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Inverse scattering for nonlocal reverse-space multicomponent nonlinear Schrödinger equations

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The paper aims to discuss nonlocal reverse-space multicomponent nonlinear Schrödinger equations and their inverse scattering transforms. The inverse scattering problems are analyzed by means of Riemann–Hilbert problems, and Gelfand–Levitan–Marchenko-type integral equations for generalized matrix Jost solutions are determined by the Sokhotski–Plemelj formula. Soliton solutions are generated from the reflectionless transforms associated with zeros of the Riemann–Hilbert problems.

Keywords: Matrix spectral problem; nonlocal reduction; inverse scattering; Riemann–Hilbert problem; soliton solution.

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1. Introduction

Nonlocal integrable nonlinear Schrödinger (NLS) equations arise while taking specific reductions.[1] The corresponding inverse scattering transforms have been recently established under zero or nonzero boundary conditions[2–4] and N-soliton solutions have been constructed from the Riemann–Hilbert problems with the identity

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jump matrix[5] and by the Hirota bilinear method.[6] Some multicomponent generalizations[7–9] and other nonlocal integrable equations[10] were also presented. We would like to present a class of general nonlocal reverse-space multicomponent NLS equations and analyze their inverse scattering transforms and soliton solutions through formulating and solving Riemann–Hilbert problems.

The Riemann–Hilbert approach is one of the most powerful techniques to study integrable equations and particularly generate soliton solutions.[11] Many integrable equations, including the multiple wave interaction equations,[11] the general coupled NLS equations,[12] the Harry Dym equation,[13] the generalized Sasa–Satsuma equation[14] and the Ablowitz–Kaup–Newell–Segur (AKNS) soliton hierarchies,[15] have been studied by analyzing the associated Riemann–Hilbert problems.

The standard procedure for establishing Riemann–Hilbert problems is to start from a pair of matrix spectral problems, let us say,

$$\begin{gathered} -i\phi_x = U\phi, \quad -i\phi_t = V\phi, \\ U = A(\lambda) + P(u,\lambda), \quad V = B(\lambda) + Q(u,\lambda), \end{gathered} \tag{1.1}$$

where i is the unit imaginary number, λ is a spectral parameter, u is a potential and ϕ is an $m \times m$ matrix eigenfunction. The zero-curvature equation, i.e., the compatibility condition of the above two matrix spectral problems,

$$U_t - V_x + i[U,V] = 0, \tag{1.2}$$

where $[\cdot,\cdot]$ is the matrix commutator, presents an integrable equation. To establish an associated Riemann–Hilbert problem for this integrable equation, we use the following equivalent pair of matrix spectral problems:

$$\psi_x = i[A(\lambda),\psi] + \check{P}(u,\lambda)\psi, \quad \psi_t = i[B(\lambda),\psi] + \check{Q}(u,\lambda)\psi, \tag{1.3}$$

where ψ is an $m \times m$ matrix eigenfunction, $\check{P} = iP$ and $\check{Q} = iQ$. We often assume that A, B are constant commuting $m \times m$ matrices, and P, Q are trace-less $m \times m$ matrices. The equivalence between (1.1) and (1.3) comes from the commutativity of A and B, and $(\det\psi)_x = (\det\psi)_t = 0$ are two consequences of $\mathrm{tr}P = \mathrm{tr}Q = 0$. There exists a direct relation between (1.1) and (1.3):

$$\phi = \psi E_g, \quad E_g = \mathrm{e}^{iA(\lambda)x + iB(\lambda)t}. \tag{1.4}$$

For the pair of matrix spectral problems in (1.3), we can impose the asymptotic conditions:

$$\psi^{\pm} \to I_m, \quad \text{when } x \text{ or } t \to \pm\infty, \tag{1.5}$$

where I_m stands for the identity matrix of size m. From these two matrix eigenfunctions $\psi^{\pm}$, we need to pick the entries and build two generalized matrix Jost solutions $T^{\pm}(x,t,\lambda)$, which are analytical in the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous in the closed upper and lower half-planes $\bar{\mathbb{C}}^+$ and $\bar{\mathbb{C}}^-$, respectively, to formulate a Riemann–Hilbert problem on the real line:

$$G^+(x,t,\lambda) = G^-(x,t,\lambda)G_0(x,t,\lambda), \quad \lambda \in \mathbb{R}, \tag{1.6}$$

where two unimodular generalized matrix Jost solutions, G^+ and G^-, and the jump matrix, G_0, are determined from T^+ and T^-.

Recall that the scattering matrix $S_g(\lambda)$ of the matrix spectral problems is defined through

$$\psi^- E_g = \psi^+ E_g S_g(\lambda). \tag{1.7}$$

Normally, the jump matrix G_0 carries basic scattering data from $S_g(\lambda)$. Solutions to the associated Riemann–Hilbert problems provide the required generalized matrix Jost solutions in recovering the potential of the matrix spectral problems, which solves the corresponding integrable equation. Such solutions could be presented by using the Sokhotski–Plemelj formula, upon computing their difference. Then, a recovery of the potential finishes the inverse scattering transforms, through observing asymptotic behaviors of the generalized matrix Jost solutions $G^\pm$ at infinity of λ. Soliton solutions are presented from solutions to the Riemann–Hilbert problems with the identity jump matrix G_0, or equivalently, the zero reflection coefficients.

In this paper, we first present a class of nonlocal reverse-space multicomponent NLS equations by making a specific group of nonlocal reductions, and analyze their inverse scattering transforms and soliton solutions, based on associated Riemann–Hilbert problems. One example with two components is

$$\begin{cases} ip_{1,t}(x,t) = p_{1,xx}(x,t) - 2[c_1 p_1(x,t)p_1^*(-x,t) + c_2 p_2(x,t)p_2^*(-x,t)]p_1(x,t), \\ ip_{2,t}(x,t) = p_{2,xx}(x,t) - 2[c_1 p_1(x,t)p_1^*(-x,t) + c_2 p_2(x,t)p_2^*(-x,t)]p_2(x,t), \end{cases} \tag{1.8}$$

where c_1 and c_2 are arbitrary nonzero real constants. The rest of the paper is structured as follows. In Sec. 2, within the zero-curvature formulation, we recall the AKNS integrable hierarchy with multiple potentials, and make a group of nonlocal reductions to construct nonlocal reverse-space multicomponent NLS equations. In Sec. 3, we analyze the inverse scattering transforms through Riemann–Hilbert problems associated with higher-order matrix spectral problems. In Sec. 4, we construct soliton solutions to the presented nonlocal reverse-space multicomponent NLS equations from special associated Riemann–Hilbert problems on the real axis where an identity jump matrix is taken. In Sec. 5, we give a conclusion, together with some concluding remarks.

