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Bäcklund Transformations of Soliton Systems from Symmetry Constraints

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ABSTRACT. Binary symmetry constraints are applied to constructing Bäcklund transformations of soliton systems, both continuous and discrete. Construction of solutions to soliton systems is split into finding solutions to lower-dimensional Liouville integrable systems, which also paves a way for separation of variables and exhibits integrability by quadratures for soliton systems. Illustrative examples are provided for the KdV equation, the AKNS system of nonlinear Schrödinger equations, the Toda lattice, and the Langmuir lattice.

1. Introduction

Symmetry constraints [**7, 12**] play an important role in showing integrability by quadratures for soliton systems, both continuous and discrete. They help to generate finite-dimensional integrable systems [**4, 5, 12**] and integrable symplectic mappings [**14**], and further provide a way of constructing finite-gap solutions to soliton systems by means of Riemann-theta functions [**2**]. Based on Lax pairs, symmetries themselves can also be applied to constructing Bäcklund transformations of soliton systems [**15**]. However, there is little work showing the importance of symmetry constraints in the study of Bäcklund transformations. In this paper we focus on the construction of Bäcklund transformations by using symmetry constraints, and show in certain cases that symmetry constraints can break up soliton systems into lower-dimensional Liouville integrable systems.

Let us recall some fundamental concepts. A system of continuous equations $u_t = K(u, u_x, \dots)$ is said to have a continuous Lax pair

$$\phi_x = U(u,\lambda)\phi, \quad \phi_t = V(u, u_x, \dots; \lambda)\phi, \tag{1.1}$$

if it is equivalent to the compatibility condition $U_t - V_x + [U, V] = 0$ of (1.1) under the isospectral condition $\lambda_t = 0$, and a system of discrete equations $u_t =$

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This is the final form of the paper.

$K(u, E^{-1}u, Eu, \dots)$ (where E is the shift operator, i.e., $(Eu)(n) = u(n+1)$) is said to have a discrete Lax pair

$$E\phi = U(u,\lambda)\phi, \quad \phi_t = V(u, E^{-1}u, Eu, \dots; \lambda)\phi, \tag{1.2}$$

if it is equivalent to the compatibility condition $U_t - (EV)U + UV = 0$ of (1.2) under the isospectral condition $\lambda_t = 0$. The corresponding adjoint Lax pairs presenting the same compatibility conditions read as

$$\psi_x = -U^T(u,\lambda)\psi, \qquad \psi_t = -V^T(u, u_x, \dots; \lambda)\psi, \tag{1.3}$$

$$E^{-1}\psi = \big(E^{-1}U^T(u,\lambda)\big)\psi, \quad \psi_t = -V^T(u, E^{-1}u, Eu, \dots; \lambda)\psi, \tag{1.4}$$

where $(\cdot)^T$ denotes matrix transpose. Adjoint Lax pairs can help us determine the variational derivative of the spectral parameter with respect to the potential u (see, for example, [**6, 10**] for the continuous case).

A soliton hierarchy of continuous or discrete systems

$$u_{t_n} = K_n = \Phi^n K_0 = JG_n = J\frac{\delta \widetilde{H}_n}{\delta u}, \quad n \geq 0, \tag{1.5}$$

can be generated through the isospectral ($\lambda_{t_n} = 0$) compatibility conditions of continuous Lax pairs

$$\phi_x = U(u,\lambda)\phi, \quad \phi_{t_n} = V^{(n)}(u, u_x, \dots; \lambda)\phi, \tag{1.6}$$

or discrete Lax pairs

$$E\phi = U(u,\lambda)\phi, \quad \phi_{t_n} = V^{(n)}(u, E^{-1}u, Eu, \dots; \lambda)\phi, \tag{1.7}$$

where $V^{(n)}$ are Laurent polynomials in λ, Φ is a hereditary recursion operator to map symmetries into symmetries, and J is a Hamiltonian operator to map conserved covariants to symmetries.

In this paper, we would like to show that symmetry constraints can be applied to constructing Bäcklund transformations of soliton systems from Lax pairs. The resulting Bäcklund transformations separate each soliton system (in a hierarchy) into two lower-dimensional integrable systems. Thus, symmetry constraints are shown to be very useful in exposing integrability by quadratures for soliton systems. Illustrative examples will be given in both continuous and discrete cases.

