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COUPLING INTEGRABLE COUPLINGS

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Integrable couplings are presented by coupling given integrable couplings. It is shown that such coupled integrable couplings can possess zero curvature representations and recursion operators, which yield infinitely many commuting symmetries. The presented zero curvature equations are associated with Lie algebras, each of which has two sub-Lie algebras in form of semi-direct sums of Lie algebras.

Keywords: Integrable coupling; zero curvature equation; symmetry; recursion operator.

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1. Introduction

Integrable couplings are coupled systems of integrable equations which contain given integrable equations as their sub-systems.[1,2] There are much richer mathematical structures behind integrable couplings than scalar integrable equations.[1–8] Moreover, the study of integrable couplings generalizes the symmetry problem and provides clues towards complete classification of integrable equations.[1,2,9]

It is also shown that zero curvature representations on semi-direct sums of Lie algebras result in integrable couplings,[10,11] and a kind of variational identity associated with general matrix spectral problems has been established to present Hamiltonian structures of the resulting integrable couplings.[12,13] Based on special semi-direct sums of Lie algebras, Lax pairs of block form and with several spectral parameters bring diverse interesting integrable couplings.[2,14,15]

Let us consider an integrable evolution equation

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots)\,, \tag{1}$$

where u is a column vector of dependent variables. Assume that it has a zero curvature representation[16]:

$$U_t - V_x + [U, V] = 0\,, \tag{2}$$

where the Lax pair, U and V, belongs to a matrix loop algebra, let us say, g. We recall that the Gateaux derivative of a function or an operator P along a direction X is given by

$$P'(u)[X] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} P(u + \varepsilon X)\,. \tag{3}$$

Any vector field $S = S(x, t, u)$ is a symmetry of Eq. (1), if it satisfies

$$\frac{\partial S}{\partial t} = K'(u)[S] - S'(u)[K]\,. \tag{4}$$

If S is time-independent, the above condition (4) can be reduced to a commutativity condition between K and S:

$$[K, S] = K'(u)[S] - S'(u)[K] = 0\,. \tag{5}$$

An operator $\Phi = \Phi(x, t, u)$ mapping vector fields to vector fields is called a recursion operator of Eq. (1),[17] if it satisfies:

$$\frac{\partial \Phi}{\partial t} X + \Phi'(u)[K]X - K'(u)[\Phi X] + \Phi K'(u)[X] = 0 \tag{6}$$

for any vector field $X = X(x, t, u)$. A recursion operator maps symmetries to symmetries.

Given two integrable couplings of the integrable equation (1) (see Refs. 1 and 2 for definition):

$$\bar{u}_{1,t} = \bar{K}_1(\bar{u}_1) = \begin{bmatrix} K(u) \\ S(u, v) \end{bmatrix}, \qquad \bar{u}_1 = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{7}$$

and

$$\bar{u}_{2,t} = \bar{K}_2(\bar{u}_2) = \begin{bmatrix} K(u) \\ T(u, w) \end{bmatrix}, \qquad \bar{u}_2 = \begin{bmatrix} u \\ w \end{bmatrix}, \tag{8}$$

we can form a new bigger system

$$\hat{u}_t = \hat{K}(\hat{u}) = \begin{bmatrix} K(u) \\ S(u, v) \\ T(u, w) \end{bmatrix}, \qquad \hat{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{9}$$

Obviously, this is a degenerate system, since the second and third dependent variables are separate. A possible choice of S and T, which leads to integrable couplings, is the first-order perturbation.[1–3,9] Naturally, we would like to ask here whether the above coupled system of two given integrable couplings is still integrable or not.

This paper is organized as follows. In Secs. 2–4, zero curvature representations, symmetry algebras and recursion operators are analyzed and presented for the coupled system of two given integrable couplings. In Sec. 5, a few concluding remarks are given, along with a question on Hamiltonian structures of the coupled system (9).