2. Nonlocal Reverse-space NLS Equations

2.1. *Multicomponent AKNS hierarchy*

Let $n \in \mathbb{N}$ be arbitrary, and α_1 and α_2, different real constants. We consider the following matrix spectral problem[16]:

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad U = (U_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} \alpha_1\lambda & p \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{2.1}$$

where λ is a spectral parameter and u is a $2n$-dimensional potential

$$u = (p, q^T)^T, \quad p = (p_1, p_2, \ldots, p_n), \quad q = (q_1, q_2, \ldots, q_n)^T. \tag{2.2}$$

When $p_j = q_j = 0, 2 \leq j \leq n$, (2.1) becomes the standard AKNS spectral problem.[17] Thus, we call it a multicomponent AKNS matrix spectral problem, and its associated hierarchy, a multicomponent AKNS integrable hierarchy. On account of the existence of a multiple eigenvalue of $\frac{\partial U}{\partial \lambda}$, the matrix spectral problem (2.1) is degenerate.

To derive an associated multicomponent AKNS integrable hierarchy, we first solve the stationary zero-curvature equation

$$W_x = i[U, W], \tag{2.3}$$

corresponding to (2.1). We look for a solution W of the form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.4}$$

where a is a scalar, b^T and c are n-dimensional columns, and d is an $n \times n$ matrix. It is direct to show that the stationary zero-curvature equation (2.3) reads

$$\begin{aligned} &a_x = i(pc - bq), \quad b_x = i(\alpha\lambda b + pd - ap), \\ &c_x = i(-\alpha\lambda c + qa - dq), \quad d_x = i(qb - cp), \end{aligned} \tag{2.5}$$

where $\alpha = \alpha_1 - \alpha_2$. We take W as a formal series:

$$\begin{aligned} W &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m}, \\ W_m &= W_m(u) = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & d^{[m]} \end{bmatrix}, \quad m \geq 0, \end{aligned} \tag{2.6}$$

where $b^{[m]}, c^{[m]}$ and $d^{[m]}$ are expressed as

$$\begin{aligned} &b^{[m]} = (b_1^{[m]}, b_2^{[m]}, \ldots, b_n^{[m]}), \quad c^{[m]} = (c_1^{[m]}, c_2^{[m]}, \ldots, c_n^{[m]})^T, \\ &d^{[m]} = (d_{jl}^{[m]})_{n\times n}, \quad m \geq 0. \end{aligned} \tag{2.7}$$

Then, the system (2.5) exactly presents the following recursion relations:

$$b^{[0]} = 0, \quad c^{[0]} = 0, \quad a_x^{[0]} = 0, \quad d_x^{[0]} = 0, \tag{2.8a}$$

$$b^{[m+1]} = \frac{1}{\alpha}(-ib_x^{[m]} - pd^{[m]} + a^{[m]}p), \quad m \geq 0, \tag{2.8b}$$

$$c^{[m+1]} = \frac{1}{\alpha}(ic_x^{[m]} + qa^{[m]} - d^{[m]}q), \quad m \geq 0, \tag{2.8c}$$

$$a_x^{[m]} = i(pc^{[m]} - b^{[m]}q), \quad d_x^{[m]} = i(qb^{[m]} - c^{[m]}p), \quad m \geq 1. \tag{2.8d}$$

Next, we fix the initial values:

$$a^{[0]} = \beta_1, \quad d^{[0]} = \beta_2 I_n, \tag{2.9}$$

where β_1, β_2 are arbitrary but different constants, and take zero constants of integration in (2.8d), which means that we require

$$W_m|_{u=0} = 0, \quad m \geq 1. \tag{2.10}$$

Then, with $a^{[0]}$ and $d^{[0]}$ given by (2.9), all matrices $W_m, m \geq 1$, defined recursively, are uniquely determined. For instance, a direct calculation, based on (2.8), yields that

$$b_j^{[1]} = \frac{\beta}{\alpha} p_j, \quad c_j^{[1]} = \frac{\beta}{\alpha} q_j, \quad a^{[1]} = 0, \quad d_{jl}^{[1]} = 0; \tag{2.11a}$$

$$b_j^{[2]} = -\frac{\beta}{\alpha^2} i p_{j,x}, \quad c_j^{[2]} = \frac{\beta}{\alpha^2} i q_{j,x}, \quad a^{[2]} = -\frac{\beta}{\alpha^2} pq, \quad d_{jl}^{[2]} = \frac{\beta}{\alpha^2} p_l q_j; \tag{2.11b}$$

$$\begin{cases} b_j^{[3]} = -\dfrac{\beta}{\alpha^3}[p_{j,xx} + 2pqp_j], \quad c_j^{[3]} = -\frac{\beta}{\alpha^3}[q_{j,xx} + 2pqq_j], \\ a^{[3]} = -\dfrac{\beta}{\alpha^3} i(pq_x - p_x q), \qquad d_{jl}^{[3]} = -\frac{\beta}{\alpha^3} i(p_{l,x} q_j - p_l q_{j,x}); \end{cases} \tag{2.11c}$$

$$\begin{cases} b_j^{[4]} = \dfrac{\beta}{\alpha^4} i[p_{j,xxx} + 3pqp_{j,x} + 3p_x q p_j], \\ c_j^{[4]} = -\dfrac{\beta}{\alpha^4} i[q_{j,xxx} + 3pqq_{j,x} + 3pq_x q_j], \\ a^{[4]} = \dfrac{\beta}{\alpha^4}[3(pq)^2 + pq_{xx} - p_x q_x + p_{xx} q], \\ d_{jl}^{[4]} = -\dfrac{\beta}{\alpha^4}[3p_l pqq_j + p_{l,xx} q_j - p_{l,x} q_{j,x} + p_l q_{j,xx}]; \end{cases} \tag{2.11d}$$

where $\beta = \beta_1 - \beta_2$ and $1 \leq j, l \leq n$. Based on (2.8d), we can obtain, from (2.8b) and (2.8c), a recursion relation for $b^{[m]}$ and $c^{[m]}$:

$$\begin{bmatrix} c^{[m+1]} \\ b^{[m+1]T} \end{bmatrix} = \Psi \begin{bmatrix} c^{[m]} \\ b^{[m]T} \end{bmatrix}, \quad m \geq 1, \tag{2.12}$$

where Ψ is a $2n \times 2n$ matrix operator

$$\Psi = \frac{i}{\alpha} \begin{bmatrix} \left(\partial + \sum_{j=1}^{n} q_j \partial^{-1} p_j\right) I_n + q\partial^{-1} p & -q\partial^{-1} q^T - (q\partial^{-1} q^T)^T \\ p^T \partial^{-1} p + (p^T \partial^{-1} p)^T & -\left(\partial + \sum_{j=1}^{n} p_j \partial^{-1} q_j\right) I_n - p^T \partial^{-1} q^T \end{bmatrix}. \tag{2.13}$$