2. Symmetry Constraints

Let us consider the space parts and the time parts of Lax pairs and adjoint Lax pairs:

$$\begin{cases} \phi_x = U(u,\lambda)\phi, \\ \psi_x = -U^T(u,\lambda)\psi, \end{cases} \qquad \begin{cases} \phi_{t_n} = V^{(n)}(u, u_x, \dots; \lambda)\phi, \\ \psi_{t_n} = -V^{(n)T}(u, u_x, \dots; \lambda)\psi, \end{cases}$$

or

$$\begin{cases} E\phi = U(u,\lambda)\phi, \\ E^{-1}\psi = \big(E^{-1}U^T(u,\lambda)\big)\psi, \end{cases} \qquad \begin{cases} \phi_{t_n} = V^{(n)}(u, E^{-1}u, Eu, \dots; \lambda)\phi, \\ \psi_{t_n} = -V^{(n)T}(u, E^{-1}u, Eu, \dots; \lambda)\psi. \end{cases}$$

By using the space parts, we can work out

$$\frac{\delta\lambda}{\delta u} = \alpha\psi^T\frac{\partial U(u,\lambda)}{\partial u}\phi, \quad \text{or} \quad \frac{\delta\lambda}{\delta u} = \beta(E\psi^T)\frac{\partial U(u,\lambda)}{\partial u}\phi, \tag{2.1}$$

where α and β are two constants. Note that the Lie homomorphism $J\delta/(\delta u)$ transforms conserved functionals to symmetries. Therefore, $J\big((\delta\lambda)/(\delta u)\big)$ is a symmetry of each system $u_{t_n} = K_n$, since λ is a conserved functional, i.e., $\lambda_{t_n} = \big(\lambda(u)\big)_{t_n} = 0$ when $u_{t_n} = K_n$.

Now for all $m_0 \geq 0$, we can make symmetry constraints

$$\text{(2.2)}\qquad \begin{aligned} K_{m_0} &= \frac{1}{\alpha} J \frac{\delta \lambda}{\delta u} = J\psi^T \frac{\partial U(u,\lambda)}{\partial u}\phi, \quad \text{or} \\ K_{m_0} &= \frac{1}{\beta} J \frac{\delta \lambda}{\delta u} = J(E\psi^T) \frac{\partial U(u,\lambda)}{\partial u}\phi. \end{aligned}$$

If we take distinct eigenvalues $\lambda_1, \ldots, \lambda_N$, and suppose that

$$\text{(2.3)}\qquad \phi_x^{(j)} = U(u,\lambda_j)\phi^{(j)}, \quad \psi_x^{(j)} = -U^T(u,\lambda_j)\psi^{(j)},$$

or

$$\text{(2.4)}\qquad E\phi^{(j)} = U(u,\lambda_j)\phi^{(j)}, \quad E^{-1}\psi^{(j)} = \big(E^{-1}U^T(u,\lambda_j)\big)\psi^{(j)},$$

where

$$\phi^{(j)} = (\phi_{1j}, \ldots, \phi_{sj})^T, \quad \psi^{(j)} = (\psi_{1j}, \ldots, \psi_{sj})^T,$$

we can make more systematical symmetry constraints

$$\text{(2.5)}\qquad K_{m_0} = \sum_{j=1}^{N} \frac{1}{\alpha_j} J \frac{\delta \lambda_j}{\delta u}, \quad \text{or} \quad K_{m_0} = \sum_{j=1}^{N} \frac{1}{\beta_j} J \frac{\delta \lambda_j}{\delta u},$$

namely,

$$\text{(2.6)}\qquad \begin{aligned} K_{m_0} &= \sum_{j=1}^{N} J\psi^{(j)T} \frac{\partial U(u,\lambda_j)}{\partial u}\phi^{(j)}, \quad \text{or} \\ K_{m_0} &= \sum_{j=1}^{N} J(E\psi^{(j)T}) \frac{\partial U(u,\lambda_j)}{\partial u}\phi^{(j)}, \end{aligned}$$

where α_j and β_j are the constants defined as in (2.1). These symmetry constraints suggest

$$\text{(2.7)}\qquad G_{m_0} = \sum_{j=1}^{N} \psi^{(j)T} \frac{\partial U(u,\lambda_j)}{\partial u}\phi^{(j)}, \quad \text{or} \quad G_{m_0} = \sum_{j=1}^{N} (E\psi^{(j)T}) \frac{\partial U(u,\lambda_j)}{\partial u}\phi^{(j)}.$$

Among those constraints between the potential, u, and the eigenfunctions and adjoint eigenfunctions, $\phi^{(j)}$ and $\psi^{(j)}$, the Bargmann constraint will be applied to constructing Bäcklund transformations between soliton systems and lower-dimensional integrable systems.