2. Zero Curvature Representations

Zero curvature representations are one of the keys to study nonlinear integrable equations by Darboux transformation, Bäcklund transformation, the Hamiltonian method, etc.[18–20] Let us assume that two given integrable couplings (7) and (8) have the following kind of Lax pairs for their zero curvature representations:

$$\bar{U}_i(\bar{u}_i) = \begin{bmatrix} U(u) & U_i(\bar{u}_i) \\ 0 & U(u) \end{bmatrix}, \qquad \bar{V}_i(\bar{u}_i) = \begin{bmatrix} V(u) & V_i(\bar{u}_i) \\ 0 & V(u) \end{bmatrix}, \; i = 1, 2, \tag{10}$$

respectively (see Ref. 10 for the continuous case and Ref. 11 for the discrete case), where $\bar{u}_1 = (u^T, v^T)^T$ and $\bar{u}_2 = (u^T, w^T)^T$. The enlarged zero curvature equations

$$\bar{U}_{i,t} - \bar{V}_{i,x} + [\bar{U}_i, \bar{V}_i] = 0, \quad i = 1, 2, \tag{11}$$

equivalently yield

$$\begin{cases} U_t - V_x + [U, V] = 0 \\ U_{i,t} - V_{i,x} + [U, V_i] + [U_i, V] = 0, \quad i = 1, 2, \end{cases} \tag{12}$$

which exactly present the integrable couplings (7) and (8), respectively.

Let us now form a matrix Lie algebra $\hat{g}$ consisting of square matrices of the following block form:

$$\hat{P} = \begin{bmatrix} P & 0 & P_1 \\ 0 & P & P_2 \\ 0 & 0 & P \end{bmatrix}, \tag{13}$$

where P, P_1 and P_2 are the same size square sub-matrices as U and V. This Lie algebra $\hat{g}$ has two sub-Lie algebras

$$\bar{g}_1 = \{\hat{P} \,|\, P_2 = 0\}, \qquad \bar{g}_2 = \{\hat{P} \,|\, P_1 = 0\}. \tag{14}$$

They can be written as semi-direct sums of sub-Lie algebras:

$$\bar{g}_1 = \bar{g}_1|_{P_1=0} \Subset \bar{g}_1|_{P=0}, \qquad \bar{g}_2 = \bar{g}_2|_{P_2=0} \Subset \bar{g}_2|_{P=0},$$

and thus, the Lie algebra $\hat{g}$ is non-semi-simple. The two given integrable couplings (7) and (8) are associated with those two sub-Lie algebras $\bar{g}_i$, $i = 1, 2$, respectively.

Theorem 1. *Assume that two integrable couplings* (7) *and* (8) *have the zero curvature representations* (11) *with Lax pairs* $\bar{U}_i$ *and* $\bar{V}_i$, $i = 1, 2$, *being defined by Eq.* (10). *Then the coupled system of two integrable couplings* (9), *has an enlarged zero curvature representation*

$$\hat{U}_t - \hat{V}_x + [\hat{U}, \hat{V}] = 0, \tag{15}$$

with the Lax pair being defined by

$$\hat{U}(\hat{u}) = \begin{bmatrix} U(u) & 0 & U_1(u,v) \\ 0 & U(u) & U_2(u,w) \\ 0 & 0 & U(u) \end{bmatrix} \in \hat{g}\,, \quad \hat{V}(\hat{u}) = \begin{bmatrix} V(u) & 0 & V_1(u,v) \\ 0 & V(u) & V_2(u,w) \\ 0 & 0 & V(u) \end{bmatrix} \in \hat{g}\,, \tag{16}$$

where $\hat{u} = (u^T, v^T, w^T)^T$.

Proof. Obviously, the enlarged zero curvature equation (15) equivalently engenders a coupling system:

$$\begin{cases} U_t - V_x + [U,V] = 0\,, \\ U_{1,t} - V_{1,x} + [U,V_1] + [U_1,V] = 0\,, \\ U_{2,t} - V_{2,x} + [U,V_2] + [U_2,V] = 0\,, \end{cases} \tag{17}$$

on the basis of the special form of Eq. (16). They imply exactly the coupled system of the zero curvature representations (11) of two integrable couplings (7) and (8). Therefore, the coupled system of two integrable couplings (9) has a zero curvature representation (15) with a Lax pair given by Eq. (16). This completes the proof. □

Example 1. The AKNS system of nonlinear Schrödinger equations[21] has the following two integrable couplings[12]:

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q,\ q_t = \frac{1}{2}q_{xx} - pq^2, \\ v_{2,t} = -\frac{1}{2}(p+v_2)_{xx} + p(pq + v_3 p + v_2 q) + v_2 pq, \\ v_{3,t} = \frac{1}{2}(p+v_3)_{xx} - (pq + v_3 p + v_2 q)q - v_3 pq, \end{cases} \tag{18}$$

and

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q,\ q_t = \frac{1}{2}q_{xx} - pq^2, \\ w_{1,t} = \frac{1}{4}(pq_{xx} - p_{xx}q), \\ w_{2,t} = -\frac{1}{2}(p+w_2)_{xx} - (w_1 p)_x + p(pq + w_3 p + w_2 q) \\ \qquad - w_1 p_x + w_2 pq - \frac{1}{2}p(pq_x - p_x q), \\ w_{3,t} = \frac{1}{2}(p+w_3)_{xx} - (w_1 q)_x - (pq + w_3 p + w_2 q)q \\ \qquad - w_3 pq - w_1 q_x + \frac{1}{2}(pq_x - p_x q)q\,. \end{cases} \tag{19}$$