The multicomponent AKNS integrable hierarchy is associated with the following temporal matrix spectral problems:

$$\begin{aligned} -i\phi_t &= V^{[r]}\phi = V^{[r]}(u,\lambda)\phi, \\ V^{[r]} &= (V^{[r]}_{jl})_{(n+1)\times(n+1)} = \sum_{m=0}^{r} W_m \lambda^{r-m}, \quad r \geq 0. \end{aligned} \tag{2.14}$$

The compatibility conditions of (2.1) and (2.14), i.e., the zero-curvature equations,

$$U_t - V^{[r]}_x + i[U, V^{[r]}] = 0, \quad r \geq 0, \tag{2.15}$$

generate the so-called multicomponent AKNS integrable hierarchy:

$$u_t = \begin{bmatrix} p^T \\ q \end{bmatrix}_t = K_r = i \begin{bmatrix} \alpha b^{[r+1]T} \\ -\alpha c^{[r+1]} \end{bmatrix}, \quad r \geq 0. \tag{2.16}$$

The first nonlinear integrable system in the above hierarchy (2.16) consists of the standard NLS equations:

$$\begin{cases} p_{j,t} = -\dfrac{\beta}{\alpha^2} i \left[p_{j,xx} + 2\left(\displaystyle\sum_{l=1}^{n} p_l q_l\right) p_j \right], & 1 \leq j \leq n, \\ q_{j,t} = \dfrac{\beta}{\alpha^2} i \left[q_{j,xx} + 2\left(\displaystyle\sum_{l=1}^{n} p_l q_l\right) q_j \right], & 1 \leq j \leq n. \end{cases} \tag{2.17}$$

When $n = 2$, under a special kind of symmetric reductions, the multicomponent NLS equations (2.17) can be reduced to the Manakov system[18] and a decomposition into finite-dimensional integrable Hamiltonian systems was made for that reduced system in Ref. 19.

2.2. *Nonlocal reverse-space NLS equations*

Let us take a specific group of nonlocal reductions for the spectral matrix:

$$U^\dagger(-x,t,-\lambda^*) = -CU(x,t,\lambda)C^{-1}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix}, \quad \Sigma^\dagger = \Sigma, \tag{2.18}$$

which implies that

$$P^\dagger(-x,t) = -CP(x,t)C^{-1}. \tag{2.19}$$

Henceforth, $\dagger$ stands for the Hermitian transpose, $*$ denotes the complex conjugate, Σ is a constant invertible Hermitian matrix, and for brevity, we adopt

$$\begin{cases} A(x,t,\lambda) = A(u(x,t),\lambda), \\ A^\dagger(f(x,t,\lambda)) = (A(f(x,t,\lambda)))^\dagger, \\ A^{-1}(f(x,t,\lambda)) = (A(f(x,t,\lambda)))^{-1}, \end{cases} \tag{2.20}$$

for a matrix A and a function f.

The matrix spectral problems of the multicomponent NLS equations (2.17) read

$$-i\phi_x = U\phi = U(u,\lambda)\phi, \quad -i\phi_t = V^{[2]}\phi = V^{[2]}(u,\lambda)\phi, \tag{2.21}$$

where the Lax pair is given by

$$U = \lambda\Lambda + P, \quad V^{[2]} = \lambda^2\Omega + Q, \tag{2.22}$$

with $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n)$, $\Omega = \mathrm{diag}(\beta_1, \beta_2 I_n)$, and

$$\begin{aligned} P &= \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \\ Q &= \begin{bmatrix} a^{[1]}\lambda + a^{[2]} & b^{[1]}\lambda + b^{[2]} \\ c^{[1]}\lambda + c^{[2]} & d^{[1]}\lambda + d^{[2]} \end{bmatrix} \\ &= \frac{\beta}{\alpha}\lambda \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2} \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}. \end{aligned} \tag{2.23}$$

In the above matrices P and Q, u, p, q are defined by (2.2), and $a^{[m]}, b^{[m]}, c^{[m]}, d^{[m]}$, $1 \le m \le 2$, are determined in (2.11).

From (2.19), we obtain

$$q(x,t) = -\Sigma^{-1}p^\dagger(-x,t). \tag{2.24}$$

The vector function c in (2.5) under such a nonlocal reduction could be taken as

$$c(x,t,\lambda) = \Sigma^{-1}b^\dagger(-x,t,-\lambda^*). \tag{2.25}$$

Those nonlocal reduction relations guarantee that

$$a^*(-x,t,-\lambda^*) = a(x,t,\lambda), \quad d^\dagger(-x,t,-\lambda^*) = \Sigma d(x,t,\lambda)\Sigma^{-1}, \tag{2.26}$$

where a and d satisfy (2.5). For example, under (2.24) and (2.25), we can compute that

$$\begin{aligned} (a^*(-x,t,-\lambda^*))_x &= -a_x^*(-x,t,-\lambda^*) \\ &= i[c^\dagger(-x,t,-\lambda^*)p^\dagger(-x,t) - q^\dagger(-x,t)b^\dagger(-x,t,-\lambda^*)] \\ &= i\{[b(x,t,\lambda)\Sigma^{-1}][-\Sigma q(x,t)] - [-p(x,t)\Sigma^{-1}][\Sigma c(x,t,\lambda)]\} \\ &= -i[b(x,t,\lambda)q(x,t) - p(x,t)c(x,t,\lambda)] = a_x(x,t,\lambda), \end{aligned}$$

from which the first relation in (2.26) follows. Therefore, upon using the Laurent expansions for a, b, c and d, we obtain

$$\begin{cases} (a^{[m]})^*(-x,t) = (-1)^m a^{[m]}(x,t), \\ (b^{[m]})^\dagger(-x,t) = (-1)^m \Sigma c^{[m]}(x,t), \\ (d^{[m]})^\dagger(-x,t) = (-1)^m \Sigma d^{[m]}(x,t)\Sigma^{-1}, \end{cases} \tag{2.27}$$

where $m \geq 0$. This implies that

$$\begin{aligned}(V^{[2]})^{\dagger}(-x,t,-\lambda^*) &= CV^{[2]}(x,t,\lambda)C^{-1},\\ Q^{\dagger}(-x,t,-\lambda^*) &= CQ(x,t,\lambda)C^{-1},\end{aligned} \tag{2.28}$$

where $V^{[2]}$ and Q are defined in (2.22) and (2.23), respectively.

