierarchy with multiple potentials

We recall the AKNS hierarchy with multiple potentials in this subsection for ease of reference. Let n be an arbitrarily given natural number, and α_1 and α_2, arbitrary but different constants.

The AKNS matrix eigenvalue problem with multiple potentials reads

$$-i\phi_x = U\phi,\ U = U(u,\lambda) = (U_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{8}$$

where λ is an eigenvalue, $I_n = \mathrm{diag}(\underbrace{1,1,\dots,1}_{n})$, and u is a potential of $2n$-dimension:

$$u = (p, q^T)^T,\ p = (p_1, p_2, \dots, p_n),\ q = (q_1, q_2, \dots, q_n)^T. \tag{9}$$

When $p_j = q_j = 0,\ 2 \le j \le n$, (8) becomes the original AKNS eigenvalue problem.[22] We call an associated soliton hierarchy with (8) an AKNS soliton hierarchy with multiple potentials.[23] Since there exists a multiple eigenvalue of $\frac{\partial U}{\partial \lambda}$, the matrix eigenvalue problem (8) is degenerate.

To work out an associated AKNS soliton hierarchy with multiple potentials, as always, we start to determine a solution of the stationary zero curvature equation

$$iW_x = [W, U], \tag{10}$$

corresponding to the matrix eigenvalue problem (8). Let us consider a solution W of the following form:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{11}$$

in which a is a scalar, b and c are a row and a column of dimension n, respectively, and d is a square matrix of size n. A direct computation shows that the corresponding stationary zero curvature equation (10) reads

$$ia_x = bq - pc,\ ib_x = -\alpha\lambda b - pd + ap, ic_x = \alpha\lambda c - qa + dq, id_x = cp - qb, \tag{12}$$

in which $\alpha = \alpha_1 - \alpha_2$. We make an expansion for W as follows:

$$W = \sum_{m=0}^{\infty} W_m \lambda^{-m},\ W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix},\ m \ge 0, \tag{13}$$

$b^{[m]}, c^{[m]}$, and $d^{[m]}$ being defined by

$$b^{[m]} = (b_1^{[m]}, b_2^{[m]}, \dots, b_n^{[m]}),\ c^{[m]T} = (c_1^{[m]}, c_2^{[m]}, \dots, c_n^{[m]}),\ d^{[m]} = (d_{jl}^{[m]})_{n\times n},\ m \ge 0. \tag{14}$$

It then follows that the system (12) precisely yields the following recursion relations:

$$b^{[0]} = 0,\ c^{[0]} = 0,\ a_x^{[0]} = 0,\ d_x^{[0]} = 0, \tag{15a}$$

$$b^{[m]} = \frac{1}{\alpha}(-ib_x^{[m-1]} - pd^{[m-1]} + a^{[m-1]}p),\ m \ge 1, \tag{15b}$$

$$c^{[m]} = \frac{1}{\alpha}(ic_x^{[m-1]} + qa^{[m-1]} - d^{[m-1]}q),\ m \geq 1, \tag{15c}$$

$$a_x^{[m]} = i(pc^{[m]} - b^{[m]}q),\ d_x^{[m]} = i(qb^{[m]} - c^{[m]}p),\ m \geq 1. \tag{15d}$$

To compute the associated soliton hierarchy explicitly, let us now take the initial values:

$$a^{[0]} = \beta_1,\ d^{[0]} = \beta_2 I_n, \tag{16}$$

where β_1, β_2 are arbitrary but different constants as well, and take zero constants of integration in (15d), which is equivalent to the following conditions:

$$W_m|_{u=0} = 0,\ m \geq 1. \tag{17}$$

All this guarantees the uniqueness of the matrices $W_m,\ m \geq 1$. As soon as $a^{[0]}$ and $d^{[0]}$ are determined by (16), all those matrices $W_m,\ m \geq 1$, defined recursively, can be worked out. Especially, we can have

$$b_j^{[1]} = \frac{\beta}{\alpha}p_j,\ c_j^{[1]} = \frac{\beta}{\alpha}q_j,\ a^{[1]} = 0,\ d_{jl}^{[1]} = 0; \tag{18a}$$

$$b_j^{[2]} = -\frac{\beta}{\alpha^2}ip_{j,x},\ c_j^{[2]} = \frac{\beta}{\alpha^2}iq_{j,x},\ a^{[2]} = -\frac{\beta}{\alpha^2}pq,\ d_{jl}^{[2]} = \frac{\beta}{\alpha^2}p_l q_j; \tag{18b}$$

$$\begin{cases} b_j^{[3]} = -\frac{\beta}{\alpha^3}[p_{j,xx} + 2pqp_j],\ c_j^{[3]} = -\frac{\beta}{\alpha^3}[q_{j,xx} + 2pqq_j], \\ a^{[3]} = -\frac{\beta}{\alpha^3}i(pq_x - p_x q),\ d_{jl}^{[3]} = -\frac{\beta}{\alpha^3}i(p_{l,x}q_j - p_l q_{j,x}); \end{cases} \tag{18c}$$

$$\begin{cases} b_j^{[4]} = \frac{\beta}{\alpha^4}i[p_{j,xxx} + 3pqp_{j,x} + 3p_x qp_j], \\ c_j^{[4]} = -\frac{\beta}{\alpha^4}i[q_{j,xxx} + 3pqq_{j,x} + 3pq_x q_j], \\ a^{[4]} = \frac{\beta}{\alpha^4}[3(pq)^2 + pq_{xx} - p_x q_x + p_{xx}q], \\ d_{jl}^{[4]} = -\frac{\beta}{\alpha^4}(3p_l pqq_j + p_{l,xx}q_j + p_l q_{j,xx} - p_{l,x}q_{j,x}); \end{cases} \tag{18d}$$

where $\beta = \beta_1 - \beta_2$ and $1 \leq j, l \leq n$. By using (15d), we can work out, from (15b) and (15c), the following recursion relation for $b^{[m]}$ and $c^{[m]}$:

$$\begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m-1]} \\ b^{[m-1]T} \end{bmatrix},\ m \geq 2, \tag{19}$$

in which Ψ is the following $2n \times 2n$ matrix operator

$$\Psi = \frac{i}{\alpha}\begin{bmatrix} \left(\partial + \sum_{j=1}^{n} q_j\partial^{-1}p_j\right)I_n + q\partial^{-1}p & -q\partial^{-1}q^T - \left(q\partial^{-1}q^T\right)^T \\ p^T\partial^{-1}p + \left(p^T\partial^{-1}p\right)^T & -\left(\partial + \sum_{j=1}^{n} p_j\partial^{-1}q_j\right)I_n - p^T\partial^{-1}q^T \end{bmatrix}. \tag{20}$$

The AKNS soliton hierarchy with multiple potentials is associated with the following temporal matrix eigenvalue problems:

$$-i\phi_t = V^{[r]}\phi,\ V^{[r]} = V^{[r]}(u,\lambda) = (V_{jl}^{[r]})_{(n+1)\times(n+1)} = \sum_{m=0}^{r} W_m\lambda^{r-m},\ r \geq 0. \tag{21}$$