Let $u = (p, q)^T$, $\bar{u}_1 = (p, q, v_2, v_3)^T$ and $\bar{u}_2 = (p, q, w_1, w_2, w_3)^T$. The above two integrable couplings have their Lax pairs

$$\bar{U}_i(\bar{u}_i) = \begin{bmatrix} U(u) & U_i(\bar{u}_i) \\ 0 & U(u) \end{bmatrix}, \qquad \bar{V}_i(\bar{u}_i) = \begin{bmatrix} V(u) & V_i(\bar{u}_i) \\ 0 & V(u) \end{bmatrix}, \quad i = 1, 2, \tag{20}$$

where

$$U(u) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix}, \qquad U_1(\bar{u}_1) = \begin{bmatrix} 0 & v_2 \\ v_3 & 0 \end{bmatrix}, \qquad U_2(\bar{u}_2) = \begin{bmatrix} -w_1 & w_2 \\ w_3 & w_1 \end{bmatrix},$$

and

$$V(u) = \begin{bmatrix} -\lambda^2 + \frac{1}{2}pq & \lambda p - \frac{1}{2}p_x \\ \lambda q + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq \end{bmatrix},$$

$$V_1(\bar{u}_1) = \begin{bmatrix} -\lambda^2 + \frac{1}{2}(pq + v_3 p + v_2 q) & \lambda(p + v_2) - \frac{1}{2}(p + v_2)_x - \frac{1}{2}p \\ \lambda(q + v_3) + \frac{1}{2}(q + v_3)_x - \frac{1}{2}q & \lambda^2 - \frac{1}{2}(pq + v_3 p + v_2 q) \end{bmatrix},$$

$$V_2(\bar{u}_2) = \begin{bmatrix} -\lambda^2 + \frac{1}{2}(pq + w_3 p + w_2 q) - a_3 & \lambda(p + w_2) - \frac{1}{2}(p + w_2)_x - w_1 p \\ \lambda(q + w_3) + \frac{1}{2}(q + w_3)_x - w_1 q & \lambda^2 - \frac{1}{2}(pq + w_3 p + w_2 q) + a_3 \end{bmatrix}$$

with $a_3 = \frac{1}{4}(pq_x - p_x q)$. Now, the coupled system of two integrable couplings (18) and (19) reads

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q, \ q_t = \frac{1}{2}q_{xx} - pq^2, \\ v_{2,t} = -\frac{1}{2}(p + v_2)_{xx} - p_x + p(pq + v_3 p + v_2 q) + v_2 pq, \\ v_{3,t} = \frac{1}{2}(p + v_3)_{xx} - q_x - (pq + v_3 p + v_2 q)q - v_3 pq, \\ w_{1,t} = \frac{1}{4}(pq_{xx} - p_{xx}q), \\ w_{2,t} = -\frac{1}{2}(p + w_2)_{xx} - (w_1 p)_x + p(pq + w_3 p + w_2 q) \\ \qquad - w_1 p_x + w_2 pq - \frac{1}{2}p(pq_x - p_x q), \\ w_{3,t} = \frac{1}{2}(p + w_3)_{xx} - (w_1 q)_x - (pq + w_3 p + w_2 q)q \\ \qquad - w_3 pq - w_1 q_x + \frac{1}{2}(pq_x - p_x q)q, \end{cases} \tag{21}$$

and by Theorem 1, it has a Lax pair

$$\hat{U}(\hat{u}) = \begin{bmatrix} U(u) & 0 & U_1(\bar{u}_1) \\ 0 & U(u) & U_2(\bar{u}_2) \\ 0 & 0 & U(u) \end{bmatrix}, \qquad \hat{V}(\hat{u}) = \begin{bmatrix} V(u) & 0 & V_1(\bar{u}_1) \\ 0 & V(u) & V_2(\bar{u}_2) \\ 0 & 0 & V(u) \end{bmatrix}, \tag{22}$$