Therefore, it is direct to see that the nonlocal reduction (2.19) does not present any new condition for the compatibility of the spatial and temporal matrix spectral problems in (2.21). The multicomponent standard NLS equations (2.17) are then reduced to the following nonlocal reverse-space multicomponent NLS equations:

$$ip_t(x,t) = \frac{\beta}{\alpha^2}[p_{xx}(x,t) - 2p(x,t)\Sigma^{-1}p^{\dagger}(-x,t)p(x,t)], \tag{2.29}$$

where Σ is an arbitrary invertible Hermitian matrix.

When $n = 1$, we can obtain two well-known scalar examples[1]:

$$ip_t(x,t) = p_{xx}(x,t) - 2\sigma p^2(x,t)p^*(-x,t), \quad \sigma = \mp 1. \tag{2.30}$$

When $n = 2$, we can get a system of nonlocal reverse-space two-component NLS equations (1.8).

3. Inverse Scattering Transforms

3.1. *Distribution of eigenvalues*

Let q be defined by (2.24). In what follows, we discuss the scattering and inverse scattering for the nonlocal reverse-space multicomponent NLS equations (2.29) through the Riemann–Hilbert approach[11] (see also Refs. 20 and 21). The results will lay the groundwork for soliton solutions in the following section. Assume that all the potentials sufficiently rapidly vanish when $x \to \pm\infty$ or $t \to \pm\infty$. For the matrix spectral problems in (2.21), we can impose the asymptotic behavior: $\phi \sim \mathrm{e}^{i\lambda\Lambda x + i\lambda^2\Omega t}$, when $x, t \to \pm\infty$. Therefore, if we make the variable transformation

$$\phi = \psi E_g, \quad E_g = \mathrm{e}^{i\lambda\Lambda x + i\lambda^2\Omega t},$$

then we can have the canonical asymptotic conditions: $\psi \to I_{n+1}$, when $x, t \to \infty$ or $-\infty$. Upon setting $\check{P} = iP$ and $\check{Q} = iQ$, the equivalent pair of matrix spectral problems to (2.21) reads

$$\psi_x = i\lambda[\Lambda, \psi] + \check{P}\psi, \tag{3.1}$$

$$\psi_t = i\lambda^2[\Omega, \psi] + \check{Q}\psi. \tag{3.2}$$

Applying a generalized Liouville's formula,[22] we can have

$$\det \psi = 1, \tag{3.3}$$

since $(\det \psi)_x = 0$ due to $\mathrm{tr}\check{P} = \mathrm{tr}\check{Q} = 0$.

Recall that the adjoint equation of the x-part of (2.21) and the adjoint equation of (3.1) are given by

$$i\tilde{\phi}_x = \tilde{\phi}U, \tag{3.4}$$

and

$$i\tilde{\psi}_x = \lambda[\tilde{\psi}, \Lambda] + \tilde{\psi}P, \tag{3.5}$$

respectively, for which there are the links: $\tilde{\phi} = \phi^{-1}$ and $\tilde{\psi} = \psi^{-1}$. Each pair of adjoint matrix spectral problems and equivalent adjoint matrix spectral problems do not bring any new condition, either, except the nonlocal reverse-space multi-component NLS equations (2.29).

Let $\psi(\lambda)$ be a matrix eigenfunction of the spatial spectral problem (3.1) associated with an eigenvalue λ. Then, $C\psi^{-1}(x,t,\lambda)$ is a matrix adjoint eigenfunction associated with the same eigenvalue λ. Under the nonlocal reduction in (2.19), we can have

$$\begin{aligned} i[\psi^\dagger(-x,t,-\lambda^*)C]_x &= i[-(\psi_x)^\dagger(-x,t,-\lambda^*)C] \\ &= -i\{(-i)(-\lambda)[\psi^\dagger(-x,t,-\lambda^*),\Lambda] \\ &\quad + (-i)\psi^\dagger(-x,t,-\lambda^*)P^\dagger(-x,t)\}C \\ &= \lambda[\psi^\dagger(-x,t,-\lambda^*),\Lambda]C + \psi^\dagger(-x,t,-\lambda^*)C[-C^{-1}P^\dagger(-x,t)C] \\ &= \lambda[\psi^\dagger(-x,t,-\lambda^*)C,\Lambda] + \psi^\dagger(-x,t,-\lambda^*)CP(x,t), \end{aligned}$$

and so

$$\tilde{\psi}(x,t,\lambda) := \psi^\dagger(-x,t,-\lambda^*)C, \tag{3.6}$$

presents another matrix adjoint eigenfunction associated with the same original eigenvalue λ, i.e., $\psi^\dagger(-x,t,-\lambda^*)C$ solves the adjoint spectral problem (3.5).

Now, we observe the asymptotic conditions for ψ, and find that by the uniqueness of solutions, we have

$$\psi^\dagger(-x,t,-\lambda^*) = C\psi^{-1}(x,t,\lambda)C^{-1}, \tag{3.7}$$

when $\psi \to I_{n+1},\ x$ or $t \to \infty$ or $-\infty$. This implies that if λ is an eigenvalue of (3.1) (or (3.5)), then $-\lambda^*$ will be another eigenvalue of (3.1) (or (3.5)), and the property (3.7) holds.

3.2. *Riemann–Hilbert problems*

Let us now formulate a class of associated Riemann–Hilbert problems with the variable x. In order to facilitate the concrete expression, we also make the following assumptions:

$$\alpha = \alpha_1 - \alpha_2 < 0, \quad \beta = \beta_1 - \beta_2 < 0. \tag{3.8}$$

In the scattering problem, we first introduce the two matrix eigenfunctions $\psi^{\pm}(x,\lambda)$ of (3.1) with the asymptotic conditions:

$$\psi^{\pm} \to I_{n+1}, \quad \text{when } x \to \pm\infty, \tag{3.9}$$

respectively. It follows from (3.3) that $\det \psi^{\pm} = 1$ for all $x \in \mathbb{R}$. Since

$$\phi^{\pm} = \psi^{\pm} E, \quad E = \mathrm{e}^{i\lambda\Lambda x}, \tag{3.10}$$

are both matrix eigenfunctions of (2.21), they must be linearly dependent, and consequently, one has

$$\psi^{-} E = \psi^{+} E S(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.11}$$

where $S(\lambda) = (s_{jl})_{(n+1)\times(n+1)}$ is the corresponding scattering matrix. Note that $\det S(\lambda) = 1$, thanks to $\det \psi^{\pm} = 1$.