The following zero curvature equations, namely, the compatibility conditions of (8) and (21),

$$U_t - V_x^{[r]} + i[U, V^{[r]}] = 0,\ r \geq 0, \tag{22}$$

lead to the so-called AKNS soliton hierarchy with multiple potentials:

$$u_t = (p, q^T)_t^T = K_r = i(\alpha b^{[r+1]}, -\alpha c^{[r+1]T})^T,\ r \geq 0. \tag{23}$$

The first nontrivial integrable system in the soliton hierarchy (23) consists of the standard NLS equations:

$$\left\{\begin{aligned} p_{j,t} &= -\frac{\beta}{\alpha^2}i\left[p_{j,xx} + 2\left(\sum_{l=1}^{n} p_l q_l\right)p_j\right],\ 1 \leq j \leq n, \\ q_{j,t} &= \frac{\beta}{\alpha^2}i\left[q_{j,xx} + 2\left(\sum_{l=1}^{n} p_l q_l\right)q_j\right],\ 1 \leq j \leq n. \end{aligned}\right) \tag{24}$$

In the case of $n = 2$, under a special kind of local group reductions,[24] the NLS equations (24) reduce to the Manokov system.[25] An integrable decomposition into integrable Hamiltonian systems of ordinary differential equations was made for that reduced system in Ref. 26.

The AKNS soliton hierarchy with multiple potentials (23) possesses a bi-Hamiltonian formulation.[27] This can be achieved by using the trace identity,[28] or more generally, the variational identity.[29] The process of determining the bi-Hamiltonian structure is as follows. A direct computation yields

$$-i\,\mathrm{tr}\left(W\frac{\partial U}{\partial\lambda}\right) = \alpha_1 a + \alpha_2\mathrm{tr}\,(d) = \sum_{m=0}^{\infty}\left(\alpha_1 a^{[m]} + \alpha_2\sum_{j=1}^{n} d_{jj}^{[m]}\right)\lambda^{-m},$$

and

$$-i\,\mathrm{tr}\left(W\frac{\partial U}{\partial u}\right) = \begin{bmatrix} c \\ b^T \end{bmatrix} = \sum_{m\geq 0} G_{m-1}\lambda^{-m}.$$

Now applying the trace identity

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(\frac{\partial U}{\partial u}W\right),\ \gamma=-\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\operatorname{tr}\left(W^2\right)|, \tag{25}$$

with $\gamma=0$, we obtain

$$\frac{\delta\tilde{H}_m}{\delta u}=iG_{m-1},\ \tilde{H}_m=-\frac{i}{m}\int\left(\alpha_1 a^{[m+1]}+\alpha_2\sum_{j=1}^{n}d_{jj}^{[m+1]}\right)dx,\ G_{m-1}=\begin{bmatrix}c^{[m]}\\ b^{[m]T}\end{bmatrix},\ m\geq 1. \tag{26}$$

This generates the following bi-Hamiltonian structure:

$$u_t=K_r=iJ_1G_r=J_1\frac{\delta\tilde{H}_{r+1}}{\delta u}=J_2\frac{\delta\tilde{H}_r}{\delta u},\ r\geq 1 \tag{27}$$

for the AKNS equations with multiple potentials in (23), where the Hamiltonian pair $(J_1, J_2 = J_1\Psi)$ is determined by

$$J_1=\begin{bmatrix}0 & \alpha I_n\\ -\alpha I_n & 0\end{bmatrix}, \tag{28a}$$

$$J_2=i\begin{bmatrix}p^T\partial^{-1}p+\left(p^T\partial^{-1}p\right)^T & -\left(\partial+\sum_{j=1}^{n}p_j\partial^{-1}q_j\right)I_n-p^T\partial^{-1}q^T\\ -\left(\partial+\sum_{j=1}^{n}p_j\partial^{-1}q_j\right)I_n-q\partial^{-1}p & q\partial^{-1}q^T+\left(q\partial^{-1}q^T\right)^T\end{bmatrix}. \tag{28b}$$

The integrodifferential operator $\Phi=\Psi^{\dagger}=J_2J_1^{-1}$ provides a hereditary recursion operator for the AKNS soliton hierarchy (23). For each $r\geq 1$, adjoint symmetry constraints (or a little bit loosely, symmetry constraints) can decompose the rth AKNS equations with multiple potentials into two commuting Hamiltonian systems of ordinary differential equations, which are integrable in the Liouville sense.[23,27]

2.2 | Nonlocal reverse-space NLS hierarchies

Motivated by the classical local reductions,[24] we introduce a specific kind of nonlocal group reductions for the eigenvalue matrix U:

$$U^{\dagger}(-x,t,-\lambda^*)=-CU(x,t,\lambda)C^{-1}, \tag{29}$$

where

$$C=\begin{bmatrix}1 & 0\\ 0 & \Sigma\end{bmatrix},\ \Sigma^{\dagger}=\Sigma.$$

This means that

$$P^{\dagger}(-x,t) = -CP(x,t)C^{-1} \tag{30}$$

in which the potential matrix P is defined by

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}. \tag{31}$$

Here and in what follows, † stands for the Hermitian transpose, $*$ denotes the complex conjugate, and Σ is an invertible constant Hermitian matrix. For the sake of convenience, we also denote

$$\left\{ \begin{array}{l} M(x,t,\lambda) = M(u(x,t),\lambda), \\ M^{\dagger}(f(x,t,\lambda)) = (M(f(x,t,\lambda)))^{\dagger}, \\ M^{-1}(f(x,t,\lambda)) = (M(f(x,t,\lambda)))^{-1} \end{array} \right) \tag{32}$$

for a matrix M and a function f.

Equivalently, (30) leads to

$$q(x,t) = -\Sigma^{-1} p^{\dagger}(-x,t). \tag{33}$$

The vector function c in (12) under such a kind of reductions may be taken as

$$c(x,t,\lambda) = -\Sigma^{-1} b^{\dagger}(-x,t,-\lambda^*), \tag{34}$$

and those reduction relations guarantee that

$$a^*(-x,t,-\lambda^*) = -a(x,t,\lambda),\ d^{\dagger}(-x,t,-\lambda^*) = -\Sigma d(x,t,\lambda)\Sigma^{-1}, \tag{35}$$

where a and d satisfy (12). Therefore, we have

$$\left\{ \begin{array}{l} (a^{[m]})^{*}(-x,t) = (-1)^{m+1} a^{[m]}(x,t), \\ (b^{[m]})^{\dagger}(-x,t) = (-1)^{m+1} \Sigma c^{[m]}(x,t), \\ (d^{[m]})^{\dagger}(-x,t) = (-1)^{m+1} \Sigma d^{[m]}(x,t)\Sigma^{-1}, \end{array} \right) \tag{36}$$

where $m \geq 1$. This implies that for all $m \geq 1$, we have

$$(V^{[2m]})^{\dagger}(-x,t,-\lambda^*) = CV^{[2m]}(x,t,\lambda)C^{-1}, \tag{37}$$

$V^{[2m]}$ being defined as in (21).

