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# EXPLICIT AND EXACT SOLUTIONS TO A KOLMOGOROV–PETROVSKII–PISKUNOV EQUATION

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**Abstract**—Some explicit traveling wave solutions to a Kolmogorov–Petrovskii–Piskunov equation are presented through two ansätze. By a Cole–Hopf transformation, this Kolmogorov–Petrovskii–Piskunov equation is also written as a bilinear equation and two solutions to describe non-linear interaction of traveling waves are further generated. Bäcklund transformations of the linear form and some special cases are considered. Copyright © 1996 Elsevier Science Ltd.

## 1. INTRODUCTION

The Cauchy problem of the Kolmogorov–Petrovskii–Piskunov equation [1]

$$\begin{cases} u_t - u_{xx} = f(u), & f \text{ non-linear}, \quad f(0) = 0, \\ u(x,0) = \phi(x), & x \in R^1 \end{cases}$$

has been extensively investigated both by analytic techniques [2, 3], and by probabilistic methods [4, 5], and the existence of traveling wave solutions with various velocities has also been proved. A special case is the Fisher equation:

$$u_t - u_{xx} = u(1-u),$$

which was originally proposed [6] as a model for the propagation of a favored gene. An explicit and exact solitary solution of the Fisher equation may be presented by Painlevé analysis [7]. As an example of the above Kolmogorov–Petrovskii–Piskunov equation, Kametaka [3] considered the Cauchy problem of a generalized Fisher equation

$$\begin{cases} u_t - u_{xx} = \lambda_0^2 u(1-u^n), \\ u(x,0) = \{1 + (2^{n/2}-1)e^{-(n/2)\sigma x}\}^{-2/n}, & x \in R^1, \end{cases}$$

where $\lambda_0 > 0$, $n \in N$, $\sigma > 0$, and gave an explicit solution

$$u(x,t) = \left[1 + (2^{n/2}-1)\exp\left\{-\frac{n}{2}\sigma_1(x+2\lambda_1 t)\right\}\right]^{-2/n}$$

when

$$\sigma = \sigma_1 = \lambda_1 - \sqrt{\lambda_1^2 - \lambda_0^2}, \qquad \lambda_1 = \frac{1}{2}\left\{\left(\frac{n}{2}+1\right)^{1/2} + \left(\frac{n}{2}+1\right)^{-1/2}\right\}\lambda_0.$$

Abdelkader [8] and Wang [9] extended the integer $n$ to a real number $\alpha$ satisfying $\alpha \geqslant 1$ and Wang successfully obtained a class of explicit traveling wave solutions by introducing a special non-linear transformation.

In this paper, we consider the following Kolmogorov–Petrovskii–Piskunov equation:

$$u_t - u_{xx} + \mu u + \nu u^2 + \delta u^3 = 0 \tag{1}$$

where $\mu$, $\nu$, $\delta$ are three real constants. Some special cases with $\mu + \nu + \delta = 0$ of this equation have been studied (see for instance [10–13]). Our purpose is to look for explicit and exact

Contributed by W. F. Ames.

solutions for the general case of (1). Note that a more general equation

$$u_{t'} - \alpha u_{x'x'} + \mu u + \nu u^2 + \delta u^3 = 0, \quad \alpha \neq 0$$

may be mapped into the Kolmogorov–Petrovskii–Piskunov equation (1) by a time scaling $t' \to \alpha t$ and therefore the Kolmogorov–Petrovskii–Piskunov equation (1) is no loss of generality. In Section 2, we analyze the possibilities of solutions which correspond to two Riccati equations and further give explicitly a number of exact solutions to (1). The essential point is to break the non-linear equation (1) into two smaller problems and then to solve these two smaller problems. In Section 3, we change the Kolmogorov–Petrovskii–Piskunov (1) into a bilinear equation, like the Hirota bilinear one, by making the well-known Cole–Hopf transformation, and present two explicit solutions to describe non-linear coalescence of traveling wave solutions. In Section 4, Bäcklund transformations of the linear form are discussed along with some explicit relations of Bäcklund transformations for the obtained solutions. In Section 5, additional discussion is given.

## 2. TRAVELING WAVE SOLUTIONS

We consider traveling wave solutions to the Kolmogorov–Petrovskii–Piskunov (1)

$$u(x, t) = u(\xi) = u(kx - \omega t),$$

where the wavenumber $k$ and the frequency $\omega$ are required to be non-zero for generating non-trivial solutions. The resulting ordinary differential equation from (1) reads as

$$-\omega u_\xi - k^2 u_{\xi\xi} + \mu u + \nu u^2 + \delta u^3 = 0. \tag{2}$$

In the following we generate traveling wave solutions to (1) starting with two ansätze.