We turn the x-part of (2.21) into the following Volterra integral equations for $\psi^{\pm}$[11]:

$$\psi^{-}(\lambda,x) = I_{n+1} + \int_{-\infty}^{x} \mathrm{e}^{i\lambda\Lambda(x-y)} \check{P}(y)\psi^{-}(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\, dy, \tag{3.12}$$

$$\psi^{+}(\lambda,x) = I_{n+1} - \int_{x}^{\infty} \mathrm{e}^{i\lambda\Lambda(x-y)} \check{P}(y)\psi^{+}(\lambda,y)\mathrm{e}^{i\lambda\Lambda(y-x)}\, dy, \tag{3.13}$$

where the asymptotic conditions (3.9) have been imposed. Now, the theory of Volterra integral equations tells that by the Neumann series,[23] one can show that the eigenfunctions $\psi^{\pm}$ exist and allow analytical continuations off the real axis $\lambda \in \mathbb{R}$ as long as the integrals on their right-hand sides converge. From the diagonal form of Λ and the first assumption in (3.8), we can observe that the integral equation for the first column of ψ^{-} contains only the exponential factor $\mathrm{e}^{-i\alpha\lambda(x-y)}$, which decays because of $y < x$ in the integral, if λ takes values in the upper half-plane $\mathbb{C}^{+}$, and the integral equation for the last n columns of ψ^{+} contains only the exponential factor $\mathrm{e}^{i\alpha\lambda(x-y)}$, which also decays because of $y > x$ in the integral, when λ takes values in the upper half-plane $\mathbb{C}^{+}$. Thus, these $n+1$ columns are analytical in the upper half-plane $\mathbb{C}^{+}$ and continuous in the closed upper half-plane $\bar{\mathbb{C}}^{+}$. In a similar manner, we can know that the last n columns of ψ^{-} and the first column of ψ^{+} are analytical in the lower half-plane $\mathbb{C}^{-}$ and continuous in the closed lower half-plane $\bar{\mathbb{C}}^{-}$.

First, to determine two generalized matrix Jost solutions T^{+} and T^{-}, we express

$$\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm}, \ldots, \psi_{n+1}^{\pm}), \tag{3.14}$$

that is, $\psi_j^{\pm}$ denotes the jth column of $\phi^{\pm}$ $(1 \le j \le n+1)$, and then we can take the generalized matrix Jost solution T^{+} as

$$\begin{aligned} T^{+} = T^{+}(x,\lambda) &= (\psi_1^{-}, \psi_2^{+}, \ldots, \psi_{n+1}^{+}) \\ &= \psi^{-} H_1 + \psi^{+} H_2, \end{aligned} \tag{3.15}$$

which is analytic in $\lambda \in \mathbb{C}^+$ and continuous in $\lambda \in \bar{\mathbb{C}}^+$. The generalized matrix Jost solution

$$(\psi_1^+, \psi_2^-, \ldots, \psi_{n+1}^-) = \psi^+ H_1 + \psi^- H_2 \tag{3.16}$$

is analytic in $\lambda \in \mathbb{C}^-$ and continuous in $\lambda \in \bar{\mathbb{C}}^-$. Here, we introduce

$$H_1 = \mathrm{diag}(1, \underbrace{0, \ldots, 0}_{n}), \quad H_2 = \mathrm{diag}(0, \underbrace{1, \ldots, 1}_{n}). \tag{3.17}$$

Second, to determine the other generalized matrix Jost solution T^-, we construct the analytic counterpart of T^+ in the lower half-plane $\mathbb{C}^-$ from the adjoint counterparts of the matrix spectral problems. Note that the inverse matrices $\tilde{\phi}^\pm = (\phi^\pm)^{-1}$ and $\tilde{\psi}^\pm = (\psi^\pm)^{-1}$ solve those two adjoint equations, respectively. Upon expressing $\tilde{\psi}^\pm$ by

$$\tilde{\psi}^\pm = (\tilde{\psi}^{\pm,1}, \tilde{\psi}^{\pm,2}, \ldots, \tilde{\psi}^{\pm,n+1})^T, \tag{3.18}$$

that is, $\tilde{\psi}^{\pm,j}$ denotes the jth row of $\tilde{\psi}^\pm$ $(1 \le j \le n+1)$, we can prove by similar arguments that we can take the generalized matrix Jost solution T^- as the adjoint matrix solution of (3.5), i.e.,

$$\begin{aligned} T^- &= (\tilde{\psi}^{-,1}, \tilde{\psi}^{+,2}, \ldots, \tilde{\psi}^{+,n+1})^T = H_1 \tilde{\psi}^- + H_2 \tilde{\psi}^+ \\ &= H_1 (\psi^-)^{-1} + H_2 (\psi^+)^{-1}, \end{aligned} \tag{3.19}$$

which is analytic for $\lambda \in \mathbb{C}^-$ and continuous for $\lambda \in \bar{\mathbb{C}}^-$, and the other generalized matrix Jost solution of (3.5),

$$(\tilde{\psi}^{+,1}, \tilde{\psi}^{-,2}, \ldots, \tilde{\psi}^{-,n+1})^T = H_1 \tilde{\psi}^+ + H_2 \tilde{\psi}^- = H_1 (\psi^+)^{-1} + H_2 (\psi^-)^{-1}, \tag{3.20}$$

is analytic for $\lambda \in \mathbb{C}^+$ and continuous for $\lambda \in \bar{\mathbb{C}}^+$.