where $\hat{u} = (p, q, v_2, v_3, w_1, w_2, w_3)^T$. ◊

Integrable couplings can have another kind of Lax pair for their zero curvature representations (see Refs. 22 and 23). Let us now assume that two given integrable couplings (7) and (8) have the following kind of Lax pair for their zero curvature representations:

$$\bar{U}_i(\bar{u}_i) = \begin{bmatrix} U(u) & U_i(\bar{u}_i) \\ 0 & 0 \end{bmatrix}, \qquad \bar{V}_i(\bar{u}_i) = \begin{bmatrix} V(u) & V_i(\bar{u}_i) \\ 0 & 0 \end{bmatrix}, \quad i = 1, 2, \tag{23}$$

respectively. Two additional matrices U_1 and U_2 (or V_1 and V_2) may not have the same size. If so, we add zero columns to the left of a smaller one to make them all the same size. This modification does not affect our discussion, and so, without loss of generality, we can assume that U_1 and U_2 have the same size from now on.

The enlarged zero curvature equations

$$\bar{U}_{i,t} - \bar{V}_{i,x} + [\bar{U}_i, \bar{V}_i] = 0, \quad i = 1, 2, \tag{24}$$

equivalently give rise to

$$\begin{cases} U_t - V_x + [U, V] = 0\,, \\ U_{i,t} - V_{i,x} + UV_i - VU_i = 0, \quad i = 1, 2\,, \end{cases} \tag{25}$$

which exactly present the integrable couplings (7) and (8), respectively.

We can similarly form a matrix Lie algebra $\hat{g}$ consisting of square matrices of the following block form:

$$\hat{Q} = \begin{bmatrix} Q & 0 & Q_1 \\ 0 & Q & Q_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

where Q is the same size square sub-matrix as U and V, and Q_1 and Q_2 are the same size square sub-matrices as U_1 and U_2. This Lie algebra $\hat{g}$ also has two sub-Lie algebras

$$\bar{g}_1 = \{\hat{Q} \mid Q_2 = 0\}, \qquad \bar{g}_2 = \{\hat{Q} \mid Q_1 = 0\} \tag{27}$$

of semi-direct sum type, and so, $\hat{g}$ is non-semi-simple. Actually, the two given integrable couplings (7) and (8) are generated from those two sub-Lie algebras $\bar{g}_i,\ i = 1, 2$, respectively.

A similar computation can show the following consequence on another zero curvature representation for the coupled system of two integrable couplings (9).

Theorem 2. *Assume that two integrable couplings* (7) *and* (8) *have the zero curvature representations* (24) *with Lax pairs* $\bar{U}_i$ *and* $\bar{V}_i$, $i = 1,2$, *being defined by Eq.* (23). *Then the coupled system of two integrable couplings* (9) *has another enlarged zero curvature representation*

$$\hat{U}_t - \hat{V}_x + [\hat{U}, \hat{V}] = 0, \tag{28}$$

with the Lax pair being defined by

$$\hat{U}(\hat{u}) = \begin{bmatrix} U(u) & 0 & U_1(u,v) \\ 0 & U(u) & U_2(u,w) \\ 0 & 0 & 0 \end{bmatrix} \in \hat{g}, \quad \hat{V}(\hat{u}) = \begin{bmatrix} V(u) & 0 & V_1(u,v) \\ 0 & V(u) & V_2(u,w) \\ 0 & 0 & 0 \end{bmatrix} \in \hat{g}\,, \tag{29}$$

where $\hat{u} = (u^T, v^T, w^T)^T$.

Example 2. The AKNS system of nonlinear Schrödinger equations has the following integrable coupling[22]:

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q, \ q_t = \frac{1}{2}q_{xx} - pq^2, \\ r_t = -r_{xx} - ps_x + \frac{1}{2}pqr - \frac{1}{2}p_x s, \\ s_t = s_{xx} + qr_x + \frac{1}{2}q_x r - \frac{1}{2}pqs. \end{cases} \tag{30}$$

It has a Lax pair

$$\bar{U} = \begin{bmatrix} -\lambda & p & r \\ q & \lambda & s \\ 0 & 0 & 0 \end{bmatrix}, \qquad \bar{V} = \begin{bmatrix} -\lambda^2 + \frac{1}{2}pq & \lambda p - \frac{1}{2}p_x & \lambda r - r_x \\ \lambda q + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq & \lambda s + s_x \\ 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

Therefore, we can form a coupled system of two such integrable couplings:

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2 q, \ q_t = \frac{1}{2}q_{xx} - pq^2\,, \\ r_t = -r_{xx} - ps_x + \frac{1}{2}pqr - \frac{1}{2}p_x s\,, \\ s_t = s_{xx} + qr_x + \frac{1}{2}q_x r - \frac{1}{2}pqs\,, \\ v_t = -v_{xx} - pw_x + \frac{1}{2}pqv - \frac{1}{2}p_x w\,, \\ w_t = w_{xx} + qv_x + \frac{1}{2}q_x v - \frac{1}{2}pqw\,, \end{cases} \tag{32}$$

and by Theorem 2, this coupled system has a Lax pair

$$\hat{U}(\hat{u}) = \begin{bmatrix} -\lambda & p & 0 & 0 & r \\ q & \lambda & 0 & 0 & s \\ 0 & 0 & -\lambda & p & v \\ 0 & 0 & q & \lambda & w \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

$$\hat{V}(\hat{u}) = \begin{bmatrix} -\lambda^2 + \frac{1}{2}pq & \lambda p - \frac{1}{2}p_x & 0 & 0 & \lambda r - r_x \\ \lambda q + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq & 0 & 0 & \lambda s + s_x \\ 0 & 0 & -\lambda^2 + \frac{1}{2}pq & \lambda p - \frac{1}{2}p_x & \lambda v - v_x \\ 0 & 0 & \lambda q + \frac{1}{2}q_x & \lambda^2 - \frac{1}{2}pq & \lambda w + w_x \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{34}$$

where $\hat{u} = (p, q, r, s, v, w)^T$. ◊

3. Symmetry Algebra

When two given integrable couplings have symmetry algebras, the following theorem guarantees that there will be a symmetry algebra for the coupled system of two integrable couplings as well.

Theorem 3. *Let* $\bar{u}_1 = (u^T, v^T)^T$ *and* $\bar{u}_2 = (u^T, w^T)^T$ *be defined as in Eqs.* (7) *and* (8). *If two integrable couplings* (7) *and* (8) *have their Abelian symmetry algebras*

$$\bar{A}_i = \{(X^T(u), Y_i^T(\bar{u}_i))^T \,|\, X \in A(u),\ Y_i \in B_i(\bar{u}_i)\}, \quad i = 1, 2\,, \tag{35}$$

then the coupled system of two integrable couplings (9), *has an Abelian symmetry algebra*

$$\hat{A} = \{(X^T(u), Y_1^T(\bar{u}_1), Y_2^T(\bar{u}_2))^T \,|\, X \in A(u),\ Y_1 \in B_1(\bar{u}_1),\ Y_2 \in B_2(\bar{u}_2)\}\,. \tag{36}$$

Proof. We only consider the time-independent case. The proof of the general case is completely similar.

Let $X \in A(u)$ and $Y_i \in B_i(\bar{u}_i)$, $i = 1, 2$. Then, it follows from the assumption of the theorem that $\bar{X}_1(\bar{u}_1) = (X^T(u), Y_1^T(\bar{u}_1))^T$ and $\bar{X}_2(\bar{u}_2) = (X^T(u), Y_2^T(\bar{u}_2))^T$ satisfy

$$[\bar{K}_i, \bar{X}_i] = \bar{K}_i'(\bar{u}_i)[\bar{X}_i] - \bar{X}_i'(\bar{u}_i)[\bar{K}_i] = 0,\ i = 1, 2,$$

where $\bar{K}_1(\bar{u}_1)$ and $\bar{K}_2(\bar{u}_2)$ are defined by Eqs. (7) and (8), respectively. Furthermore, for any enlarged vector field

$$\hat{X}(\hat{u}) = (X^T(u), Y_1^T(\bar{u}_1), Y_2^T(\bar{u}_2))^T \in \hat{A},$$

it follows from the above two equalities that

$$\begin{aligned}[\hat{K},\hat{X}] &= \hat{K}'(\hat{u})[\hat{X}] - \hat{X}'(\hat{u})[\hat{K}] \\ &= \begin{bmatrix} K'(u)[X(u)] - X'(u)[K(u)] \\ (S(u,v))'(u,v)[\bar{X}_1(u,v)] - (Y_1(u,v))'(u,v)[\bar{K}_1(u,v)] \\ (T(u,w))'(u,w)[\bar{X}_2(u,w)] - (Y_2(u,w))'(u,w)[\bar{K}_2(u,w)] \end{bmatrix} = 0\,,\end{aligned}$$

where $\hat{u} = (u^T, v^T, w^T)^T$ and $\hat{K}$ is defined by Eq. (9). Therefore, any enlarged vector field $\hat{X} \in \hat{A}$ is a symmetry of the coupled system (9). The same argument can show that $\hat{A}$ is Abelian, since $\bar{A}_i,\ i = 1, 2$, are Abelian. This completes the proof. □