Now, based on (29) and (37), it is direct to see that the reductions in (30) do not present any additional conditions on the previous spatial and temporal matrix eigenvalue problems, when $r = 2m$. Therefore, under the nonlocal group reductions in (29), the half hierarchy of the equations in (23) with $r = 2m$ reduces to the following nonlocal reverse-space NLS hierarchies:

$$p_t = X_m = K_{2m,1}|_{q=-\Sigma^{-1}p^{\dagger}(-x,t)},\ m \geq 0, \tag{38}$$

where $K_r = (K_{r,1}^T, K_{r,2}^T)^T = i(\alpha b^{(r+1)}, -\alpha c^{(r+1)T})^T,\ \ r \geq 0$. Those hierarchies are associated the matrix eigenvalue problems

$$\begin{cases} -i\varphi_x = U\varphi = U(u,\lambda)\varphi, \\ -i\varphi_t = V^{[2m]}\varphi = V^{[2m]}(u,\lambda)\varphi, \end{cases} \quad m \geq 0, \tag{39}$$

in which the Lax pairs read

$$U = \lambda\Lambda + P,\ V^{[2m]} = \lambda^{2m}\Omega + Q_{2m}, \tag{40}$$

with $\Lambda = \text{diag}(\alpha_1, \alpha_2 I_n)$, $\Omega = \text{diag}(\beta_1, \beta_2 I_n)$, and

$$Q_{2m} = \sum_{l=1}^{2m} \lambda^{2m-l} \begin{bmatrix} a^{[l]} & b^{[l]} \\ c^{[l]} & d^{[l]} \end{bmatrix}. \tag{41}$$

Moreover, they possess an infinite hierarchy of commuting symmetries $\{X_k\}_{k=0}^{\infty}$ and an infinite hierarchy of commuting conserved functionals $\{\tilde{H}_{2k+1}|_{q=-\Sigma^{-1}p^{\dagger}(-x,t)}\}_{k=0}^{\infty}$.

When $m = 1$, we obtain the multicomponent nonlocal reverse-space NSL equations:

$$ip_t(x,t) = \frac{\beta}{\alpha^2}[p_{xx}(x,t) - 2p(x,t)\Sigma^{-1}p^{\dagger}(-x,t)p(x,t)], \tag{42}$$

where Σ is an arbitrary invertible constant Hermitian matrix, which can exhibit mixed focusing and defocusing nonlinearities. When $n = 1$, we can obtain two well-known examples:[2]

$$ip_t(x,t) = p_{xx}(x,t) + 2\sigma p^2(x,t)p^*(-x,t), \tag{43}$$

where $\sigma = \pm 1$; and when $n = 2$, we can get

$$\begin{cases} ip_{1,t}(x,t) = p_{1,x,x}(x,t) + (\gamma_1 p_1(x,t)p_1^{\dagger}(-x,t) + \gamma_2 p_2(x,t)p_2^{\dagger}(-x,t))p_1(x,t), \\ ip_{2,t}(x,t) = p_{2,x,x}(x,t) + (\gamma_1 p_1(x,t)p_1^{\dagger}(-x,t) + \gamma_2 p_2(x,t)p_2^{\dagger}(-x,t))p_2(x,t), \end{cases} \tag{44}$$

where γ_1 and γ_2 are arbitrary nonzero real constants.

3 | RIEMANN-HILBERT PROBLEMS

We would now like to construct a class of associated Riemann-Hilbert problems from the matrix eigenvalue problems with respect to the spatial variable x. The results will lay the basic foundation for building the inverse scattering transforms and soliton solutions in the following two sections.

3.1 | Property of eigenfunctions

Let q be determined by (33). Assume that each potential rapidly vanishes as $x \to \pm\infty$ or $t \to \pm\infty$. For the matrix eigenvalue problems in (39), we observe, under the integrable conditions on the potentials:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^k |t|^l \sum_{j=1}^{n} |p_j|\, dxdt < \infty,\ 0 \le k, l \le 1, \tag{45}$$

that one has the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda\Lambda x + i\lambda^{2m}\Omega t}$, as $x, t \to \pm\infty$. Therefore, if one takes the transformation

$$\phi = \psi E_g,\ E_g = \mathrm{e}^{i\lambda\Lambda x + i\lambda^{2m}\Omega t},$$

one can obtain the canonical normalization condition $\psi \to I_{n+1}$, as $x, t \to \pm\infty$. Upon defining $\check{P} = iP$ and $\check{Q}_{2m} = iQ_{2m}$, the required equivalent pair of matrix eigenvalue problems to (39) reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{46}$$

$$\psi_t = i\lambda^{2m}[\Omega, \psi] + \check{Q}_{2m}\psi. \tag{47}$$

Based on a generalized Liouville's formula,[30] one can obtain

$$\det \psi = 1, \tag{48}$$

since $\mathrm{tr}(\check{P}) = \mathrm{tr}(\check{Q}_{2m}) = 0$.

The adjoint counterpart of the x-part of (39) and the adjoint counterpart of (46) are given by

$$i\tilde{\phi}_x = \tilde{\phi}U, \tag{49}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P, \tag{50}$$

respectively. Each pair of adjoint matrix eigenvalue problems and equivalent adjoint matrix eigenvalue problems does not bring any new conditions, except the nonlocal reverse-space NLS hierarchies of equations in (38).

Assume that $\psi(\lambda)$ solves the spatial eigenvalue problem (46) with a given eigenvalue λ. Then, obviously, $C\psi^{-1}(x, t, \lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. Taking the nonlocal reductions in (30) into consideration, we can compute that

$$\begin{aligned}
i[\psi^\dagger(-x, t, -\lambda^*)C]_x &= i[-(\psi_x)^\dagger(-x, t, -\lambda^*)C] \\
&= -i\{(-i)(-\lambda)[\psi^\dagger(-x, t, -\lambda^*), \Lambda] + (-i)\psi^\dagger(-x, t, -\lambda^*)P^\dagger(-x, t)\}C \\
&= \lambda[\psi^\dagger(-x, t, -\lambda^*), \Lambda]C + \psi^\dagger(-x, t, -\lambda^*)C[-C^{-1}P^\dagger(-x, t)C]
\end{aligned}$$

$$= \lambda[\psi^{\dagger}(-x,t,-\lambda^*)C,\Lambda] + \psi^{\dagger}(-x,t,-\lambda^*)CP(x,t),$$

and thus,

$$\tilde{\psi}(x,t,\lambda) = \psi^{\dagger}(-x,t,-\lambda^*)C, \tag{51}$$

presents a new matrix adjoint eigenfunction associated with the same original eigenvalue λ, which means that $\psi^{\dagger}(-x,t,-\lambda^*)C$ solves the adjoint eigenvalue problem (50).

Now, using the asymptotic properties for ψ, we see that the uniqueness of solutions guarantees that

$$\psi^{\dagger}(-x,t,-\lambda^*) = C\psi^{-1}(x,t,\lambda)C^{-1}, \tag{52}$$

if $\psi \to I_{n+1}$, x or $t \to \infty$ or $-\infty$. It therefore follows that when λ is an eigenvalue of (46) (or (50)), $-\lambda^*$ will be another eigenvalue of (46) (or (50)), and the property (52) holds.

3.2 | Riemann-Hilbert problems

We point out that the procedure to establish Riemann-Hilbert problems is actually the same as the one in the local case for the mKdV equations,[21,24] but we present it below for subsequent discussions.