3. Bäcklund Transformations

Let us take the Bargmann constraint, i.e., the constraint (2.7) with $G_{m_0} = G_{m_0}(u)$ not involving any $\partial^i u$ $\big(\partial = \partial/(\partial x)\big)$, $i > 0$, or any $E^i u$, $i \neq 0$. Note that the discrete constraint defined by (2.7) can be rewritten as

$$\text{(3.1)}\qquad G_{m_0}(u) = \frac{\delta \widetilde{H}_{m_0}(u)}{\delta u} = \sum_{j=1}^{N} \frac{1}{\beta_j} \frac{\delta \lambda_j}{\delta u} = \sum_{j=1}^{N} \psi^{(j)T} U^{-1}(u,\lambda_j) \frac{\partial U(u,\lambda_j)}{\partial u}\phi^{(j)},$$

and therefore, in each of continuous and discrete cases, the Bargmann constraint defined by (2.7) is an algebraic equation on u, $\phi^{(j)}$, and $\psi^{(j)}$. Assume that solving the corresponding algebraic equation for u gives rise to an explicit expression of u:

$$u = f(\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(N)}; \psi^{(1)}, \psi^{(2)}, \dots, \psi^{(N)}). \tag{3.2}$$

Substituting this expression of u into Lax pairs and adjoint Lax pairs leads to two systems, called binary constrained Lax pairs. Binary constrained continuous Lax pairs read as

$$\begin{cases} \phi_x^{(j)} = U(f, \lambda_j)\phi^{(j)}, & 1 \le j \le N, \\ \psi_x^{(j)} = -U^T(f, \lambda_j)\psi^{(j)}, & 1 \le j \le N, \end{cases} \tag{3.3}$$

$$\begin{cases} \phi_{t_n}^{(j)} = V^{(n)}(f, f_x, \dots; \lambda_j)\phi^{(j)}, & 1 \le j \le N, \\ \psi_{t_n}^{(j)} = -V^{(n)T}(f, f_x, \dots; \lambda_j)\psi^{(j)}, & 1 \le j \le N, \end{cases} \tag{3.4}$$

the first of which is a system of ordinary differential equations, but the second of which is a system of partial differential equations since $V^{(n)}$ contains some derivatives of u with respect to x. Binary constrained discrete Lax pairs read as

$$\begin{cases} E\phi^{(j)} = U(f, \lambda_j)\phi^{(j)}, & 1 \le j \le N, \\ E^{-1}\psi^{(j)} = \big(E^{-1}U^T(f, \lambda_j)\big)\psi^{(j)}, & 1 \le j \le N, \end{cases} \tag{3.5}$$

$$\begin{cases} \phi_{t_n}^{(j)} = V^{(n)}(f, E^{-1}f, Ef, \dots; \lambda_j)\phi^{(j)}, & 1 \le j \le N, \\ \psi_{t_n}^{(j)} = -V^{(n)T}(f, E^{-1}f, Ef, \dots; \lambda_j)\psi^{(j)}, & 1 \le j \le N, \end{cases} \tag{3.6}$$

the first of which is a system of difference equations, but the second of which is a system of difference-differential equations since $V^{(n)}$ contains some of $E^i u$, $i \ne 0$. However, the second systems can be transformed into systems of ordinary differential equations by using the first systems. Furthermore, it can be shown by r-matrix formulation that all binary constrained Lax pairs, both continuous and discrete, are integrable in the Liouville sense [**1, 3**].

Therefore, (3.2) provides Bäcklund transformations between soliton systems and integrable binary constrained Lax pairs, and construction of solutions $u = f(\phi^{(j)}, \psi^{(j)})$ to soliton systems is split into finding solutions $\phi^{(j)}$ and $\psi^{(j)}$ to two lower-dimensional integrable systems.

4. Examples of Continuous Systems

EXAMPLE 4.1. Let us consider the KdV Equation

$$u_{t_1} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x = J\frac{\delta \widetilde{H}_1}{\delta u}, \quad J = \partial, \quad \widetilde{H}_1 = \int \left(\frac{1}{8}uu_{xx} + \frac{3}{12}u^3\right) dx, \tag{4.1}$$

which can be written as $U_t - V_x + [U, V] = 0$ with

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{4}u_x & \lambda + \frac{1}{2}u \\ \lambda^2 - \frac{1}{2}u\lambda - \frac{1}{4}u_{xx} - \frac{1}{2}u^2 & \frac{1}{4}u_x \end{pmatrix}. \tag{4.2}$$

Take the Bargmann symmetry constraint

$$K_{m_0} = 2\partial_x \sum_{j=1}^N \psi^{(j)T} \frac{\partial U(u, \lambda_j)}{\partial u}\phi^{(j)}, \quad \text{where } K_{m_0} = K_0 = u_x, \tag{4.3}$$

which implies the following equation

$$u_x = 2\partial_x \sum_{j=1}^{N} (\psi_{1j}, \psi_{2j}) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} = 2\partial_x \sum_{j=1}^{N} \phi_{1j}\psi_{2j}.$$

Integrating this equation with respect to x yields a Bäcklund transformation

$$u = f(\phi_{ij}; \psi_{ij}) = 2\sum_{j=1}^{N} \phi_{1j}\psi_{2j} = 2\langle P_1, Q_2\rangle, \tag{4.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^N$, and P_i and Q_i are defined by

$$P_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{iN})^T, \quad Q_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{iN})^T, \qquad i = 1, 2.$$

A general Bäcklund transformation $u = 2\langle P_1, Q_2\rangle + c$ with an arbitrary constant c can also result from the above symmetry constraint, but it will not generate essentially new integrable systems from Lax pair of the KdV equation and so we omit to discuss it.