Now we have constructed the two generalized matrix Jost solutions, T^+ and T^-. Directly from $\det \psi^\pm = 1$ and the scattering relation (3.11) between ψ^+ and ψ^-, we have

$$\det T^+(x, \lambda) = s_{11}(\lambda), \quad \det T^-(x, \lambda) = \hat{s}_{11}(\lambda), \tag{3.21}$$

where $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jl})_{(n+1)\times(n+1)}$. It also follows that

$$\begin{aligned} \lim_{x\to\infty} T^+(x, \lambda) &= \begin{bmatrix} s_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \quad \lambda \in \bar{\mathbb{C}}^+; \\ \lim_{x\to\infty} T^-(x, \lambda) &= \begin{bmatrix} \hat{s}_{11}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, \quad \lambda \in \bar{\mathbb{C}}^-. \end{aligned} \tag{3.22}$$

This way, we can introduce the following two unimodular generalized matrix Jost solutions:

$$\begin{cases} G^+(x,\lambda) = T^+(x,\lambda)\begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix}, & \lambda \in \bar{\mathbb{C}}^+; \\ (G^-)^{-1}(x,\lambda) = \begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} T^-(x,\lambda), & \lambda \in \bar{\mathbb{C}}^-. \end{cases} \tag{3.23}$$

Those two generalized matrix Jost solutions form the required matrix Riemann–Hilbert problems on the real line for the nonlocal reverse-space multicomponent NLS equations (2.29):

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{3.24}$$

where by (3.11), the jump matrix G_0 reads

$$G_0(x,\lambda) = E\begin{bmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} \tilde{S}(\lambda) \begin{bmatrix} s_{11}^{-1}(\lambda) & 0 \\ 0 & I_n \end{bmatrix} E^{-1}. \tag{3.25}$$

In the above jump matrix G_0, $\tilde{S}(\lambda)$ has the factorization:

$$\tilde{S}(\lambda) = (H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda)H_2), \tag{3.26}$$

which can be worked out as follows:

$$\tilde{S}(\lambda) = (\tilde{s}_{jl})_{(n+1)\times(n+1)} = \begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \cdots & 0 \\ s_{31} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ s_{n+1,1} & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.27}$$

Based on the Volterra integral equations (3.12) and (3.13), we can get the canonical normalization conditions:

$$G^\pm(x,\lambda) \to I_{n+1}, \quad \text{when } \lambda \in \bar{\mathbb{C}}^\pm \to \infty, \tag{3.28}$$

for the presented Riemann–Hilbert problems. By the property (3.7), we see also that

$$(G^+)^\dagger(-x,t,-\lambda^*) = C(G^-)^{-1}(x,t,\lambda)C^{-1}, \tag{3.29}$$

and hence, the jump matrix G_0 satisfies the following involution property:

$$G_0^\dagger(-x,t,-\lambda^*) = CG_0(x,t,\lambda)C^{-1}. \tag{3.30}$$

3.3. *Evolution of the scattering data*

To complete the direct scattering transforms, we take the derivative of (3.11) with time t and use the temporal matrix spectral problems:

$$\psi_t^{\pm} = i\lambda^2[\Omega, \psi^{\pm}] + \check{Q}\psi^{\pm}.$$

It then follows that the scattering matrix S satisfies an evolution law:

$$S_t = i\lambda^2[\Omega, S], \tag{3.31}$$

which tells the time evolution of the time-dependent scattering coefficients:

$$\begin{cases} s_{12} = s_{12}(0,\lambda)\mathrm{e}^{i\beta\lambda^2 t}, \quad s_{13} = s_{13}(0,\lambda)\mathrm{e}^{i\beta\lambda^2 t}, \ldots, \\ \quad s_{1,n+1} = s_{1,n+1}(0,\lambda)\mathrm{e}^{i\beta\lambda^2 t}, \\ s_{21} = s_{21}(0,\lambda)\mathrm{e}^{-i\beta\lambda^2 t}, \quad s_{31} = s_{31}(0,\lambda)\mathrm{e}^{-i\beta\lambda^2 t}, \ldots, \\ \quad s_{n+1,1} = s_{n+1,1}(0,\lambda)\mathrm{e}^{-i\beta\lambda^2 t}, \end{cases}$$

and all other scattering coefficients are independent of the time variable t.

3.4. *Gelfand–Levitan–Marchenko-type equations*

To get Gelfand–Levitan–Marchenko-type integral equations to determine the generalized matrix Jost solutions, we transform the associated Riemann–Hilbert problem (3.24) into

$$\begin{cases} G^+ - G^- = G^- v, \quad v = G_0 - I_{n+1}, \text{ on } \mathbb{R}, \\ G^{\pm} \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty, \end{cases} \tag{3.32}$$

where the jump matrix G_0 is given by (3.25) and (3.26).

Let $G(\lambda) = G^{\pm}(\lambda)$ if $\lambda \in \mathbb{C}^{\pm}$. Suppose that G has simple poles off $\mathbb{R}$: $\{\mu_j\}_{j=1}^R$, where R is an arbitrary integer. Set

$$\tilde{G}^{\pm}(\lambda) = G^{\pm}(\lambda) - \sum_{j=1}^{R} \frac{G_j}{\lambda - \mu_j}, \quad \lambda \in \bar{\mathbb{C}}^{\pm}; \quad \tilde{G}(\lambda) = \tilde{G}^{\pm}(\lambda), \quad \lambda \in \mathbb{C}^{\pm}, \tag{3.33}$$

where G_j is the residue of G at $\lambda = \mu_j$, i.e.,

$$G_j = \mathrm{res}(G(\lambda), \lambda_j) = \lim_{\lambda \to \mu_j} (\lambda - \mu_j) G(\lambda). \tag{3.34}$$

Then, we have

$$\begin{cases} \tilde{G}^+ - \tilde{G}^- = G^+ - G^- = G^- v, \text{ on } \mathbb{R}, \\ \tilde{G}^{\pm} \to I_{n+1} \text{ as } \lambda \in \bar{\mathbb{C}}^{\pm} \to \infty. \end{cases} \tag{3.35}$$

By the Sokhotski–Plemelj formula,[24] we obtain the solution

$$\tilde{G}(\lambda) = I_{n+1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \lambda} \, d\xi. \tag{3.36}$$

Now, taking the limit as $\lambda \to \mu_l$ yields

$$\text{LHS} = \lim_{\lambda \to \lambda_l} \tilde{G} = F_l - \sum_{j \neq l}^{R} \frac{G_j}{\mu_l - \mu_j},$$

$$\text{RHS} = I_{n+1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \mu_l} \, d\xi,$$

where

$$F_l = \lim_{\lambda \to \mu_l} \frac{(\lambda - \mu_l)G(\lambda) - G_l}{\lambda - \mu_l}, \quad 1 \leq l \leq R, \tag{3.37}$$

and accordingly, the required Gelfand–Levitan–Marchenko-type integral equations are given by

$$I_{n+1} - F_l + \sum_{j \neq l}^{R} \frac{G_j}{\mu_l - \mu_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(G^- v)(\xi)}{\xi - \mu_l} \, d\xi = 0, \quad 1 \leq l \leq R. \tag{3.38}$$

These equations are used to determine solutions to the associated Riemann–Hilbert problems and thus the generalized matrix Jost solutions. The existence and uniqueness of solutions are yet to be investigated. In the case of soliton solutions, a formulation of solutions will be presented for nonlocal integrable equations in the following section.