Example 3. Let us consider the following two integrable couplings of the AKNS system of nonlinear Schrödinger equations[12]:

$$\begin{cases} p_t = -\dfrac{1}{2}p_{xx} + p^2 q,\ \ q_t = \dfrac{1}{2}q_{xx} - pq^2\,, \\ v_{2,t} = -\dfrac{1}{2}(p+v_2)_{xx} + p(pq + v_3 p + v_2 q) + v_2 pq\,, \\ v_{3,t} = \dfrac{1}{2}(p+v_3)_{xx} - (pq + v_3 p + v_2 q)q - v_3 pq\,, \end{cases} \tag{37}$$

and

$$\begin{cases} p_t = -\dfrac{1}{2}p_{xx} + p^2 q,\ \ q_t = \dfrac{1}{2}q_{xx} - pq^2\,, \\ w_{2,t} = -\dfrac{1}{2}(p+w_2)_{xx} - 2p_x + p(pq + w_3 p + w_2 q) + w_2 pq\,, \\ w_{3,t} = \dfrac{1}{2}(p+w_3)_{xx} - 2q_x - (pq + w_3 p + w_2 q)q - w_3 pq\,. \end{cases} \tag{38}$$

Let $u = (p,q)^T,\ \bar{u}_1 = (p,q,v_2,v_3)^T,\ \bar{u}_2 = (p,q,w_2,w_3)^T$ and $\partial = \frac{\partial}{\partial x}$. The integrable couplings (37) and (38) have infinitely many commuting symmetries[12]:

$$\bar{K}_{i,n}(\bar{u}_i) = \bar{\Phi}_i^n(\bar{u}_i)\bar{K}_{i,1}(\bar{u}_i) = \begin{bmatrix} \Phi(u) & 0 \\ \Phi_i(\bar{u}_i) & \Phi(u) \end{bmatrix}^n \begin{bmatrix} p_x \\ q_x \\ S_i \end{bmatrix}, \quad n \ge 0, \quad i = 1, 2\,, \tag{39}$$

where

$$\begin{cases} \Phi(u) = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \\ \Phi_1(\bar{u}_1) = \begin{bmatrix} v_2\partial^{-1}q + p\partial^{-1}v_3 & v_2\partial^{-1}p + p\partial^{-1}v_2 \\ -v_3\partial^{-1}q - q\partial^{-1}v_3 & -v_3\partial^{-1}p - q\partial^{-1}v_2 \end{bmatrix}, \\ \Phi_2(\bar{u}_2) = \begin{bmatrix} w_2\partial^{-1}q + p\partial^{-1}w_3 - 1 & w_2\partial^{-1}p + p\partial^{-1}w_2 \\ -w_3\partial^{-1}q - q\partial^{-1}w_3 & -w_3\partial^{-1}p - q\partial^{-1}w_2 - 1 \end{bmatrix}, \end{cases} \tag{40}$$

and

$$S_1 = \begin{bmatrix} (p + v_2)_x \\ (q + v_3)_x \end{bmatrix}, \qquad S_2 = \begin{bmatrix} (p + w_2)_x + 2p \\ (q + w_3)_x - 2q \end{bmatrix}. \tag{41}$$

Those two hierarchies of commuting symmetries defined by Eq. (39) span two Abelian symmetry algebras described in Theorem 3. Now, the coupled system of the two integrable couplings (37) and (38) reads

$$\begin{cases} p_t = -\frac{1}{2}p_{xx} + p^2q, \; q_t = \frac{1}{2}q_{xx} - pq^2, \\ v_{2,t} = -\frac{1}{2}(p + v_2)_{xx} + p(pq + v_3p + v_2q) + v_2pq, \\ v_{3,t} = \frac{1}{2}(p + v_3)_{xx} - (pq + v_3p + v_2q)q - v_3pq, \\ w_{2,t} = -\frac{1}{2}(p + w_2)_{xx} - 2p_x + p(pq + w_3p + w_2q) + w_2pq, \\ w_{3,t} = \frac{1}{2}(p + w_3)_{xx} - 2q_x - (pq + w_3p + w_2q)q - w_3pq, \end{cases} \tag{42}$$