Keeping (4.4) in mind, the corresponding constrained Lax pairs (3.3) and (3.4), where two matrices U and $V^{(n)} = V$ are defined by (4.2), can simultaneously be rewritten as

$$P_{ix} = -\frac{\partial H^c}{\partial Q_i}, \quad Q_{ix} = \frac{\partial H^c}{\partial P_i}, \quad H^c = -F_3, \qquad i = 1, 2, \tag{4.5}$$

$$P_{it} = -\frac{\partial H_1^c}{\partial Q_i}, \quad Q_{it} = \frac{\partial H_1^c}{\partial P_i}, \quad H_1^c = -F_4, \qquad i = 1, 2, \tag{4.6}$$

where the functions F_m are given by

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 0,$$

$$F_3 = \langle P_2, Q_1\rangle - \langle P_1, Q_2\rangle^2 + \langle AP_1, Q_2\rangle,$$

$$F_m = \sum_{i=0}^{m-4} (\bar{a}_i\bar{a}_{m-i-4} + \bar{b}_i\bar{c}_{m-i-4}) + \bar{c}_{m-3} + \bar{b}_{m-2} - \frac{1}{2}f\bar{b}_{m-3}, \quad m \geq 4,$$

$$\bar{a}_i = \langle A^iP_1, Q_1\rangle - \langle A^iP_2, Q_2\rangle, \quad \bar{b}_i = \langle A^iP_1, Q_2\rangle, \quad \bar{c}_i = \langle A^iP_2, Q_1\rangle,$$

through $F = \det V|_{u=f} = \sum_{m=0}^{\infty} F_m\lambda^{-m}$, $V = \sum_{i=0}^{\infty} V_i\lambda^{-i}$ satisfying $V_x = [U, V]$. Note that we always accept

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_N).$$

These two systems (4.5) and (4.6) are Liouville integrable [**1**], since they have involutive integrals of motion F_m, $m \geq 0$, and

$$\overline{F}_j = \phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}, \quad 1 \leq j \leq N, \tag{4.7}$$

among which $F_3, F_4, \ldots, F_{N+2}$ and $\overline{F}_1, \ldots, \overline{F}_N$ are functionally independent. All functions

$$u(x, t_1) = 2\langle P_1(x, t_1), Q_2(x, t_1)\rangle = \langle g_H^x g_{H_1}^{t_1} P_{10}, g_H^x g_{H_1}^{t_1} Q_{20}\rangle, \tag{4.8}$$

where g_H^x and $g_{H_1}^{t_1}$ are the Hamiltonian flows associated with (4.5) and (4.6) respectively, and P_{10} and Q_{20} are two arbitrary constant vectors, will determine solutions to the KdV equation (4.1).

EXAMPLE 4.2. Let us now consider the coupled nonlinear Schrödinger system:

$$u_{t_2} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_2} = \begin{pmatrix} -\frac{1}{2}q_{xx} + q^2 r \\ \frac{1}{2}r_{xx} - qr^2 \end{pmatrix} = J\frac{\delta \widetilde{H}_2}{\delta u}, \tag{4.9}$$

with the Hamiltonian operator J and the Hamiltonian functional $\widetilde{H}_2$:

$$J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \widetilde{H}_2 = \frac{1}{12}\int (qr_{xx} - q_x r_x + q_{xx} r - 3q^2 r^2)\, dx. \tag{4.10}$$

It has a Lax pair

$$U = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \quad V = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr & q\lambda - \frac{1}{2}q_x \\ r\lambda + \frac{1}{2}r_x & \lambda^2 - \frac{1}{2}qr \end{pmatrix}. \tag{4.11}$$

The Bargmann symmetry constraint (2.6) reads as

$$K_{m_0} = J\sum_{j=1}^N \psi^{(j)T}\frac{\partial U(u,\lambda_j)}{\partial u}\phi^{(j)}, \text{ where } K_{m_0} = K_0 = J\begin{pmatrix} r \\ q \end{pmatrix}. \tag{4.12}$$

This implies the following equation

$$J\begin{pmatrix} r \\ q \end{pmatrix} = J\sum_{j=1}^N \begin{pmatrix} (\psi_{1j}, \psi_{2j})\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} \\ (\psi_{1j}, \psi_{2j})\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} \end{pmatrix} = J\sum_{j=1}^N \begin{pmatrix} \phi_{2j}\psi_{1j} \\ \phi_{1j}\psi_{2j} \end{pmatrix}, \tag{4.13}$$

which equivalently engenders a Bäcklund transformation

$$u = \begin{pmatrix} q \\ r \end{pmatrix} = f(\phi_{ij}; \psi_{ij}) = \sum_{j=1}^N \begin{pmatrix} \phi_{1j}\psi_{2j} \\ \phi_{2j}\psi_{1j} \end{pmatrix} = \begin{pmatrix} \langle P_1, Q_2 \rangle \\ \langle P_2, Q_1 \rangle \end{pmatrix}. \tag{4.14}$$