First we make the ansätze for (2)

$$u(\xi) = \sum_{i=0}^{M} a_i v^i = \sum_{i=0}^{M} a_i (v(\xi))^i, \quad v_\xi = \varepsilon(1 - v^2), \quad \varepsilon = \pm 1. \tag{3}$$

It is easy to show that $M$ must be 1 if the functions $1, v, v^2, \ldots, v^M$ ($M \in N$) are linear independent and $\delta \neq 0$. So, without loss of generality, we may choose

$$u = a_0 + a_1 v,$$

and thus

$$u_\xi = \varepsilon a_1 - \varepsilon a_1 v^2,$$

$$u_{\xi\xi} = -2a_1 v + 2a_1 v^3.$$

The substitution into (2) yields the following conditions for determining $a_0, a_1, k, \omega$:

$$\begin{cases} -\varepsilon\omega a_1 + \mu a_0 + \nu a_0^2 + \delta a_0^3 = 0, \\ 2k^2 a_1 + \mu a_1 + 2\nu a_0 a_1 + 3\delta a_0^2 a_1 = 0, \\ \varepsilon\omega a_1 + \nu a_1^2 + 3\delta a_0 a_1^2 = 0, \\ -2k^2 a_1 + \delta a_1^3 = 0. \end{cases} \tag{4}$$

We need to assume $a_1 \neq 0$ for non-trivial solutions, and thereby we obtain

$$k^2 = \frac{1}{2}\delta a_1^2, \qquad \omega = -\varepsilon(\nu a_1 + 3\delta a_0 a_1), \tag{5}$$

$$a_1^2 = -\frac{\mu}{\delta} - \frac{2\nu}{\delta} a_0 - 3a_0^2, \tag{6}$$

$$f(a_0) := \frac{\mu\nu}{\delta} + 2\left(\frac{\nu^2}{\delta} + \mu\right) a_0 + 8\nu a_0^2 + 8\delta a_0^3 = 0, \tag{7}$$

in which (5) shows that $\delta > 0$. The equation (7) is a cubic equation. Krishnan [13] analyzed the case of (7) with $\mu = a$, $\nu = -(a+1)$, $\delta = 1$, but he failed to give any solution. Actually, this equation has a solution $a_0 = -\nu/2\delta$ by inspection. We can accordingly decompose

$$f(a_0) = \left(a_0 + \frac{\nu}{2\delta}\right)(8\delta a_0^2 + 4\nu a_0 + 2\mu),$$

from which we acquire three roots

$$a_{01} = -\frac{\nu}{2\delta}, \qquad a_{02,3} = \frac{-\nu \pm \sqrt{\Delta}}{4\delta}, \qquad \Delta = \nu^2 - 4\mu\delta. \tag{8}$$

Of course, we can also solve the cubic equation (7) by computer algebra tools, for example, Mathematica, Maple and MuPAD. Further, we can obtain by (6) and (5)

$$a_{11} = \varepsilon_1 \frac{\sqrt{\Delta}}{2\delta}, \qquad a_{12,3} = \varepsilon_{2,3} a_{02,3}; \tag{9}$$

$$k_1 = \varepsilon_4 \frac{\sqrt{\Delta}}{2\sqrt{2\delta}}, \qquad k_2 = \varepsilon_5 \frac{\nu - \sqrt{\Delta}}{4\sqrt{2\delta}}, \qquad k_3 = \varepsilon_6 \frac{\nu + \sqrt{\Delta}}{4\sqrt{2\delta}}; \tag{10}$$

$$\omega_1 = \varepsilon\varepsilon_1 \frac{\nu\sqrt{\Delta}}{4\delta}, \qquad \omega_{2,3} = -\varepsilon\varepsilon_{2,3}\left[\frac{(\nu \mp \sqrt{\Delta})^2}{16\delta} - \frac{\nu}{2}\right]. \tag{11}$$

Now we may conclude that only when $\delta > 0$, $\Delta \geqslant 0$, there are real solutions $a_0$, $a_1$, $k$, $\omega$ and we acquire the following three exact solutions for the Kolmogorov–Petrovskii–Piskunov equation (1) with $\delta > 0$, $\Delta \geqslant 0$:

$$\begin{aligned} u_1(x,t) &= u_1(x,t;\varepsilon_1,\varepsilon_4) = a_{01} + a_{11} v(k_1 x - \omega_1 t) \\ &= -\frac{\nu}{2\delta} + \varepsilon_1 \frac{\sqrt{\Delta}}{2\delta} v\left(\varepsilon_4 \frac{\sqrt{\Delta}}{2\sqrt{2\delta}} x - \varepsilon_1 \frac{\nu\sqrt{\Delta}}{4\delta} t\right), \end{aligned} \tag{12}$$

$$\begin{aligned} u_2(x,t) &= u_2(x,t;\varepsilon_2,\varepsilon_5) = a_{02} + a_{12} v(k_2 x - \omega_2 t) \\ &= \frac{-\nu + \sqrt{\Delta}}{4\delta} + \varepsilon_2 \frac{-\nu + \sqrt{\Delta}}{4\delta} v\left(\varepsilon_5 \frac{\nu - \sqrt{\Delta}}{4\sqrt{2\delta}} x \right. \\ &\quad \left. + \varepsilon_2\left[\frac{(\nu - \sqrt{\Delta})^2}{16\delta} - \frac{\mu}{2}\right] t\right), \end{aligned} \tag{13}$$