To express Riemann-Hilbert problems concretely, we assume that

$$\alpha = \alpha_1 - \alpha_2 < 0,\ \beta = \beta_1 - \beta_2 < 0, \tag{53}$$

so that we will know what entries of matrix eigenfunctions to pick. In the direct scattering problem, we first consider the two matrix eigenfunctions $\psi^{\pm}(x,\lambda)$ of (46) possessing the boundary conditions

$$\psi^{\pm} \to I_{n+1},\ \text{as } x \to \pm\infty, \tag{54}$$

respectively. From (48), we can readily find that $\det\psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Because

$$\phi^{\pm} = \psi^{\pm}E,\ E = \mathrm{e}^{i\lambda\Lambda x}, \tag{55}$$

are both matrix eigenfunctions of (39), they must be linearly dependent, and as a consequence, we have

$$\psi^{-}E = \psi^{+}ES(\lambda),\ \lambda \in \mathbb{R}, \tag{56}$$

in which $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is traditionally called the scattering matrix. We point out that owing to $\det\psi^{\pm} = 1$, we have $\det S(\lambda) = 1$.

We can transform the x-part of (39) into the following Volterra integral equations of the second kind for $\psi^{\pm}$:[17,31]

$$\psi^{-}(x,\lambda) = I_{n+1} + \int_{-\infty}^{x} \mathrm{e}^{-i\lambda\Lambda(y-x)}\check{P}(y)\psi^{-}(y,\lambda)\mathrm{e}^{-i\lambda\Lambda(x-y)}\,dy, \tag{57}$$

$$\psi^{+}(x,\lambda) = I_{n+1} - \int_{x}^{\infty} \mathrm{e}^{-i\lambda\Lambda(y-x)}\check{P}(y)\psi^{+}(y,\lambda)\mathrm{e}^{-i\lambda\Lambda(x-y)}\,dy, \tag{58}$$

where we have used the boundary conditions (54). Under the conditions (45), the analytic Fredholm theory (or more precisely, the Volterra theory on integral equations) guarantees that the two eigenfunctions $\psi^{\pm}$ exist, and allow analytical continuations off the real line $\lambda \in \mathbb{R}$ as soon as the both integrals on their right-hand sides converge. Noting the diagonal form of Λ and the first assumption in (53), one can observe that the integral equation for the last n columns of ψ^{+} contains only the exponential factor $\mathrm{e}^{-i\alpha\lambda(y-x)}$, which also decays due to $y > x$ in the integral, when λ takes values in $\mathbb{C}^{+}$, and the integral equation for the first column of ψ^{-} contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(y-x)}$, which decays due to $y < x$ in the integral, if λ takes values in $\mathbb{C}^{+}$. Thus, these $n+1$ columns are analytical with respect to λ in $\mathbb{C}^{+}$ and they are continuous with respect to λ in $\bar{\mathbb{C}}^{+}$. By similar arguments, we can find that the first column of $\psi^{+}$and the last n columns of ψ^{-} are analytical with respect to λ in $\mathbb{C}^{-}$ and they are continuous with respect to λ in $\bar{\mathbb{C}}^{-}$.

On one hand, to determine the generalized matrix Jost eigenfunctions, we will denote

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \ldots, \psi_{n+1}^{\pm}), \tag{59}$$

that is, for each $1 \leq j \leq n+1$, $\psi_j^{\pm}$ represents the jth column of $\phi^{\pm}$. This way, we can set the generalized matrix Jost eigenfunction T^{+} to be

$$T^{+}(x,\lambda) = (\psi_1^{-}, \psi_2^{+}, \ldots, \psi_{n+1}^{+}) = \psi^{-}H_1 + \psi^{+}H_2, \tag{60}$$

which is continuous with respect to λ in $\bar{\mathbb{C}}^{+}$ and analytic with respect to λ in $\mathbb{C}^{+}$. The other generalized matrix Jost eigenfunction

$$(\psi_1^{+}, \psi_2^{-}, \ldots, \psi_{n+1}^{-}) = \psi^{+}H_1 + \psi^{-}H_2 \tag{61}$$

is continuous with respect to λ in $\bar{\mathbb{C}}^{-}$ and analytic with respect to λ in $\mathbb{C}^{-}$. Here the two matrices H_1 and H_2 are defined by

$$H_1 = \mathrm{diag}(1, \underbrace{0, 0, \ldots, 0}_{n}),\ H_2 = \mathrm{diag}(0, \underbrace{1, 1, \ldots, 1}_{n}). \tag{62}$$

On the other hand, to build the partner generalized Jost eigenfunction T^{-}, we consider the analytic counterpart of T^{+} in the lower half-plane $\mathbb{C}^{-}$, based on the adjoint matrix eigenvalue problems. Notice that the two inverse matrices $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ can solve those two adjoint eigenvalue problems, respectively. Once denoting $\tilde{\psi}^{\pm}$ by

$$\tilde{\psi}^{\pm} = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \dots, \tilde{\psi}^{\pm,n+1})^T, \tag{63}$$

namely, for each $1 \leq j \leq n+1$, $\tilde{\psi}^{\pm,j}$ represents the jth row of $\tilde{\psi}^{\pm}$, we can show by a quite similar argument that the generalized matrix Jost eigenfunction T^- can be taken as the adjoint matrix solution of (50), ie,

$$T^-(x,\lambda) = (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \dots, \tilde{\psi}^{+,n+1})^T = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}, \tag{64}$$

which is continuous with respect to λ in $\bar{\mathbb{C}}^-$ and analytic with respect to λ in $\mathbb{C}^-$, and the other generalized matrix Jost solution of (50),

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \dots, \tilde{\psi}^{-,n+1})^T = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1}, \tag{65}$$

is continuous for $\lambda \in \bar{\mathbb{C}}^+$ and analytic for $\lambda \in \mathbb{C}^+$.

Based on $\det \psi^{\pm} = 1$, the definitions of $T^{\pm}$, and the scattering relation (56) between ψ^+ and ψ^-, we can obtain

$$\det T^+(x,\lambda) = s_{11}(\lambda),\ \det T^-(x,\lambda) = \hat{s}_{11}(\lambda), \tag{66}$$

where $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. This implies that

$$\lim_{x\to\infty} T^+(x,\lambda) = \begin{bmatrix} s_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \lambda \in \bar{\mathbb{C}}^+;\ \lim_{x\to\infty} T^-(x,\lambda) = \begin{bmatrix} \hat{s}_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \lambda \in \bar{\mathbb{C}}^-. \tag{67}$$

Therefore, we can now introduce the following two unimodular generalized matrix Jost solutions:

$$\begin{cases} G^+(x,\lambda) = T^+(x,\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix},\ \lambda \in \bar{\mathbb{C}}^+; \\ (G^-)^{-1}(x,\lambda) = \begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x,\lambda),\ \lambda \in \bar{\mathbb{C}}^-. \end{cases} \tag{68}$$

Those two generalized matrix Jost solutions generate the required matrix Riemann-Hilbert problems on the real line for the nonlocal reverse-space NLS hierarchies (38):