Keeping (4.14) in mind, the corresponding constrained Lax pairs (3.3) and (3.4), where two matrices U and $V^{(n)} = V$ are defined by (4.11), can simultaneously be transformed into the following

$$P_{ix} = -\frac{\partial H^c}{\partial Q_i}, \quad Q_{ix} = \frac{\partial H^c}{\partial P_i}, \quad H^c = F_2 - \frac{1}{4}F_1^2, \qquad i = 1, 2, \tag{4.15}$$

$$P_{it_2} = -\frac{\partial H_2^c}{\partial Q_i}, \quad Q_{it_2} = \frac{\partial H_2^c}{\partial P_i}, \quad H_2^c = F_3 - \frac{1}{2}F_1F_2 + \frac{3}{24}F_1^3, \qquad i = 1, 2, \tag{4.16}$$

where the functions F_m are given by

$$F_1 = \langle P_2, Q_2 \rangle - \langle P_1, Q_1 \rangle,$$

$$F_m = \sum_{i=0}^{m-2}\Big[\frac{1}{4}(\langle A^i P_1, Q_1 \rangle - \langle A^i P_2, Q_2 \rangle)$$
$$(\langle A^{m-i-2}P_1, Q_1 \rangle - \langle A^{m-i-2}P_2, Q_2 \rangle) + \langle A^i P_1, Q_2 \rangle\langle A^{m-i-2}P_2, Q_1 \rangle\Big]$$
$$+ \langle A^{m-1}P_2, Q_2 \rangle - \langle A^{m-1}P_1, Q_1 \rangle, \quad m \geq 2.$$

These two systems are completely integrable in the Liouville sense [**1**], since they have involutive integrals of motion F_m, $m \geq 0$, defined above, and $\overline{F}_j$, $1 \leq j \leq N$, defined by (4.7), among which $F_1, F_2, \ldots, F_N$ and $\bar{F}_1, \ldots, \bar{F}_N$ are functionally independent. The Bäcklund transformation (4.14) determines solutions to the coupled nonlinear Schrödinger system (4.9):

$$(4.17)\qquad \begin{cases} q(x,t_2) = \langle P_1(x,t_2), Q_2(x,t_2)\rangle = \sum_{j=1}^N \phi_{1j}(x,t_2)\psi_{2j}(x,t_2), \\ r(x,t_2) = \langle P_2(x,t_2), Q_1(x,t_2)\rangle = \sum_{j=1}^N \phi_{2j}(x,t_2)\psi_{1j}(x,t_2), \end{cases}$$

if $\phi_{ij}(x,t_2)$ and $\psi_{ij}(x,t_2)$ simultaneously solve two integrable finite-dimensional Hamiltonian systems (4.15) and (4.16).

5. Examples of Discrete Systems

EXAMPLE 5.1. Let us consider the Toda lattice [**13**]:

$$(5.1)\qquad a_t(n,t) = a(n,t)\big(b(n+1,t) - b(n,t)\big), \quad b_t(n,t) = a(n,t) - a(n-1,t),$$

which associates with the discrete spectral problem

$$(5.2)\qquad E\phi = U(u,\lambda)\phi, \quad U(u,\lambda) = \begin{pmatrix} 0 & a \\ -1 & \lambda - b \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

where $u = (a,b)^T$ and λ is a spectral parameter. In order to derive a hierarchy of lattice equations associated with (5.2), we first solve the stationary discrete zero-curvature equation:

$$(5.3)\qquad (EV)U - UV = 0, \quad V = (V_{ij})_{2\times 2},$$

by assuming

$$V_{11} = aB + (b-\lambda)C, \quad V_{12} = E^{-1}aC, \quad V_{21} = -C, \quad V_{22} = E^{-1}aB,$$

where

$$B = \sum_{i\ge 0} B_{i-1}\lambda^{-i}, \quad C = \sum_{i\ge 0} C_{i-1}\lambda^{-i}.$$

The discrete zero-curvature equation (5.3) requires

$$(5.4)\qquad JG_{-1} = 0, \quad MG_{n-1} = JG_n, \qquad n \ge 0,$$

where $G_n = (B_n, C_n)^T$, and J, M are two skew-symmetric operators:

$$(5.5)\qquad J = \begin{pmatrix} 0 & a\Delta \\ -\Delta^* a & 0 \end{pmatrix}, \quad M = \begin{pmatrix} a(\Delta - \Delta^*)a & a\Delta b \\ -b\Delta^* a & a\Delta - \Delta^* a \end{pmatrix}.$$

We choose $G_{-1} = (0,1)^T$, and assume that all terms of G_n, $n \ge 0$, do not belong to $\ker J = \{G_{-1}, G_{-2}\}$ where $G_{-2} = (a^{-1}, 0)^T$, when we uniquely determine all G_n, $n \ge 0$. For instance, the second member has to be $G_0 = (1,b)^T$. This requirement also means that we just choose the key vector fields to form systems of lattice equations.