$$\begin{aligned} u_3(x,t) &= u_3(x,t;\varepsilon_3,\varepsilon_6) = a_{03} + a_{13} v(k_3 x - \omega_3 t) \\ &= \frac{-\nu - \sqrt{\Delta}}{4\delta} + \varepsilon_3 \frac{-\nu - \sqrt{\Delta}}{4\delta} v\left(\varepsilon_6 \frac{\nu + \sqrt{\Delta}}{4\sqrt{2\delta}} x \right. \\ &\quad \left. + \varepsilon_3\left[\frac{(\nu + \sqrt{\Delta})^2}{16\delta} - \frac{\mu}{2}\right] t\right), \end{aligned} \tag{14}$$

where $v_\xi = 1 - v^2$, $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant 6$. In the above solutions, we cancel the case of $\varepsilon = -1$ due to the same solutions.

Notice that the Riccati equation $v_\xi = a(1 - v^2)$ $(a \in R^1)$ has a general solution

$$v = v(\xi) = \frac{A - Be^{-2a\xi}}{A + Be^{-2a\xi}} = \begin{cases} 1 & \text{for } B = 0, \\ -1 & \text{for } A = 0, \\ \tanh\left(a\xi - \frac{1}{2}\ln\left(\frac{B}{A}\right)\right) & \text{for } AB > 0, \\ \coth\left(a\xi - \frac{1}{2}\ln\left(-\frac{B}{A}\right)\right) & \text{for } AB < 0, \end{cases} \tag{15}$$

where $A$, $B$ are arbitrary constants satisfying $A^2 + B^2 \neq 0$. This solution may be obtained by three tricks: a Möbius transformation, a Cole–Hopf transformation or a relation [14]

$$\frac{(v_1 - v_2)(v_3 - v_4)}{(v_1 - v_3)(v_2 - v_4)} = C, \quad C = \text{const.}$$

of the solutions $v_i$, $1 \leqslant i \leqslant 4$, beginning with three known solutions $1$, $-1$, $\tanh(a\xi)$. From $v(\xi) = \pm 1$, (12), (13) and (14) result in three constant solutions of (1), but after choosing

$$v(\xi) = \tanh(\xi + \xi_0), \qquad v(\xi) = \coth(\xi + \xi_0) \quad (\xi_0 \text{ arbitrary}),$$

(12), (13) and (14) yield non-trivial solutions: three explicit traveling front solutions and three explicit singular traveling solutions, respectively.

Secondly, we make another ansätze for (2)

$$u(\xi) = \sum_{i=0}^{M} b_i v^i = \sum_{i=0}^{M} b_i (v(\xi))^i, \qquad v_\xi = \varepsilon(1 + v^2), \quad \varepsilon = \pm 1. \tag{16}$$

Similarly, $M$ must equal 1 if the functions $1, v, v^2, \ldots, v^M (M \in N)$ are linear independent and $\delta \neq 0$. So without loss of generality, we can choose

$$u = b_0 + b_1 v,$$

and further we find

$$u_\xi = \varepsilon b_1 + \varepsilon b_1 v^2,$$

$$u_{\xi\xi} = 2b_1 v + 2b_1 v^3.$$

The substitution into (2) engenders the following conditions on $b_0, b_1, k, \omega$:

$$\begin{cases} -\varepsilon\omega b_1 + \mu b_0 + \nu b_0^2 + \delta b_0^3 = 0, \\ -2k^2 b_1 + \mu b_1 + 2\nu b_0 b_1 + 3\delta b_0^2 b_1 = 0, \\ -\varepsilon\omega\beta_1 + \nu b_1^2 + 3\delta b_0 b_1^2 = 0, \\ -2k^2 b_1 + \delta b_1^3 = 0. \end{cases} \tag{17}$$

Note that there are only two terms in (4) and (17) have opposite signs. In an analogous way, we can prove that only when $\delta > 0$, $\Delta \leqslant 0$, there exists a set of real non-zero solutions $b_0, b_1, k, \omega$, which may be worked out

$$b_0 = -\frac{\nu}{2\delta}, \qquad b_1 = \varepsilon_1 \frac{\sqrt{-\Delta}}{2\delta}, \qquad k = \varepsilon_2 \frac{\sqrt{-\Delta}}{2\sqrt{2\delta}}, \qquad \omega = -\varepsilon\varepsilon_1 \frac{\nu\sqrt{-\Delta}}{4\delta}. \tag{18}$$