3.5. *Recovery of the potential*

To recover the potential matrix P from the generalized matrix Jost solutions, we make an asymptotic expansion

$$G^+(x, t, \lambda) = I_{n+1} + \frac{1}{\lambda} G_1^+(x, t) + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \infty. \tag{3.39}$$

Plugging this asymptotic expansion into the matrix spectral problem (3.1) and comparing $\mathrm{O}(1)$ terms yields

$$P = \lim_{\lambda \to \infty} \lambda [G^+(\lambda), \Lambda] = -[\Lambda, G_1^+]. \tag{3.40}$$

This equivalently presents the potential matrix:

$$P = \begin{bmatrix} 0 & -\alpha(G_1^+)_{12} & -\alpha(G_1^+)_{13} & \cdots & -\alpha(G_1^+)_{1,n+1} \\ \alpha(G_1^+)_{21} & 0 & 0 & \cdots & 0 \\ \alpha(G_1^+)_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha(G_1^+)_{n+1,1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.41}$$

where $G_1^+ = ((G_1^+)_{jl})_{(n+1)\times(n+1)}$. Namely, the $2n$ potentials p_j and q_j, $1 \leq j \leq n$, of the standard multicomponent NLS equations (2.17) are determined by

$$p_j = -\alpha(G_1^+)_{1,j+1}, \quad q_j = \alpha(G_1^+)_{j+1,1}, \quad 1 \leq j \leq n. \tag{3.42}$$

When the nonlocal reduction requirement (2.19) is satisfied, the reduced potentials p_j, $1 \leq j \leq n$, solve the nonlocal reverse-space multicomponent NLS equations (2.29).

This completes the inverse scattering procedure from the scattering matrix $S(\lambda)$, through the jump matrix $G_0(\lambda)$ and the solution $\{G^+(\lambda), G^-(\lambda)\}$ to the associated Riemann–Hilbert problems, to the potential matrix P, which solves the nonlocal reverse-space multicomponent NLS equations (2.29).

4. Soliton Solutions

4.1. *Nonreduced local case*

Let N be another arbitrary natural number. Assume that $s_{11}(\lambda)$ has N zeros $\{\lambda_k \in \mathbb{C},\ 1 \leq k \leq N\}$, and $\hat{s}_{11}(\lambda)$ has N zeros $\{\hat{\lambda}_k \in \mathbb{C},\ 1 \leq k \leq N\}$. To construct soliton solutions, we also assume that all these zeros, λ_k and $\hat{\lambda}_k$, $1 \leq k \leq N$, are geometrically simple. Then, each of $\ker T^+(\lambda_k)$, $1 \leq k \leq N$, contains only a single basis column vector, denoted by v_k, $1 \leq k \leq N$; and each of $\ker T^-(\hat{\lambda}_k)$, $1 \leq k \leq N$, a single basis row vector, denoted by $\hat{v}_k$, $1 \leq k \leq N$:

$$T^+(\lambda_k)v_k = 0, \quad \hat{v}_k T^-(\hat{\lambda}_k) = 0, \quad 1 \leq k \leq N. \tag{4.1}$$

To work out soliton solutions explicitly, we need to take $G_0 = I_{n+1}$ in each Riemann–Hilbert problem (3.24). This can be achieved if we assume that $s_{i1} = \hat{s}_{1i} = 0,\ 2 \leq i \leq n+1$, which means that only zero reflection coefficients are taken in the scattering problem. This kind of special Riemann–Hilbert problems with the canonical normalization conditions in (3.28) and the zero structures given in (4.1) can be solved exactly in the case of local integrable equations,[11,25] and in consequence, we can directly determine the potential matrix P. However, in the case of nonlocal integrable equations, we may not have

$$\{\lambda_k | 1 \leq k \leq N\} \cap \{\hat{\lambda}_k | 1 \leq k \leq N\} = \emptyset. \tag{4.2}$$

Without this condition, the solutions to the special Riemann–Hilbert problem with the identity jump matrix can be presented through (see, e.g., Ref. 8):

$$\begin{aligned} G^+(\lambda) &= I_{n+1} - \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \\ (G^-)^{-1}(\lambda) &= I_{n+1} + \sum_{k,l=1}^{N} \frac{v_k (M^{-1})_{kl} \hat{v}_l}{\lambda - \lambda_k}, \end{aligned} \tag{4.3}$$

where $M = (m_{kl})_{N \times N}$ is a square matrix whose entries are determined by

$$m_{kl} = \begin{cases} \dfrac{\hat{v}_k v_l}{\lambda_l - \hat{\lambda}_k} & \text{if } \lambda_l \neq \hat{\lambda}_k, \\ 0 & \text{if } \lambda_l = \hat{\lambda}_k, \end{cases} \quad 1 \leq k, \quad l \leq N, \tag{4.4}$$

and we need an orthogonal condition

$$\hat{v}_k v_l = 0 \quad \text{if } \lambda_l = \hat{\lambda}_k, \quad 1 \le k, \quad l \le N, \tag{4.5}$$

to guarantee that

$$(G^-)^{-1}(\lambda)G^+(\lambda) = I_{n+1}. \tag{4.6}$$

Note that the zeros λ_k and $\hat{\lambda}_k$ are constants, i.e., space–time independent, we can compute the spatial and temporal evolutions for the vectors, $v_k(x,t)$ and $\hat{v}_k(x,t)$, $1 \le k \le N$, in the kernels. For instance, let us evaluate the x-derivative of both sides of the first set of equations in (4.1). By using (3.1) first and then again the first set of equations in (4.1), we can obtain

$$P^+(x,\lambda_k)\left(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\right) = 0, \quad 1 \le k \le N. \tag{4.7}$$

This tells that for each $1 \le k \le N$, $\frac{dv_k}{dx} - i\lambda_k \Lambda v_k$ is in the kernel of $P^+(x,\lambda_k)$ and so a constant multiple of v_k. Without loss of generality, we take

$$\frac{dv_k}{dx} = i\lambda_k \Lambda v_k, \quad 1 \le k \le N. \tag{4.8}$$

The time dependence of v_k,

$$\frac{dv_k}{dt} = i\lambda_k^2 \Omega v_k, \quad 1 \le k \le N, \tag{4.9}$$

can be obtained similarly through an application of the t-part of the matrix spectral problem (3.2). As a result, we obtain

$$v_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k, \quad 1 \le k \le N, \tag{4.10}$$

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t}, \quad 1 \le k \le N, \tag{4.11}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively, but need to satisfy

$$\hat{w}_k w_l = 0 \quad \text{if } \lambda_l = \hat{\lambda}_k, \quad 1 \le k, \quad l \le N, \tag{4.12}$$

which is a consequence of the orthogonal condition (4.5).