Let $\lambda_1, \ldots, \lambda_N$ be distinct eigenvalues. Then we have

$$(5.6)\qquad (E\phi_{1j}, E\phi_{2j}) = (\phi_{1j}, \phi_{2j})U(u,\lambda_j)^T, \quad 1 \le j \le N,$$

$$(5.7)\qquad (E\psi_{1j}, E\psi_{2j}) = (\psi_{1j}, \psi_{2j})U(u,\lambda_j)^{-1}, \quad 1 \le j \le N.$$

It is easy to see that

$$(5.8)\qquad M\frac{\delta\lambda_j}{\delta u} = \lambda_j J \frac{\delta\lambda_j}{\delta u},$$

where $(\delta\lambda_j)/(\delta u)$ is determined by

$$\frac{\delta\lambda_j}{\delta u} = \begin{pmatrix} \frac{\delta\lambda_j}{\delta a} \\ \frac{\delta\lambda_j}{\delta b} \end{pmatrix} = \frac{\beta_j}{a}\begin{pmatrix} (\lambda_j - b)\phi_{2j}\psi_{1j} + \phi_{2j}\psi_{2j} \\ a\phi_{2j}\psi_{1j} \end{pmatrix}, \quad \beta_j = \text{const.} \tag{5.9}$$

Now the Bargmann constraint $G_0 = \sum_{j=1}^{N}\beta_j^{-1}(\delta\lambda_j)/(\delta u)$ leads to a Bäcklund transformation

$$a = \langle AP_2, P_1\rangle + \langle P_2, Q_2\rangle - \langle P_2, Q_1\rangle^2, \quad b = \langle P_2, Q_1\rangle, \tag{5.10}$$

where $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$, $P_i = (\phi_{i1}, \ldots, \phi_{iN})^T$, $Q_i = (\psi_{i1}, \ldots, \psi_{iN})^T$, and $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{R}^N$, as defined before.

Substituting (5.10) into (5.6) and (5.7) yields a discrete Bargmann system

$$\begin{cases} EP_1 = \left(\langle AP_2, Q_1\rangle + \langle P_2, Q_2\rangle - \langle P_2, Q_1\rangle^2\right)P_2, \\ EP_2 = -P_1 - \langle P_2, Q_1\rangle P_2 + AP_2, \\ EQ_1 = \frac{Q_2 - \langle P_2, Q_1\rangle Q_1 + AQ_1}{\langle AP_2, Q_1\rangle + \langle P_2, Q_2\rangle - \langle P_2, Q_1\rangle^2}, \\ EQ_2 = -Q_1, \end{cases} \tag{5.11}$$

which determines a symplectic mapping H:

$$(EP_1, EP_2, EQ_1, EQ_2) = H(P_1, P_2, Q_1, Q_2), \tag{5.12}$$

since we have by a direct calculation

$$\sum_{j=1}^{N} d(E\phi^{(j)}) \wedge d(E\psi^{(j)}) = \sum_{j=1}^{N} d\phi^{(j)} \wedge d\psi^{(j)}.$$

The generating function $\mathcal{F}_\lambda = \det V|_{u=f}$:

$$\mathcal{F}_\lambda = -Q_\lambda(AP_1, Q_1) - Q_\lambda(P_1, Q_2) + \langle P_1, Q_1\rangle Q_\lambda(P_2, Q_1) + \begin{vmatrix} Q_\lambda(P_1, Q_1) & Q_\lambda(P_1, Q_2) \\ Q_\lambda(P_2, Q_1) & Q_\lambda(P_2, Q_2) \end{vmatrix} = \sum_{m\geq 0} F_m\lambda^{-m-1},$$

where

$$Q_\lambda(\xi, \eta) = \sum_{j=1}^{N} \frac{\xi_j\eta_j}{\lambda - \lambda_j} = \sum_{m\geq 0}\langle A^m\xi, \eta\rangle\lambda^{-m-1},$$

generates a hierarchy of invariants of (5.11):

$$F_0 = \langle AP_1, Q_1\rangle - \langle P_1, Q_2\rangle + \langle P_1, Q_1\rangle\langle P_2, Q_1\rangle,$$

$$F_m = -\langle A^{m+1}P_1, Q_1\rangle - \langle A^m P_1, Q_2\rangle + \langle P_1, Q_1\rangle\langle A^m P_2, Q_1\rangle + \sum_{i=1}^{m}\begin{vmatrix} \langle A^{i-1}P_1, Q_1\rangle & \langle A^{m-i}P_2, Q_1\rangle \\ \langle A^{i-1}P_1, Q_2\rangle & \langle A^{m-i}P_2, Q_2\rangle \end{vmatrix}, \quad m \geq 1.$$

A direct computation can show the involutivity

$$\{F_m, \overline{F}_l\} = 0, \quad m, l \geq 0,$$

where the variants $\overline{F}_j = \phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}$, $1 \leq j \leq N$, defined as before. Now we can easily see that the symplectic mapping (5.11) is Liouville integrable [**3**].