In this case, notice that the corresponding Riccati equation $v_\xi = a(1 + v^2)$, $a \in R^1$ has the solutions

$$v(\xi) = \tan(a\xi + \xi_0), \qquad v(\xi) = -\cot(a\xi + \xi_0)$$

with an arbitrary constant $\xi_0$. Accordingly, we obtain two explicit exact solutions for the Kolmogorov–Petrovskii–Piskunov equation (1) with $\delta > 0$, $\Delta \leqslant 0$:

$$u_1' = -\frac{\nu}{2\delta} + \varepsilon_1 \frac{\sqrt{-\Delta}}{2\delta} \tan\left(\varepsilon_2 \frac{\sqrt{-\Delta}}{2\sqrt{2\delta}} x + \varepsilon_1 \frac{\nu\sqrt{-\Delta}}{4\delta} t + \xi_0\right), \tag{19}$$

$$u_2' = -\frac{\nu}{2\delta} - \varepsilon_1 \frac{\sqrt{-\Delta}}{2\delta} \cot\left(\varepsilon_2 \frac{\sqrt{-\Delta}}{2\sqrt{2\delta}} x + \varepsilon_1 \frac{\nu\sqrt{-\Delta}}{4\delta} t + \xi_0\right), \tag{20}$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$ and $\xi_0$ is arbitrary. In the above solutions, $\varepsilon$ is again incorporated into $\varepsilon_1$ due to the same solution.

## 3. NON-LINEAR INTERACTIONS OF TRAVELING WAVES

We make a Cole–Hopf transformation

$$u = \alpha(\ln f)_x = \alpha f_x / f, \quad \alpha = \text{const.} \tag{21}$$

for the Kolmogorov–Petrovski–Piskunov equation (1). We have

$$(f_{xt}f - f_x f_t)f^2 - (f_{xxx}f - 3f_x f_{xx})f^2 - 2f_x^3 f + \mu f_x f^3 + \nu\alpha f_x^2 f^2 + \delta\alpha^2 f_x^3 f = 0. \quad (22)$$

Therefore if we choose $\alpha = \pm\sqrt{2/\delta} = \varepsilon\sqrt{2/\delta}$, we get a bilinear equation

$$f_{xt}f - f_x f_t - f_{xxx}f + 3f_x f_{xx} + \mu f_x f + \nu\alpha f_x^2 = 0. \quad (23)$$

After assuming a kind of solutions to be expressed by an exponential function

$$f = A_1 e^{k_1 x + \omega_1 t} + A_2 e^{k_2 x + \omega_2 t} + A_3 e^{k_3 x + \omega_3 t},$$

we find the conditions

$$\begin{cases} 2k_i^3 + \nu\alpha k_i^2 + \mu k_i = 0, \quad 1 \leqslant i \leqslant 3, \\ (\omega_i - \omega_j)(k_i - k_j) - (k_i^3 + k_j^3 - 3k_i k_j^2 - 3k_i^2 k_j) \\ + \mu(k_i + k_j) + 2\nu\alpha k_i k_j = 0, \quad 1 \leqslant i < j \leqslant 3. \end{cases} \quad (24)$$

By solving this equation, we get a non-trivial solution to the equation (23)

$$f = A_1 + A_2 e^{k_+ x + (k_+^2 - \mu)t} + A_3 e^{k_- x + (k_-^2 - \mu)t}, \quad (25)$$

where $A_i$, $1 \leqslant i \leqslant 3$, are arbitrary constants and

$$k_\pm = \frac{-\varepsilon\nu \pm \sqrt{\Delta}}{2\sqrt{2\delta}} = \frac{-\varepsilon\nu \pm \sqrt{\nu^2 - 4\mu\delta}}{2\sqrt{2\delta}}.$$

Finally, we obtain two different solutions to the Kolmogorov–Petrovskii–Piskunov equation (1) with $\delta > 0$, $\Delta \geqslant 0$:

$$u_4(x, t) = u_4(x, t; A_1, A_2, A_3) = \sqrt{\frac{2}{\delta}}\frac{A_2 k_+ e^{\eta_+} + A_3 k_- e^{\eta_-}}{A_1 + A_2 e^{\eta_+} + A_3 e^{\eta_-}}, \quad (26)$$

with

$$k_\pm = \frac{-\nu \pm \sqrt{\Delta}}{2\sqrt{2\delta}}, \qquad \eta_\pm = k_\pm x + (k_\pm^2 - \mu)t;$$

$$u_5(x, t) = u_5(x, t; A_1, A_2, A_3) = -\sqrt{\frac{2}{\delta}}\frac{A_2 k_+ e^{\eta_+} + A_3 k_- e^{\eta_-}}{A_1 + A_2 e^{\eta_+} + A_3 e^{\eta_-}}, \quad (27)$$

with

$$k_\pm = \frac{\nu \pm \sqrt{\Delta}}{2\sqrt{2\delta}}, \qquad \eta_\pm = k_\pm x + (k_\pm^2 - \mu)t,$$

where $A_1, A_2, A_3$ are three arbitrary constants. We note that $u_4(-x, t, A_1, A_2, A_3) = u_5(x, t, A_1, A_3, A_2)$. Therefore for the case of FitzHugh–Nagumo equation, $u_5$ is exactly a solution lost in ref. [11]. The solutions $u_4, u_5$ describe the coalescence of two traveling fronts or two singular traveling waves of the same sense. Direct numerical calculations of non-linear interactions for the FitzHugh–Nagumo case of (26) were done in ref. [11]. These two solutions are analytic on the whole plane of $(x, t)$ when $A_i A_j > 0$, $i \neq j$, and they blow up at some points of $(x, t)$ when $A_i$, $1 \leqslant i \leqslant 3$ do not possess the same signs.