and by Theorem 3, it has infinitely many commuting symmetries:

$$\hat{K}_n(\hat{u}) = \hat{\Phi}^n(\hat{u})\hat{K}_1(\hat{u}) = \begin{bmatrix} \Phi(u) & 0 & 0 \\ \Phi_1(\bar{u}_1) & \Phi(u) & 0 \\ \Phi_2(\bar{u}_2) & 0 & \Phi(u) \end{bmatrix}^n \begin{bmatrix} p_x \\ q_x \\ (p + v_2)_x \\ (q + v_3)_x \\ (p + w_2)_x + 2p \\ (q + w_3)_x - 2q \end{bmatrix}, \; n \geq 0\,, \tag{43}$$

where $\hat{u} = (p, q, v_2, v_3, w_2, w_3)^T$. The space $\text{span}\{\hat{K}_n(\hat{u}) | n \geq 0\}$ is exactly the Abelian symmetry algebra generated according to Theorem 3 for the coupled system (42). ◊

4. Recursion Operator

Let us now consider how to generate symmetries for the coupled system of two integrable couplings (9), by a recursion operator.

Theorem 4. *Let* $\bar{u}_1 = (u^T, v^T)^T$ *and* $\bar{u}_2 = (u^T, w^T)^T$ *be defined as in Eqs.* (7) *and* (8). *Assume that two integrable couplings* (7) *and* (8) *possess the following recursion operators*

$$\bar{\Phi}_1(\bar{u}_1) = \begin{bmatrix} \Phi(u) & 0 \\ \Phi_1(u,v) & \Phi(u) \end{bmatrix}, \qquad \bar{\Phi}_2(\bar{u}_2) = \begin{bmatrix} \Phi(u) & 0 \\ \Phi_2(u,w) & \Phi(u) \end{bmatrix}, \tag{44}$$

respectively. Then, the coupled system of two integrable couplings (9), *has an enlarged recursion operator:*

$$\hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(u) & 0 & 0 \\ \Phi_1(u,v) & \Phi(u) & 0 \\ \Phi_2(u,w) & 0 & \Phi(u) \end{bmatrix}, \qquad \hat{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{45}$$

Proof. Since $\bar{\Phi}_i,\ i = 1, 2$, are recursion operators of Eqs. (7) and (8), respectively, we have

$$\frac{\partial \bar{\Phi}_i}{\partial t}\bar{X}_i + \bar{\Phi}_i'(\bar{u}_i)[\bar{K}_i]\bar{X}_i - \bar{K}_i'(\bar{u}_i)[\bar{\Phi}_i\bar{X}_i] + \bar{\Phi}_i\bar{K}_i'(\bar{u}_i)[\bar{X}_i] = 0,\ i = 1, 2,$$

for any $\bar{X}_i = (X^T(\bar{u}_i), Y_i^T(\bar{u}_i))^T,\ i = 1, 2$. The two second components of these equalities lead to the second and third components of the following equality:

$$\frac{\partial \hat{\Phi}}{\partial t}\hat{X} + \hat{\Phi}'(\hat{u})[\hat{K}]\hat{X} - \hat{K}'(\hat{u})[\hat{\Phi}\hat{X}] + \hat{\Phi}\hat{K}'(\hat{u})[\hat{X}] = 0 \tag{46}$$

for any $\hat{X} = (X^T(\hat{u}), Y_1^T(\hat{u}), Y_2^T(\hat{u}))^T$. The first component of Eq. (46) is exactly the same as the first component in the previous two equalities. We view v and w as dummy variables when needed in the computation of the above equality. The proof is finished. □

Example 4. Let $u = (p, q)^T,\ \bar{u}_1 = (p, q, v_2, v_3)^T$ and $\bar{u}_2 = (p, q, w_2, w_3)^T$. The two integrable couplings (37) and (38) of the AKNS system of nonlinear Schrödinger equations have recursion operators[12]:

$$\bar{\Phi}_i(\bar{u}_i) = \begin{bmatrix} \Phi(u) & 0 \\ \Phi_i(\bar{u}_i) & \Phi(u) \end{bmatrix}, \quad i = 1, 2\,, \tag{47}$$

respectively, where Φ, Φ_1 and Φ_2 are defined by Eq. (40). Then by Theorem 4, the coupled system of two integrable couplings (42), possesses the recursion operator $\hat{\Phi}(\hat{u})$ defined in Eq. (43), i.e.