Introduce an initial-value problem

$$P_{it} = \frac{\partial F_0}{\partial Q_i}, \quad Q_{it} = -\frac{\partial F_0}{\partial P_i}, \ (P_i(t), Q_i(t))\big|_{t=0} = (P_{i0}, Q_{i0}), \qquad i = 1, 2, \tag{5.13}$$

where P_{i0} and Q_{i0}, $i = 1, 2$, are arbitrary constant vectors. Let $\big(P_i(t), Q_i(t)\big)$, $i = 1, 2$, be a solution to the initial-value problems (5.13), and further define

$$(P_1(n,t), P_2(n,t), Q_1(n,t), Q_2(n,t)) = H^n\big(P_1(t), P_2(t), Q_1(t), Q_2(t)\big). \tag{5.14}$$

Then $a(n,t)$ and $b(n,t)$ determined by the Bäcklund transformation (5.10) solves the Toda lattice (5.1).

EXAMPLE 5.2. Let us now consider the Langmuir lattice [**16**]:

$$a_t(n,t) = a(n,t)\big(a(n+1,t) - a(n-1,t)\big), \tag{5.15}$$

which associates with a reduction of the discrete spectral problem (5.2):

$$E\phi = U(a,\lambda)\phi, \quad U(a,\lambda) = \begin{pmatrix} 0 & a \\ -1 & \lambda \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{5.16}$$

Assume that its discrete zero-curvature equation

$$(EV)U - UV = 0, \quad V = (V_{ij})_{2\times 2}, \tag{5.17}$$

has a solution

$$V_{11} = aB - \lambda^2 C, \quad V_{12} = \lambda E^{-1}aC, \quad V_{21} = -\lambda C, \quad V_{22} = E^{-1}aB,$$

where

$$B = \sum_{i\geq 0} B_i\lambda^{-2i}, \quad C = \sum_{i\geq 0} C_i\lambda^{-2i}.$$

Then upon choosing $B_0 = C_0 = 1$, we can easily find that (5.17) is equivalent to

$$\begin{aligned} MB_{n-1} &= JB_n, \quad B_0 = 1, \quad ej \geq 1, \\ C_n &= (\Delta^* a - a\Delta)^{-1}\Delta^* aB_n, \quad n \geq 1, \end{aligned} \tag{5.18}$$

where two skew-symmetric operators J and M read as

$$M = a(\Delta - \Delta^*)a, \quad J = a\Delta(\Delta^* a - a\Delta)^{-1}\Delta^* a. \tag{5.19}$$

Let $\lambda_1, \ldots, \lambda_N$ be distinct eigenvalues, then we have

$$\begin{cases} (E\phi_{1j}, E\phi_{2j}) = (\phi_{1j}, \phi_{2j})U(a,\lambda_j)^T, & 1 \leq j \leq N, \\ (E\psi_{1j}, E\psi_{2j}) = (\psi_{1j}, \psi_{2j})U(a,\lambda_j)^{-1}, & 1 \leq j \leq N, \end{cases} \tag{5.20}$$

and

$$M\frac{\delta\lambda_j}{\delta a} = \lambda_j^2 J\frac{\delta\lambda_j}{\delta a}, \text{ where } \frac{\delta\lambda_j}{\delta a} = \frac{\beta_j}{a}(\lambda_j\phi_{2j}\psi_{1j} + \phi_{2j}\psi_{2j}), \quad \beta_j = \text{const}. \tag{5.21}$$