Finally, from the solutions in (4.3), we get

$$G_1^+ = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l, \tag{4.13}$$

and further, the presentations in (3.42) lead to the following N-soliton solution to the standard multicomponent NLS equations (2.17):

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl} \hat{v}_{l,j+1}, \quad q_j = -\alpha \sum_{k,l=1}^{N} v_{k,j+1}(M^{-1})_{kl} \hat{v}_{l,1}, \quad 1 \le j \le n. \tag{4.14}$$

4.2. Reduced nonlocal case

In order to compute N-soliton solutions to the nonlocal reverse-space multicomponent NLS equations (2.29), we need to check an involution property for G_1^+ determined in (4.13):

$$(G_1^+(-x,t))^\dagger = CG_1^+(x,t)C^{-1}. \tag{4.15}$$

This equivalently tells that the potential matrix P determined through (3.41) satisfies the reduction requirement (2.19). The N-soliton solution to the standard multicomponent NLS equations (2.17) is then reduced to the N-soliton solution:

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,j+1}, \quad 1 \le j \le n, \tag{4.16}$$

for the nonlocal reverse-space multicomponent NLS equations (2.29).

Let us now state how to realize the involution property (4.15). We first take N distinct zeros of $\det T^+(\lambda)$ (or eigenvalues of the spectral problems under the zero potential): $\lambda_k \in \mathbb{C},\ 1 \le k \le N$, and define

$$\hat{\lambda}_k = \begin{cases} -\lambda_k^* & \text{if } \lambda_k \notin i\mathbb{R},\ 1 \le k \le N; \\ \text{any value } \in i\mathbb{R} & \text{if } \lambda_k \in i\mathbb{R},\ 1 \le k \le N; \end{cases} \tag{4.17}$$

which are zeros of $\det T^-(\lambda)$. Then, the $\ker T^+(\lambda_k)$, $1 \le k \le N$, are determined by

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k\Lambda x + i\lambda_k^2\Omega t}w_k, \quad 1 \le k \le N, \tag{4.18}$$

respectively, where $w_k,\ 1 \le k \le N$, are arbitrary column vectors. These column vectors in (4.18) are eigenfunctions of the spectral problems under the zero potential associated with $\lambda_k,\ 1 \le k \le N$. Further, according to the previous analysis in Sec. 3.1, the $\ker T^-(\lambda_k)$, $1 \le k \le N$, are spanned by

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(-x,t,\lambda_k)C = w_k^\dagger \mathrm{e}^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^2\Omega t}C, \quad 1 \le k \le N, \tag{4.19}$$

respectively. These row vectors are eigenfunctions of the adjoint spectral problems under the zero potential associated with $\hat{\lambda}_k,\ 1 \le k \le N$. To satisfy the orthogonal property (4.12), we require the following condition:

$$w_k^\dagger C w_l = 0 \quad \text{if } \lambda_l = \hat{\lambda}_k, \quad 1 \le k, \quad l \le N. \tag{4.20}$$

It is interesting to note that the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when $\lambda_k \in i\mathbb{R}$ and $\hat{\lambda}_k = -\lambda_k^*$.

Now, it is direct to see that if the solutions to the specific Riemann–Hilbert problems, determined by (4.3) and (4.4), satisfy the property (3.29), then G_1^+ is in agreement with the requirement (4.15) for each nonlocal reduction in (2.18), and as a consequence, the formula (4.16), together with (4.3), (4.4), (4.18) and (4.19), presents the required N-soliton solutions to the nonlocal reverse-space multicomponent NLS equations (2.29).

When $N = n = 1$, we choose $\lambda_1 = i\eta_1,\ \hat{\lambda}_1 = -i\eta_1,\ \eta_1 \in \mathbb{R}$, and denote $w_1 = (w_{1,1}, w_{1,2})^T$. Then we can obtain the following one-soliton solution to the nonlocal reverse-space scalar NLS equations in (2.30):

$$p(x,t) = \frac{2\eta_1 i w_{1,1} w_{1,2}^*}{\varepsilon |w_{1,1}|^2 \mathrm{e}^{-\eta_1 x + i\eta_1^2 t} + |w_{1,2}|^2 \mathrm{e}^{\eta_1 x + i\eta_1^2 t}}, \tag{4.21}$$

where $\varepsilon = \pm 1$, η_1 is an arbitrary real number, and $w_{1,1}$ and $w_{1,2}$ are arbitrary complex numbers but satisfy $\sigma |w_{1,1}|^2 + |w_{1,2}|^2 = 0$, which comes from the involution property (4.15). This solution may have a singularity at a certain point in space and the case of $\varepsilon = 1$ and $\sigma = -1$ can present the breather one-soliton in Ref. 2.

5. Concluding Remarks

This paper aims to present and analyze nonlocal reverse-space integrable multicomponent NLS equations and their inverse scattering transforms. The basic theory is based on Riemann–Hilbert problems associated with higher-order matrix spectral problems. The associated Riemann–Hilbert problems were transformed into Gelfand–Levitan–Marchenko-type integral equations through the the Sokhotski–Plemelj formula, and soliton solutions of the nonlocal reverse-space multicomponent NLS equations were presented in the zero reflection coefficient situation.

The Riemann–Hilbert approach is very effective in generating soliton solutions (see also, e.g., Refs. 12–14 and 26). It has been recently generalized to solve initial-boundary value problems of integrable equations on the half-line and the finite interval.[27,28] Many other approaches to soliton solutions are available in the theory of integrable equations, among which are the Hirota direct method,[29] the generalized bilinear technique,[30] the Wronskian technique[31,32] and the Darboux transformation.[33,34] It would be important to determine connections between those distinct approaches.

We also remark that it would be definitely interesting to construct different kinds of exact solutions to integrable equations, such as position and complexion solutions,[35,36] lump and interaction solutions,[37–40] Rossby wave solutions,[41] solutionless solutions[42–44] and algebro-geometric solutions,[45,46] through the Riemann–Hilbert technique. It is our hope that there will be a clear understanding about those exact solutions from a perspective of the Riemann–Hilbert technique.

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