## 4. BÄCKLUND TRANSFORMATIONS

We know there are three solutions

$$\beta = 0, \qquad \beta = C_\pm = \frac{-\nu \pm \sqrt{\Delta}}{2\delta}$$

to the equation $\mu\beta + \nu\beta^2 + \delta\beta^3 = 0$, when $\Delta = \nu^2 - 4\mu\delta \geqslant 0$. Make a linear transformation

$$u = \alpha\tilde{u} + \beta, \qquad \alpha \neq 0, \qquad \beta = 0 \quad \text{or } C_\pm. \quad (28)$$

This moment

$$\mu u + \nu u^2 + \delta u^3 = \alpha(\mu + 2\beta\nu + 3\beta^2\delta)\tilde{u} + \alpha^2(\nu + 3\beta\delta)\tilde{u}^2 + \alpha^3\delta\tilde{u}^3$$
$$= \begin{cases} \alpha\mu\tilde{u} + \alpha^2\nu\tilde{u}^2 + \alpha^3\delta\tilde{u}^3, & \text{for } \beta = 0, \\ -\alpha(2\mu + \beta\nu)\tilde{u} + \alpha^2(\nu + 3\beta\delta)\tilde{u}^2 + \alpha^3\delta\tilde{u}^3, & \text{for } \beta = C_\pm, \end{cases}$$

and thus the Kolmogorov–Petrovskii–Piskunov equation (1) is equivalent to the following new Kolmogorov–Petrovskii–Piskunov equation

$$\begin{aligned} \tilde{u}_t - \tilde{u}_{xx} &= -(\mu + 2\beta\nu + 3\beta^2\delta)\tilde{u} - \alpha(\nu + 3\beta\delta)\tilde{u}^2 - \alpha^2\delta\tilde{u}^3 \\ &:= -\tilde{\mu}\tilde{u} - \tilde{\nu}\tilde{u}^2 - \tilde{\delta}\tilde{u}^3. \end{aligned} \tag{29}$$

A direct calculation yields that

$$\tilde{\Delta} := \tilde{\nu}^2 - \tilde{\mu}\tilde{\delta} = \begin{cases} \alpha^2\Delta & \text{when } \beta = 0; \\ \frac{1}{4}\alpha^2(\nu + \sqrt{\Delta})^2, & \text{when } \beta = C_+; \\ \frac{1}{4}\alpha^2(\nu - \sqrt{\Delta})^2, & \text{when } \beta = C_-. \end{cases} \tag{30}$$

We remark that a similar equation $\tilde{u}_t - \tilde{u}_{xx} + \mu\tilde{u} - \nu\tilde{u}^2 + \delta\tilde{u}^3 = 0$ is generated under the mirror transformation $u = -\tilde{u}$, which possesses the same property as the old equation (1).

The transformation (28) maps the case: $\delta > 0$, $\Delta \geqslant 0$ into the same case: $\tilde{\delta} > 0$, $\tilde{\Delta} \geqslant 0$. Therefore a new Kolmogorov–Petrovskii–Piskunov equation (29) also has five explicit exact solutions defined by

$$\tilde{u}_i(x, t) = u_i(x, t)|_{\mu = \tilde{\mu}, \nu = \tilde{\nu}, \delta = \tilde{\delta}}, \quad 1 \leqslant i \leqslant 5,$$

and further five new exact solutions to the old Kolmogorov–Petrovskii–Piskunov equation (1) may be presented by $\alpha\tilde{u}_i(x, t) + \beta$, $1 \leqslant i \leqslant 5$. However, this transformation process has not given a new kind of solution to (1) beginning with the obtained five solutions, which will be shown below.