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda),\ \lambda \in \mathbb{R}, \tag{69}$$

where upon following (56), the jump matrix G_0 is given by

$$G_0(x,\lambda) = E\begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}\tilde{S}(\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}E^{-1}. \tag{70}$$

The matrix $\tilde{S}(\lambda)$ in the jump matrix G_0 has the factorization:

$$\tilde{S}(\lambda) = (H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda) H_2), \tag{71}$$

which can be expressed as

$$\tilde{S}(\lambda) = (\tilde{s}_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \dots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \dots & 0 \\ s_{31} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ s_{n+1,1} & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{72}$$

For the above Riemann-Hilbert problems, the standard canonical normalization conditions

$$G^{\pm}(x,\lambda) \to I_{n+1}, \text{ as } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty, \tag{73}$$

are a consequence of the Volterra integral equations (57) and (58). Moreover, the basic property of eigenfunctions in (52) tells that

$$(G^+)^{\dagger}(-x,t,-\lambda^*) = C(G^-)^{-1}(x,t,\lambda)C^{-1}, \tag{74}$$

and thus, we find that the jump matrix G_0 possesses the following characteristic property:

$$G_0^{\dagger}(-x,t,-\lambda^*) = CG_0(x,t,\lambda)C^{-1}. \tag{75}$$

We point out that the jump matrix G_0 contains basic scattering data inherited from the scattering matrix $S(\lambda)$.

4 | INVERSE SCATTERING TRANSFORMS

In this section, we analyze the direct and inverse scattering for the nonlocal reverse-space NLS hierarchies (38) through the Riemann-Hilbert technique[17] (see also Refs. 32, 33). We build the inverse scattering theory by transforming the associated Riemann-Hilbert problems with the Sokhotski-Plemelj formula.

4.1 | Evolution of the scattering data

We first develop the evolution laws for the scattering data to formulate the inverse scattering transforms. Take the derivative of (56) with respect to the temporal variable t and utilize the temporal matrix spectral problems for $\psi^{\pm}$.

This way, one can readily show that the scattering matrix S obeys an evolution law:

$$S_t = i\lambda^{2m}[\Omega, S]. \tag{76}$$

This leads to the following time evolution formulas for the time-dependent scattering coefficients:

$$\left\{\begin{array}{l} s_{12}=s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^{2m}t},\ s_{13}=s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^{2m}t},\ \dots,\ s_{1,n+1}=s_{1,n+1}(0,\lambda)\mathrm{e}^{i\beta\lambda^{2m}t}, \\ s_{21}=s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^{2m}t},\ s_{31}=s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^{2m}t},\ \dots,\ s_{n+1,1}=s_{n+1,1}(0,\lambda)\mathrm{e}^{-i\beta\lambda^{2m}t}, \end{array}\right)$$

and shows that any other scattering coefficient does not depend on the time variable t.

4.2 | Gelfand-Levitan-Marchenko type equations

To determine the generalized matrix Jost eigenfunctions, we make the transformation for the Riemann-Hilbert problems in (69) as follows:

$$\left\{\begin{array}{l} G^+-G^-=G^-v,\ v=G_0-I_{n+1},\ \text{on } \mathbb{R}, \\ G^\pm\to I_{n+1} \text{ as } \lambda\in\bar{\mathbb{C}}^\pm\to\infty, \end{array}\right) \tag{77}$$

where G_0 is the jump matrix defined by (70) and (71).

Let $G(\lambda)=G^\pm(\lambda)$ if $\lambda\in\mathbb{C}^\pm$. Suppose that G has simple poles off $\mathbb{R}$: $\{\mu_j\}_{j=1}^R$, where the integer R is arbitrary. Define

$$\tilde{G}^\pm(\lambda) := G^\pm(\lambda)-\sum_{j=1}^R\frac{G_j}{\lambda-\mu_j},\ \lambda\in\bar{\mathbb{C}};\ \tilde{G}(\lambda)=\tilde{G}^\pm(\lambda),\ \lambda\in\mathbb{C}^\pm, \tag{78}$$

where G_j denotes the residue of G at $\lambda=\mu_j$, ie, $G_j=\operatorname{res}(G(\lambda),\mu_j)=\lim_{\lambda\to\mu_j}(\lambda-\mu_j)G(\lambda),\ 1\le j\le R$. Then, we find

$$\left\{\begin{array}{l} \tilde{G}^+-\tilde{G}^-=G^+-G^-=G^-v,\ \text{on } \mathbb{R}, \\ \tilde{G}^\pm\to I_{n+1} \text{ as } \lambda\in\mathbb{C}^\pm\to\infty. \end{array}\right) \tag{79}$$

By applying the Sokhotski-Plemelj formula,[34] we determine the solutions of the transformed Riemann-Hilbert problems in (79) as follows:

$$\tilde{G}(\lambda)=I_{n+1}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(G^-v)(\xi)}{\xi-\lambda}\,d\xi. \tag{80}$$

Furthermore, computing the limit as $\lambda\to\mu_l$ yields

$$\text{lhs}=\lim_{\lambda\to\mu_l}\tilde{G}=F_l-\sum_{j\neq l}^R\frac{G_j}{\mu_l-\mu_j},$$

$$\text{rhs}=I_{n+1}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(G^-v)(\xi)}{\xi-\mu_l}\,d\xi,$$

where $F_l = \lim_{\lambda\to\mu_l}[(\lambda-\mu_l)G(\lambda)-G_l]/(\lambda-\mu_l),\ 1\le l\le R$, and consequently, we obtain the following Gelfand-Levitan-Marchenko type integral equations:

$$I_{n+1} - F_n + \sum_{j\neq l}^{R} \frac{G_j}{\mu_l-\mu_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{(G^-G_0)(\xi)}{\xi-\mu_l}\,d\xi - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{G^-(\xi)}{\xi-\mu_l}\,d\xi = 0,\ 1\le l\le R. \tag{81}$$

Once the jump matrix G_0 is given, by analyzing and solving these equations, one determines solutions of the associated Riemann-Hilbert problems, and hence, the generalized matrix Jost eigenfunctions.

4.3 | Recovery of the potentials

To recover all potentials from the generalized matrix Jost eigenfunctions, we consider an asymptotic expansion

$$G^+(x,t,\lambda) = I_{n+1} + \frac{1}{\lambda}G_1^+(x,t) + \mathrm{O}\left(\frac{1}{\lambda^2}\right),\ \text{as}\ \lambda\to\infty. \tag{82}$$

Plugging this asymptotic expansion into the matrix eigenvalue problem (46) and comparing constant (λ^0) terms engenders

$$P = \lim_{\lambda\to\infty}\lambda[G^+(\lambda),\Lambda] = -[\Lambda, G_1^+]. \tag{83}$$

This equivalently leads to the potential matrix:

$$P = \begin{bmatrix} 0 & -\alpha(G_1^+)_{12} & -\alpha(G_1^+)_{13} & \dots & -\alpha(G_1^+)_{1,n+1} \\ \alpha(G_1^+)_{21} & 0 & 0 & \dots & 0 \\ \alpha(G_1^+)_{31} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha(G_1^+)_{n+1,1} & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{84}$$

where $G_1^+ = ((G_1^+)_{jl})_{(n+1)\times(n+1)}$. Namely, the $2n$ potentials p_j and q_j, $1\le j\le n$, determined by

$$p_j = -\alpha(G_1^+)_{1,j+1},\ q_j = \alpha(G_1^+)_{j+1,1},\ 1\le j\le n, \tag{85}$$

solve the AKNS equations with $r=2m$ in (23). When the nonlocal reduction requirement (30), equivalently

$$(G_1^+(-x,t))^\dagger = CG_1^+(x,t)C^{-1} \tag{86}$$

is satisfied, the reduced potentials p_j, $1\le j\le n$, solve the nonlocal reverse-space NLS hierarchies (38).