$$\hat{\Phi}(\hat{u}) = \begin{bmatrix} \Phi(u) & 0 & 0 \\ \Phi_1(\bar{u}_1) & \Phi(u) & 0 \\ \Phi_2(\bar{u}_2) & 0 & \Phi(u) \end{bmatrix}, \tag{48}$$

where $\hat{u} = (p, q, v_2, v_3, w_2, w_3)^T$. It can be verified that this enlarged recursion operator $\hat{\Phi}(\hat{u})$ is hereditary.[24] Such hereditary recursion operators possess many nice properties.[25] ♢

5. Concluding Remarks

A problem of constructing integrable couplings by coupling known integrable couplings has been presented and discussed. Zero curvature representations, Abelian symmetry algebras and recursion operators have been analyzed and generated for the coupled system of two given integrable couplings. Other integrable properties such as bilinear forms and Bäcklund transformations[26] can be also discussed for coupled systems of integrable couplings.

There are two other possible choices for Lax pairs of the coupled system of two integrable couplings (9). The first one is

$$\hat{U}(\hat{u}) = \begin{bmatrix} U(u) & U_1(u,v) & U_2(u,w) \\ 0 & U(u) & 0 \\ 0 & 0 & U(u) \end{bmatrix}, \quad \hat{V}(\hat{u}) = \begin{bmatrix} V(u) & V_1(u,v) & V_2(u,w) \\ 0 & V(u) & 0 \\ 0 & 0 & V(u) \end{bmatrix}, \tag{49}$$

if the two integrable couplings (7) and (8) have Lax pairs in Eq. (10) for their zero curvature representations. The second one is

$$\hat{U}(\hat{u}) = \begin{bmatrix} U(u) & 0 & 0 \\ 0 & U(u) & 0 \\ U_1(u,v) & U_2(u,w) & 0 \end{bmatrix}, \qquad \hat{V}(\hat{u}) = \begin{bmatrix} V(u) & 0 & 0 \\ 0 & V(u) & 0 \\ V_1(u,v) & V_2(u,w) & 0 \end{bmatrix}, \tag{50}$$

if the two integrable couplings (7) and (8) have Lax pairs

$$\bar{U}_i(\bar{u}_i) = \begin{bmatrix} U(u) & 0 \\ U_i(\bar{u}_i) & 0 \end{bmatrix}, \qquad \bar{V}_i(\bar{u}_i) = \begin{bmatrix} V(u) & 0 \\ V_i(\bar{u}_i) & 0 \end{bmatrix}, \quad i = 1, 2, \tag{51}$$

for their zero curvature representations.

Moreover, enlarged Lax pairs of direct-sum type always hold for coupled systems. For example, an enlarged spectral matrix $\hat{U}(\hat{u})$ can be taken as either of the following matrices:

$$\begin{bmatrix} U(u) & U_1(u,v) & 0 & 0 \\ 0 & U(u) & 0 & 0 \\ 0 & 0 & U(u) & U_2(u,w) \\ 0 & 0 & 0 & U(u) \end{bmatrix}, \quad \begin{bmatrix} U(u) & U_1(u,v) & 0 & 0 \\ 0 & U(u) & 0 & 0 \\ 0 & 0 & U(u) & U_2(u,w) \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

Furthermore, from the idea of using the Kronecker product,[27,28] we can construct many different zero curvature representations.

On the other hand, we can form a general coupled system of n integrable couplings:

$$u_t = K(u),\ v_{1,t} = S_1(u, v_n), \ldots, v_{n,t} = S_n(u, v_n) \tag{53}$$

for any given natural number n. Similar properties can be verified for this large coupled system. Such coupled systems of integrable couplings can also provide examples of soliton equations sharing diversities of exact solutions (see, e.g. Refs. 29–34).

The following is a remaining question for us: Is there any Hamiltonian structure for a coupled system of integrable couplings, defined by Eq. (53), if $u_t = K(u)$ is Hamiltonian? In particular, does any Hamiltonian structure exist for the following coupled system:

$$u_t = K(u), \qquad v_t = K'(u)[v], \qquad w_t = K'(u)[w], \tag{54}$$

where $K'(u)[X] = \frac{\partial}{\partial \varepsilon}\big|_{\varepsilon=0} K(u + \varepsilon X)$ as defined by Eq. (3)? This system is related to the first-order perturbation,[1,2] and so, polynomial Virasoro algebras[35] may be useful. It is hoped that our analysis on integrable couplings could help us work towards a complete classification of integrable equations.

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