Note that we have for $\alpha > 0$, $\nu \geqslant 0$:

$$\sqrt{\tilde{\Delta}} = \tfrac{1}{2}\alpha(\nu + \sqrt{\Delta}), \quad \tilde{\nu} - \sqrt{\tilde{\Delta}} = \alpha(-\nu + \sqrt{\Delta}), \quad \tilde{\nu} + \sqrt{\tilde{\Delta}} = 2\alpha\sqrt{\Delta}, \quad \text{when } \beta = C_+;$$
$$\sqrt{\tilde{\Delta}} = \tfrac{1}{2}\alpha(\nu - \sqrt{\Delta}), \quad \tilde{\nu} - \sqrt{\tilde{\Delta}} = -\alpha(\nu + \sqrt{\Delta}), \quad \tilde{\nu} + \sqrt{\tilde{\Delta}}$$
$$= -2\alpha\sqrt{\Delta}, \quad \text{when } \beta = C_-, \quad \mu \geqslant 0;$$
$$\sqrt{\tilde{\Delta}} = \tfrac{1}{2}\alpha(\sqrt{\Delta} - \nu), \quad \tilde{\nu} - \sqrt{\tilde{\Delta}} = -2\alpha\sqrt{\Delta}, \quad \tilde{\nu} + \sqrt{\tilde{\Delta}}$$
$$= -\alpha(\nu + \sqrt{\Delta}), \quad \text{when } \beta = C_-, \quad \mu \leqslant 0.$$

The concrete results of the Bäcklund transformation

$$(BT)_\beta\colon \quad u(x, t) \mapsto \alpha u(x, t)|_{\mu = \tilde{\mu}, \nu = \tilde{\nu}, \delta = \tilde{\delta}} + \beta \tag{31}$$

may be given out for $\alpha > 0$, $\nu \geqslant 0$ as follows:

$$\begin{aligned} (BT)_{C_+}\colon \quad & u_1(x, t; \varepsilon_1, \varepsilon_4) \mapsto u_3(x, t; \varepsilon_3 = -\varepsilon_1, \varepsilon_6 = -\varepsilon_4), \\ & u_2(x, t; \varepsilon_2, \varepsilon_5) \mapsto u_2(x, t; -\varepsilon_2, -\varepsilon_5), \\ & u_3(x, t; \varepsilon_3, \varepsilon_6) \mapsto u_1(x, t; \varepsilon_1 = -\varepsilon_3, \varepsilon_4 = \varepsilon_6), \\ & u_4(x, t; A_1, A_2, A_3) \mapsto u_4(x, t; A_2, A_1, A_3), \\ & u_5(x, t; A_1, A_2, A_3) \mapsto u_5(x, t; A_3, A_2, A_1); \\ (BT)_{C_-, \mu \geqslant 0}\colon \quad & u_1(x, t; \varepsilon_1, \varepsilon_4) \mapsto u_2(x, t; \varepsilon_2 = -\varepsilon_1, \varepsilon_5 = \varepsilon_4), \\ & u_2(x, t; \varepsilon_2, \varepsilon_5) \mapsto u_3(x, t; \varepsilon_3 = -\varepsilon_2, \varepsilon_6 = -\varepsilon_5), \\ & u_3(x, t; \varepsilon_3, \varepsilon_6) \mapsto u_1(x, t; \varepsilon_1 = \varepsilon_3, \varepsilon_4 = -\varepsilon_6), \\ & u_4(x, t; A_1, A_2, A_3) \mapsto u_4(x, t; A_2, A_1, A_3), \\ & u_5(x, t; A_1, A_2, A_3) \mapsto u_5(x, t; A_3, A_2, A_1); \end{aligned}$$

$$
\begin{aligned}
(BT)_{C_-,\mu\leqslant 0}\colon\quad & u_1(x,t;\varepsilon_1,\varepsilon_4)\mapsto u_2(x,t;\varepsilon_2=\varepsilon_1,\varepsilon_5=-\varepsilon_4),\\
& u_2(x,t;\varepsilon_2,\varepsilon_5)\mapsto u_1(x,t;\varepsilon_1=\varepsilon_2,\varepsilon_4=-\varepsilon_5),\\
& u_3(x,t;\varepsilon_3,\varepsilon_6)\mapsto u_3(x,t;\ -\varepsilon_3,\ -\varepsilon_6),\\
& u_4(x,t;A_1,A_2,A_3)\mapsto u_4(x,t;A_3,A_2,A_1),\\
& u_5(x,t;A_1,A_2,A_3)\mapsto u_5(x,t;A_2,A_1,A_3).
\end{aligned}
$$

When $\beta = 0$, the transformation (28) also maps the other case: $\delta > 0$, $\Delta \leqslant 0$ into the same case: $\tilde{\delta} > 0$, $\tilde{\Delta} \leqslant 0$. We may show that $(BT)_{\beta=0,\alpha>0}$ is an identity map on the set of solutions span $\{u_i, u'_j \mid 1 \leqslant i \leqslant 5, j = 1, 2\}$ and that

$$
\begin{aligned}
(BT)_{\beta=0,\alpha<0}\colon\quad & u_1(x,t;\varepsilon_1,\varepsilon_4)\mapsto u_1(x,t;\ -\varepsilon_1,\varepsilon_4),\\
& u_2(x,t;\varepsilon_2,\varepsilon_5)\mapsto u_3(x,t;\varepsilon_3=\varepsilon_2,\varepsilon_6=-\varepsilon_5),\\
& u_3(x,t;\varepsilon_3,\varepsilon_6)\mapsto u_2(x,t;\varepsilon_2=\varepsilon_3,\varepsilon_5=-\varepsilon_6),\\
& u_4(x,t;A_1,A_2,A_3)\mapsto u_5(x,t;A_1,A_2,A_3),\\
& u_5(x,t;A_1,A_2,A_3)\mapsto u_4(x,t;A_1,A_2,A_3),\\
& u'_1(x,t;\varepsilon_1,\varepsilon_2)\mapsto u'_1(x,t;\ -\varepsilon_1,\varepsilon_2),\\
& u'_2(x,t;\varepsilon_1,\varepsilon_2)\mapsto u'_2(x,t;\ -\varepsilon_1,\varepsilon_2).
\end{aligned}
$$