The above whole procedure presents the inverse scattering transforms from the scattering matrix $S(\lambda)$, through the jump matrix $G_0(\lambda)$ and the solution $\{G^+(\lambda), G^-(\lambda)\}$ of the Riemann-Hilbert problems, to the potentials that solve the nonlocal reverse-space NLS hierarchies (38).

5 | SOLITON SOLUTIONS

5.1 | Nonreduced case

On account of $\det S = 1$, one has

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & \cdots & s_{2,n+1} \\ s_{32} & s_{33} & \cdots & s_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,2} & s_{n+1,3} & \cdots & s_{n+1,n+1} \end{vmatrix}.$$

Let N be another arbitrarily given natural number. We take N zeros $\{\lambda_k \in \mathbb{C},\ 1 \le k \le N\}$, for s_{11} and N zeros $\{\hat{\lambda}_k \in \mathbb{C},\ 1 \le k \le N\}$, for $\hat{s}_{11}$. To construct soliton solutions, it is also assumed that all these zeros, λ_k and $\hat{\lambda}_k,\ 1 \le k \le N$, are geometrically simple. Then, every $\ker T^+(\lambda_k)$ $(1 \le k \le N)$ contains a single basis column vector, which is denoted by v_k; and every $\ker T^-(\hat{\lambda}_k)$ $(1 \le k \le N)$, only a single basis row vector, which is denoted by $\hat{v}_k$ $(1 \le k \le N)$. This can be presented as

$$T^+(\lambda_k)v_k = 0,\ \hat{v}_k T^-(\hat{\lambda}_k) = 0,\ 1 \le k \le N. \tag{87}$$

To work out soliton solutions, we need to take $G_0 = I_{n+1}$ in each of the Riemann-Hilbert problems in (69). This condition can be achieved if we assume that $s_{i1} = \hat{s}_{1i} = 0,\ 2 \le i \le n+1$. It means that only zero reflection coefficients are taken in the direct scattering problem.

It is known that special Riemann-Hilbert problems with the identity jump matrix, with the canonical normalization conditions in (73) and the zero structures indicated in (87), could be solved explicitly, when $\lambda_l \ne \hat{\lambda}_k,\ 1 \le k, l \le N$ (see, eg, Refs. 17, 35). Consequently, we can directly determine the potential matrix P. If this condition on pole locations of generalized Jost eigenfunctions is not satisfied, which happens in the case of nonlocal integrable equations, solutions to specical Riemann-Hilbert problems with the identity jump matrix can be presented as follows:

$$G^+(\lambda) = I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l},\ (G^-)^{-1}(\lambda) = I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \tag{88}$$

where $M = (m_{kl})_{N\times N}$ is a matrix whose entries are defined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k}, & \text{if } \lambda_l \ne \hat{\lambda}_k,\ 1 \le k, l \le N, \\ 0, & \text{if } \lambda_l = \hat{\lambda}_k,\ 1 \le k, l \le N, \end{cases} \tag{89}$$

for which an additional orthogonal condition

$$\hat{v}_k v_l = 0,\ \text{if } \lambda_l = \hat{\lambda}_k,\ 1 \le k, l \le N \tag{90}$$

is required. Note that within this solution formulation, we do not need the condition of $\lambda_l \ne \hat{\lambda}_k,\ 1 \le k, l \le N$, indeed.

Note that the zeros λ_k and $\hat{\lambda}_k$ do not depend on x and t, ie, space and time independent, and thus, one can determine the spatial and temporal evolution laws for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1 \le k \le N$, in the kernels. For instance, let us calculate the x-derivative of both sides of the first set of equations in (87). Using (46) first and then again the first set of equations in (87), one can find

$$T^+(x,\lambda_k)\Big(\frac{dv_k}{dx} - i\lambda_k\Lambda v_k\Big) = 0,\ 1 \le k \le N. \tag{91}$$

This means that for every $1 \le k \le N$, $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ will be in the kernel of $T^+(x,\lambda_k)$, and thus a constant multiple of v_k. We, without loss of generality, can assume that

$$\frac{dv_k}{dx} = i\lambda_k\Lambda v_k,\ 1 \le k \le N. \tag{92}$$

Analogously, the time dependence of v_k is determined by

$$\frac{dv_k}{dt} = i\lambda_k^{2m}\Omega v_k,\ 1 \le k \le N, \tag{93}$$

which can be obtained by applying the matrix eigenvalue problem (47). Similarly from the second equations in (87), we can obtain space and time dependence of $\hat{v}_k$, $1 \le k \le N$. It then follows that we can have

$$v_k(x,t) = \mathrm{e}^{i\lambda_k\Lambda x + i\lambda_k^{2m}\Omega t} w_k,\ 1 \le k \le N, \tag{94}$$

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^{2m}\Omega t},\ 1 \le k \le N, \tag{95}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary column and row constant vectors, respectively, but need to satisfy

$$\hat{w}_k w_l = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k,\ 1 \le k,l \le N, \tag{96}$$

which follows from (90).

Finally, let us set $\hat{M} = M^{-1} = (\hat{m}_{jl})_{N\times N}$, and then from the solutions in (88), we can obtain

$$G_1^+ = -\sum_{k,l=1}^{N} v_k \hat{m}_{kl} \hat{v}_l, \tag{97}$$

and furthermore, the presentations in (85) lead to the following N-soliton solution of the AKNS equations with multiple potentials with $r = 2m$ in (23):

$$\begin{cases} p_j = \alpha \displaystyle\sum_{k,l=1}^{N} v_{k,1}\widehat{m}_{kl}\widehat{v}_{l,j+1}, \\ q_j = -\alpha \displaystyle\sum_{k,l=1}^{N} v_{k,j+1}\widehat{m}_{kl}\widehat{v}_{l,1}, \end{cases} 1 \le j \le n. \tag{98}$$

5.2 | Nonlocal case

In order for us to work out N-soliton solutions of the nonlocal reverse-space NLS hierarchies (38), we have to satisfy an involution property (86) for G_1^+ defined by (97), which equivalently requires that the potential matrix P determined by (84) satisfies the reduction requirement (30). Then, the above N-soliton solution of the standard AKNS equations with multiple potentials (23) is reduced to the N-soliton solution:

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1} \hat{m}_{kl} \hat{v}_{l,j+1}, \; 1 \le j \le n \tag{99}$$

for the nonlocal reverse-space NLS hierarchies (38).