Now similarly, the Bargmann constraint $G_0 = B_0 = \sum_{j=1}^N \beta_j{}^{-1}(\delta\lambda_j)/(\delta a)$ leads to a Bäcklund transformation

$$a = \langle AP_2, Q_1\rangle + \langle P_2, Q_2\rangle. \tag{5.22}$$

Substituting (5.22) into (5.20), we obtain another discrete Bargmann system

$$\begin{cases} EP_1 &= \big(\langle AP_2, Q_1\rangle + \langle P_2, Q_2\rangle\big)P_2, \\ EP_2 &= -P_1 + AP_2, \\ EQ_1 &= \frac{Q_2 + AQ_1}{\langle AP_2, Q_1\rangle + \langle P_2, Q_2\rangle}, \\ EQ_2 &= -Q_1, \end{cases} \tag{5.23}$$

which determines a symplectic mapping H:

$$(EP_1, EP_2, EQ_1, EQ_2) = H(P_1, P_2, Q_1, Q_2). \tag{5.24}$$

This symplectic mapping is Liouville integrable [**3**], since we have involutive invariants: $\overline{F}_j = \phi_{1j}\psi_{1j} + \phi_{2j}\psi_{2j}$, $1 \le j \le N$, and F_m, $m \ge 0$, defined by

$$\begin{aligned} F_0 &= -\langle A^2P_1, Q_1\rangle - \langle AP_1, Q_2\rangle + \langle P_1, Q_1\rangle\big(\langle AP_2, Q_1\rangle + \langle P_2, Q_2\rangle\big), \\ F_m &= -\langle A^{2m+2}P_1, Q_1\rangle - \langle A^{2m+1}P_1, Q_2\rangle + \langle P_1, Q_1\rangle\langle A^{2m+1}P_2, Q_1\rangle \\ &\quad + \langle P_2, Q_2\rangle\langle A^{2m}P_1, Q_1\rangle + \sum_{i=1}^{m} \begin{vmatrix} \langle A^{2i-2}P_1, Q_1\rangle & \langle A^{2m-2i+1}P_2, Q_1\rangle \\ \langle A^{2i-1}P_1, Q_2\rangle & \langle A^{2m-2i+2}P_2, Q_2\rangle \end{vmatrix}. \end{aligned}$$

Let $\big(P_i(t), Q_i(t)\big)$, $i = 1, 2$, be a solution to an initial-value problem:

$$(5.25)\quad P_{it} = \frac{\partial F_0}{\partial Q_i}, \quad Q_{it} = -\frac{\partial F_0}{\partial P_i}, \quad \big(P_i(t), Q_i(t)\big)\big|_{t=0} = (P_{i0}, Q_{i0}), \qquad i = 1, 2,$$

where P_{i0} and Q_{i0}, $i = 1, 2$, are arbitrary, and similarly define

$$(5.26)\qquad \big(P_1(n,t), P_2(n,t), Q_1(n,t), Q_2(n,t)\big) = H^n\big(P_1(t), P_2(t), Q_1(t), Q_2(t)\big).$$

Then $a(n,t)$ determined by the Bäcklund transformation (5.22):

$$a(n,t) = \big\langle AP_2(n,t), Q_1(n,t)\big\rangle + \big\langle P_2(n,t), Q_2(n,t)\big\rangle,$$

provides a solution to the Langmuir lattice (5.15).

6. Concluding Remarks

It has been shown that solving symmetry constraints for u can give rise to Bäcklund transformations between soliton systems and lower-dimensional Liouville integrable systems, which supplements the study of binary nonlinearization of Lax pairs [**8–10, 12**]. Construction of solutions to soliton systems is split into finding solutions to the space and time parts of integrable constrained Lax pairs, which gives a way to separate variables for soliton systems and exhibits integrability by quadratures for soliton systems. Upon solving the Riemann–Jacobi inversion problems for constrained Lax pairs, the resulting Bäcklund transformations can generate finite-gap solutions to soliton systems in terms of Riemann-theta functions.

We remark that all symmetry constraints defined by (2.7) can put Lax pairs into integrable symplectic mappings and/or integrable finite-dimensional Hamiltonian systems. The corresponding constrained Lax pairs may have some specific properties, e.g., bi-Hamiltonian and quasi-bi-Hamiltonian structures. Therefore, symmetry constraints are very powerful in constructing lower-dimensional integrable systems from Lax pairs of soliton systems. Nevertheless, there exist symmetry constraints which do not force Lax pairs into integrable systems with constant coefficient symplectic forms [**11**], and the problem of integrability has not been solved for the time parts of the original constrained Lax pairs, i.e., systems of partial differential equations

$$(6.1)\qquad \begin{cases} \phi^{(j)}_{t_n} = V^{(n)}(f, f_x, \dots; \lambda_j)\phi^{(j)}, & 1 \le j \le N, \\ \psi^{(j)}_{t_n} = -V^{(n)T}(f, f_x, \dots; \lambda_j)\psi^{(j)}, & 1 \le j \le N, \end{cases}$$

and systems of difference-differential equations

$$(6.2)\qquad \begin{cases} \phi^{(j)}_{t_n} = V^{(n)}(f, Ef, E^{-1}f, \dots; \lambda_j)\phi^{(j)}, & 1 \le j \le N, \\ \psi^{(j)}_{t_n} = -V^{(n)T}(f, Ef, E^{-1}f, \dots; \lambda_j)\psi^{(j)}, & 1 \le j \le N. \end{cases}$$

We are curious to known whether they are good candidates for integrable systems.

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