The rest case of $(BT)_{C_\pm}$ may be computed similarly. It is interesting to note that $(BT)_{\beta=0,\alpha<0}$ casts the solution (26) into the solution (27) and vice versa.

We point out that we may also transform a more general equation

$$w_t - w_{xx} = f(w) = a + bw + cw^2 + dw^3, \quad a \neq 0, \tag{32}$$

where $a, b, c, d$ are real constants, into the Kolmogorov–Petrovskii–Piskunov equation (1) under the linear transformation $w = \alpha u + \beta$, $\alpha, \beta =$ constants. Therefore we can generate solutions to (32) by the obtained results. However, we should note that equation (32) lost the property $f(0) = 0$.

## 5. DISCUSSION

The Kolmogorov–Petrovskii–Piskunov equation (1) contains the following various equations with $\mu + \nu + \delta = 0$, for which there is always the condition $\Delta = \nu^2 - 4\mu\delta = (\mu - \delta)^2 \geqslant 0$, and thus has five explicit solutions.

(i) The non-integrable Newell–Whitehead [15] equation:

$$u_t - u_{xx} = u - u^3 \tag{33}$$

is a special case with $\mu = -1$, $\nu = 0$, $\delta = 1$. The cases of $\mu = 0$, $\nu = -1$, $\delta = 1$ and $\mu = 1$, $\nu = -2$, $\delta = 1$ engender the equations

$$u_t - u_{xx} = u^2(1 - u), \tag{34}$$

$$u_t - u_{xx} = -u(1 - u)^2, \tag{35}$$

respectively. The above three equations are all simple generalizations of the Fisher equation. Interestingly, another simple generalization of the Fisher equation $u_t - u_{xx} = u(1 - u)^2$ has no non-constant solution of the form:

$$u = \sum_{i=0}^{M} a_i \tanh^i(kx - \omega t + \xi_0), \qquad u = \sum_{i=0}^{M} b_i \coth^i(kx - \omega t + \xi_0), \quad a_i, b_i \in R^1.$$

This is in agreement with the result in ref. [10], where it was shown that this equation has no solution of the form $a_1 \tanh(kx - \omega t + \xi_0) + a_0$.

(ii) The case of $\mu = a$, $\nu = -(a+1)$, $\delta = 1$ engenders the FitzHugh–Nagumo equation

$$u_t - u_{xx} = u(u-a)(1-u), \tag{36}$$

where $a$ is arbitrary. This equation may describe nerve pulse propagation in nerve fibers and wall motion in liquid crystals. Its solutions were discussed in refs [11, 12].

Our method in Section 2 is a kind of combination of the direct method [16] and the ansätze method [17, 18] and thus we term it the combined ansätze method. The idea is to make the unknown variable $u$ to be a practicable function $g(v)$ of the ansätze unknown variable $v$, which satisfies a differential equation solvable by quadratures. This allows us to solve a large class of physically important non-linear equations including some non-integrable ones, for example, 2D–KdV–Burgers equations [19] and seventh-generalized KdV equation [20]. The crucial point is to choose the proper ansätze equations solvable by quadratures. We here list two useful ansätze equations and their solutions. The first one is the Bernoulli equation

$$v_\xi = av + bv^\alpha, \qquad a, b, \alpha \in R^1, \quad ab \neq 0, \quad \alpha \neq 1. \tag{37}$$

It has a general solution

$$v = \left[ -\frac{a}{b}\frac{1}{\xi_0 e^{a(1-\alpha)\xi} + 1}\right]^{1/(\alpha-1)}$$

$$= \begin{cases} \left(-\dfrac{a}{2b}\right)^{1/(\alpha-1)}, & \text{for } \xi_0 = 0, \\ \left\{-\dfrac{a}{2b}\left[\tanh\left(\dfrac{a(\alpha-1)}{2}\xi - \dfrac{\ln \xi_0}{2}\right) + 1\right]\right\}^{1/(\alpha-1)}, & \text{for } \xi_0 > 0, \\ \left\{-\dfrac{a}{2b}\left[\coth\left(\dfrac{a(\alpha-1)}{2}\xi - \dfrac{\ln(-\xi_0)}{2}\right) + 1\right]\right\}^{1/(\alpha-1)}, & \text{for } \xi_0 < 0, \end{cases} \tag{38}$$