Let us now analyze how to realize the involution property (86). As usual, we first take N zeros of $\det T^+(\lambda)$ (or eigenvalues of the eigenvalue problems under the zero potential): $\lambda_k \in \mathbb{C}$ for $1 \le k \le N$, and then take

$$\hat{\lambda}_k = \left\{ \begin{array}{l} -\lambda_k^*, \text{ if } \lambda_k \notin i\mathbb{R}, \; 1 \le k \le N, \\ \text{anyvalue } \in i\mathbb{R}, \text{ if } \lambda_k \in i\mathbb{R}, \; 1 \le k \le N, \end{array} \right) \tag{100}$$

which are zeros of $\det T^-(\lambda)$. Now, we see that $\ker T^+(\lambda_k)$, $1 \le k \le N$, are spanned by

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^{2m} \Omega t} w_k, \; 1 \le k \le N, \tag{101}$$

respectively, in which w_k, $1 \le k \le N$, are arbitrary column vectors. These column vectors in (101) are eigenfunctions of the eigenvalue problems under the zero potential associated with λ_k, $1 \le k \le N$. Moreover, according to the previous analysis in Subsection 3.1, $\ker T^-(\lambda_k)$, $1 \le k \le N$, are determined by

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k) C = w_k^\dagger \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^{2m} \Omega t} C, \; 1 \le k \le N, \tag{102}$$

respectively. These row vectors are eigenfunctions of the adjoint eigenvalue problems under the zero potential associated with $\hat{\lambda}_k$, $1 \le k \le N$. To satisfy the orthogonal property (96), we require the following condition on w_k, $1 \le k \le N$:

$$w_k^\dagger C w_k = 0, \text{ if } \lambda_k = \hat{\lambda}_k, \; 1 \le k \le N. \tag{103}$$

Note that for each $1 \le k \le N$, the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when $\lambda_k \in i\mathbb{R}$ and $\hat{\lambda}_k = -\lambda_k^*$.

At this moment, if the solutions of the special Riemann-Hilbert problems, defined by (88) and (89), satisfy the property (74), ie, G_1^+ satisfies the requirement (86) for our nonlocal group reductions in (29), then the formula (99), together with (88), (89), (101), and (102), provides the N-soliton solutions of the nonlocal reverse-space NLS hierarchies of equations in (38), provided that the orthogonal condition (103) holds.

When $N = m = 1$, let us fix $\Sigma = \mathrm{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$, where $\gamma_1, \gamma_2, \dots, \gamma_n$ are arbitrary nonzero real numbers. We choose $\lambda_1 = i\eta_1$, $\hat{\lambda}_1 = -i\eta_1$, $\eta_1 \in \mathbb{R}$, and denote $w_1 = (w_{1,1}, w_{1,2}, \dots, w_{1,n+1})^T$. Then, we can derive the one-soliton solution to the multicomponent nonlocal reverse-space NLS equations in (42):

$$p_j(x,t) = \frac{2\alpha\eta_1 i w_{1,1} w^*_{1,j+1}\gamma_j}{\varepsilon|w_{1,1}|^2 \,\mathrm{e}^{-\alpha\eta_1 x + i\beta\eta_1^2 t} + (|w_{1,2}|^2\gamma_1 + \cdots + |w_{1,n+1}|^2\gamma_n)\,\mathrm{e}^{\alpha\eta_1 x + i\beta\eta_1^2 t}}, \quad 1 \le j \le n, \tag{104}$$

where $\varepsilon = \pm 1$, η_1 is an arbitrary real number, and $w_{1,1}, w_{1,2}, \ldots, w_{1,2}$ are arbitrary complex numbers but satisfy $w_1^\dagger C w_1 = |w_{1,1}|^2 + \gamma_1|w_{1,2}|^2 + \cdots + \gamma_n|w_{1,n+1}|^2 = 0$. This condition comes from the involution property (86). Such one-soliton solutions can develop a singularity at a finite time, and the case of $\varepsilon = 1$ and $n = 1$ can be reduced to the breather one-soliton solution in Ref. 3.

6 | CONCLUDING REMARKS

The paper aims to propose nonlocal reverse-space NLS hierarchies and present their inverse scattering transforms first and then soliton solutions. The primary step is to establish Riemann-Hilbert problems from matrix eigenvalue problems. Solutions of the associated Riemann-Hilbert problems were generated from the Sokhotski-Plemelj formula, and the inverse scattering transforms were formulated, based on the associated Riemann-Hilbert problems. A solution formulation was further presented for special Riemann-Hilbert problems whose jumps are the identity matrix, and thus, from such special Riemann-Hilbert problems (or equivalently, the reflectionless inverse scattering transforms), the N-soliton solutions were worked out for the nonlocal reverse-space NLS hierarchies.

The Riemann-Hilbert technique is a powerful approach for formulating the inverse scattering transforms and constructing soliton solutions (see also, eg, Refs. 18–20, 36). The technique has been recently extended to deal with initial and boundary value problems for integrable equations over both a finite interval and a half-line.[37,38] One can also develop the Riemann-Hilbert technique for general multicomponent NLS equations associated with simple Lie algebras[39] and their nonlocal counterparts.[7] Solution formulations, however, vary from case to case for nonlocal integrable equations, including reverse-space, reverse-time, and reverse-space-time equations (see, eg, Refs. 3, 8, 40). There are also other effective and powerful methods to construct soliton solutions within the theory of integrable equations, which contain the Darboux transformation,[41,42] the Hirota bilinear technique,[43] the Wronskian determinant technique,[44,45] and the generalized bilinear method.[46] It would be significantly important that one could understand the relationships among those interesting distinct methods.

We also point out that it would be particularly interesting to generate different kinds of explicit and exact solutions of integrable equations, such as positon solutions and complexiton ones,[47,48] lump and lump-type solutions,[49–51] solitonless solutions,[52,53] algebrogeometric solutions,[54,55] and dromions[56,57] from a perspective of Riemann-Hilbert problems. Moreover, it is worthy for further investigation is how to establish Riemann-Hilbert problems for dealing with extended integrable counterparts including super or supersymmetric equations, integrable couplings, and fractional analogous equations.

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REFERENCES

1. Ablowitz MJ, Musslimani ZH. Integrable nonlocal nonlinear equations. *Stud Appl Math.* 2017;139:7-59.
2. Ablowitz MJ, Musslimani ZH. Integrable nonlocal nonlinear Schrödinger equation. *Phys Rev Lett.* 2013;110:064105.
3. Ablowitz MJ, Musslimani ZH. Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. *Nonlinearity* 2016;29:915-946.
4. Gerdjikov VS, Saxena A. Complete integrability of nonlocal nonlinear Schrödinger equation. *J Math Phys.* 2017;58:013502.
5. Ablowitz MJ, Luo XD, Musslimani ZH. Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions. *J Math Phys.* 2018;59:011501.
6. Ablowitz MJ, Feng BF, Luo XD, Musslimani ZH. Inverse scattering transform for the nonlocal reverse space-time nonlinear Schrödinger equation. *Theor Math Phys.* 2018;196:1241-1267.
7. Grahovski GG, Mustafa JI, Susanto H. Nonlocal reductions of the multicomponent nonlinear Schrödinger equation on symmetric spaces. *Theor Math Phys.* 2018;197:1430-1450.
8. Ma WX. Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations. *Appl Math Lett.* 2020;102:106161.
9. Yang J. General N-solitons and their dynamics in several nonlocal nonlinear Schrödinger equations. *Phys Lett A* 2019;383:328-337.
10. Ma LY, Zhu ZN. Nonlocal nonlinear Schrödinger equation and its discrete version: soliton solutions and gauge equivalence. *J Math Phys.* 2016;57:083507.
11. Huang X, Ling L. Soliton solutions for the nonlocal nonlinear Schrödinger equation. *Eur Phys J Plus* 2016;131:148.
12. Vinayagam PS, Radha R, Al Khawaja U, Ling L. New classes of solutions in the coupled PT symmetric nonlocal nonlinear Schrödinger equations with four wave mixing. *Commun Nonlinear Sci Numer Simul.* 2018;59:387-395.
13. Gürses M, Pekcan A. Nonlocal nonlinear Schrödinger equations and their soliton solutions. *J Math Phys.* 2018;59:051501.
14. Fokas AS. Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation. *Nonlinearity* 2016;29:319-324.
15. Yang J. Physically significant nonlocal nonlinear Schrödinger equation and its soliton solutions. *Phys Rev E* 2018;98:042202.
16. Ji JL, Zhu ZN. Soliton solutions of an integrable nonlocal modified Korteweg-de Vries equation through inverse scattering transform. *J Math Anal Appl.* 2017;453:973-984.
17. Novikov SP, Manakov SV, Pitaevskii LP, Zakharov VE. *Theory of Solitons: The Inverse Scattering Method*. New York: Consultants Bureau; 1984.
18. Wang DS, Zhang DJ, Yang J. Integrable properties of the general coupled nonlinear Schrödinger equations. *J Math Phys.* 2010;51:023510.
19. Geng XG, Wu JP. Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation. *Wave Motion* 2016;60:62-72.
20. Xiao Y, Fan EG. A Riemann-Hilbert approach to the Harry-Dym equation on the line. *Chin Ann Math Ser B* 2016;37:373-384.
21. Ma WX. Riemann-Hilbert problems and N-soliton solutions for a coupled mKdV system. *J Geom Phys.* 2018;132:45-54.
22. Ablowitz MJ, Kaup DJ, Newell AC, Segur H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud Appl Math.* 1974;53:249-315.

23. Ma WX, Zhou RG. Adjoint symmetry constraints of multicomponent AKNS equations. *Chin Ann Math Ser B* 2002;23:373-384.
24. Ma WX. Riemann-Hilbert problems and soliton solutions of a multicomponent mKdV system and its reduction. *Math Methods Appl Sci.* 2019;42:1099-1113.
25. Manakov SV. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. *Sov Phys JETP* 1974;38:248-253.
26. Chen ST, Zhou RG. An integrable decomposition of the Manakov equation. *Comput Appl Math.* 2012;31:1-18.
27. Ma WX, Zhou RZ. Adjoint symmetry constraints leading to binary nonlinearization. *J Nonlinear Math Phys.* 2002;9(1):106-126.
28. Tu GZ. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J Phys A Math Gen* 1989;22:2375-2392.
29. Ma WX, Chen M. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras. *J Phys A Math Gen* 2006;39:10787-10801.
30. Ma WX, Yong XL, Qin ZY, Gu X, Zhou Y. A generalized Liouville's formula. Preprint, 2017.
31. Beals R, Coifman RR. Scattering and inverse scattering for first order systems. *Commun Pure Appl Math.* 1984;37:39-90.
32. Gerdjikov VS. Geometry, integrability and quantization. In: Mladenov IM, Hirshfeld, eds. *Proceedings of the 6th International Conference* (Varna, June 3-10, 2004), Softex, Sofia, 2005: 78-125.
33. Doktorov EV, Leble SB. *A Dressing Method in Mathematical Physics*. Dordrecht: Springer; 2007.
34. Gakhov FD. *Boundary Value Problems*. London: Elsevier Science; 2014.
35. Kawata T. Riemann spectral method for the nonlinear evolution equation. In: *Advances in Nonlinear Waves, Vol. I*, Boston, MA: Pitman; 1984:210-225.
36. Yang J. *Nonlinear Waves in Integrable and Nonintegrable Systems*. Philadelphia: SIAM; 2010.
37. Fokas AS, Lenells J. The unified method: I. Nonlinearizable problems on the half-line. *J Phys A Math Theor.* 2012;45:195201.
38. Fokas AS, Lenells J. The unified method: III. Nonlinearizable problems on the interval. *J Phys A Math Theor.* 2012;45:195203.
39. Fordy AP, Kulish PP. Nonlinear Schrödinger equations and simple Lie algebras. *Commun Math Phys.* 1983;89:427-443.
40. Ma WX. Inverse scattering and soliton solutions of nonlocal reverse-spacetime nonlinear Schrödinger equations. *Proc Am Math Soc.* in press.
41. Matveev VB, Salle MA. *Darboux Transformations and Solitons*. Berlin: Springer; 1991.
42. Ma WX, Zhang YJ. Darboux transformations of integrable couplings and applications. *Rev Math Phys.* 2018;30:1850003.
43. Hirota R. *The Direct Method in Soliton Theory*. New York: Cambridge University Press; 2004.
44. Freeman NC, Nimmo JJC. Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique. *Phys Lett A* 1983;95:1-3.
45. Ma WX, You Y. Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions. *Trans Am Math Soc.* 2005;357:1753-1778.
46. Ma WX. Generalized bilinear differential equations. *Stud Nonlinear Sci.* 2011;2:140-144.
47. Matveev VB. Generalized Wronskian formula for solutions of the KdV equations: first applications. *Phys Lett A* 1992;166:205-208.
48. Ma WX. Complexiton solutions to the Korteweg-de Vries equation. *Phys Lett A* 2002;301:35-44.
49. Ma WX, Zhou Y. Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. *J Differ Equ.* 2018;264:2633-2659.
50. Tang YN, Tao SQ, Qing G. Lump solitons and the interaction phenomena of them for two classes of nonlinear evolution equations. *Comput Math Appl.* 2016;72:2334-2342.
51. Tang YN, Ma JL, Xie WX, Zhang LJ. Interaction solutions for the (2+1)-dimensional Ito equation. *Mod Phys Lett B* 2019;33:1950167.
52. Ma WX. Long-time asymptotics of a three-component coupled nonlinear Schrödinger system. *J. Geom. Phys.* 2020;153:103669.
53. Ma WX. Long-time asymptotics of a three-component coupled mKdV system. *Mathematics* 2019;7:573.

54. Belokolos ED, Bobenko AI, Enol'skii VZ, Its AR, Matveev VB. *Algebro-Geometric Approach to Nonlinear Integrable Equations*. Berlin: Springer; 1994.
55. Ma WX. Trigonal curves and algebro-geometric solutions to soliton hierarchies I, II. *Proc R Soc A* 2017;473:20170232-20170233.
56. Boiti M, Leon JJP, Martina L, Pempinelli F. Scattering of localized solitons in the plane. *Phys Lett A* 1988;132:432-439.
57. Hietarinta J. One-dromion solutions for generic classes of equations. *Phys Lett A* 1990;149:113-118.

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