where $\xi_0$ is arbitrary. The second one is the Riccati equation

$$v_\xi = a_0 + a_1 v + a_2 v^2, \quad a_i \in R^1, \quad a_2 \neq 0. \tag{39}$$

This equation has the following solutions:

$$v = -\frac{a_1}{2a_2}, \qquad -\frac{1}{a_2\xi + \xi_0} - \frac{a_1}{2a_2}, \quad (\Delta = 0); \tag{40}$$

$$v = -\frac{\varepsilon\sqrt{\Delta}}{a_2}\frac{1}{\xi_0 \exp(-\varepsilon\sqrt{\Delta}\xi) + 1} + \frac{\varepsilon\sqrt{\Delta}}{2a_2} - \frac{a_1}{2a_2} \quad (\varepsilon = \pm 1),$$

$$= \begin{cases} \dfrac{\varepsilon\sqrt{\Delta}}{2a_2} - \dfrac{a_1}{2a_2}, & \text{for } \xi_0 = 0, \\ -\dfrac{\sqrt{\Delta}}{2a_2}\tanh\left(\dfrac{\sqrt{\Delta}}{2}\xi - \dfrac{\varepsilon \ln \xi_0}{2}\right) - \dfrac{a_1}{2a_2}, & \text{for } \xi_0 > 0, \quad (\Delta > 0); \\ -\dfrac{\sqrt{\Delta}}{2a_2}\coth\left(\dfrac{\sqrt{\Delta}}{2}\xi - \dfrac{\varepsilon \ln(-\xi_0)}{2}\right) - \dfrac{a_1}{2a_2}, & \text{for } \xi_0 < 0, \end{cases} \tag{41}$$

$$v = \begin{cases} \dfrac{\sqrt{-\Delta}}{2a_2}\tan\left(\dfrac{\sqrt{-\Delta}}{2}\xi + \xi_0\right) - \dfrac{a_1}{2a_2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (\Delta < 0); \\ -\dfrac{\sqrt{-\Delta}}{2a_2}\cot\left(\dfrac{\sqrt{-\Delta}}{2}\xi + \xi_0\right) - \dfrac{a_1}{2a_2}, \end{cases} \tag{42}$$

where $\Delta = a_1^2 - 4a_0a_2$ and $\xi_0$ is arbitrary. The Bernoulli equation and the Riccati equation have appeared in [17, 18]. However, singular solutions of these two equations are all missing in their works.

For the general Fisher equation

$$u_t - u_{xx} + \mu u + \nu u^2 = 0, \quad \mu, \nu \in R^1, \tag{43}$$

the same combined ansätze as ones for the Kolmogorov–Petrovskii–Piskunov (1) may also result in solutions, but only a kind of solutions

$$u(x, t) = -\frac{\mu}{2\nu} - \frac{|\mu|}{4\nu} + \frac{\varepsilon_1\mu}{2\nu} v\left(\frac{\varepsilon_2\sqrt{6|\mu|}}{12}x + \frac{5\varepsilon_1\mu}{12}t\right) + \frac{|\mu|}{4\nu} v^2\left(\frac{\varepsilon_2\sqrt{6|\mu|}}{12}x + \frac{5\varepsilon_1\mu}{12}t\right), \tag{44}$$

where $v_\xi = 1 - v^2$, $\varepsilon_1, \varepsilon_2 = \pm 1$. In particular, the Fisher equation (43) has no solution of the form

$$u = \sum_{i=0}^{M} a_i \tan^i(kx - \omega t + \xi_0), \qquad u = \sum_{i=0}^{M} b_i \cot^i(kx - \omega t + \xi_0), \quad a_i, b_i \in R^1. \tag{45}$$

It is worth pointing out that the Fisher equation cannot be solved through the ansätze method proposed in refs [17, 18]. When we choose $v = \tanh(\xi + \xi_0)$ or $v = \coth(\xi + \xi_0)$, the solution (44) yields two explicit solutions, for instance, a traveling front solution

$$u(x, t) = -\frac{\mu}{2\nu} - \frac{|\mu|}{4\nu} + \frac{\varepsilon_1\mu}{2\nu}\tanh\left(\frac{\sqrt{6|\mu|}}{12}x + \frac{5\varepsilon_1\mu}{12}t + \xi_0\right) + \frac{|\mu|}{4\nu}\tanh^2\left(\frac{\sqrt{6|\mu|}}{12}x + \frac{5\varepsilon_1\mu}{12}t + \xi_0\right). \tag{46}$$

They contain two solutions of the standard Fisher equation $u_t - u_{xx} = u(1 - u)$.

Finally, we remark that any Riccati equation possesses an important property: given a particular solution, its general solution may be found by quadratures. This property is named the Riccati property by Fuchssteiner and Carillo [21] and a method to construct ordinary differential equations which have the Riccati property is proposed in their works [21, 22]. We may also take the differential equations proposed in refs [21, 22] as the basic ansätze equations. This may make more non-linear equations solvable by quadratures.

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