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# Do symmetry constraints yield exact solutions?

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**Abstract**

Two classes of natural symmetry constraints are introduced and analyzed for the Sharma–Tasso–Olver equation. Through those symmetry constraints, the phenomenon is exhibited that symmetry constraints do not always yield exact solutions. It is also explained why such phenomenon can happen in the symmetry theory.


## 1. Introduction

Symmetry constraints are powerful in determining integrability of differential equations. A feasible way of constructing symmetry constraints involving non-Lie symmetries was furnished by using Lax pairs of soliton equations [1]. The resulting symmetry constraints link soliton equations to finite-dimensional integrable Hamiltonian systems, thereby showing the Liouville integrability of soliton equations.

In this paper, we would like to show, through a study of symmetry constraints of the Sharma–Tasso–Olver equation, that not all symmetry constraints of differential equations yield exact solutions of the equations. This seems surprising, since symmetries generate transformation groups which transform solutions to solutions of the same equation. We will also explain why such phenomenon can happen in the symmetry theory so that mistakes can be avoided in constructing exact solutions by the symmetry constraint method.

## 2. Symmetry constraints of the Sharma–Tasso–Olver equation

Let us consider the Sharma–Tasso–Olver (STO) equation [2,3]

$$u_t + 3\alpha u_x^2 + 3\alpha u^2 u_x + 3\alpha u u_{xx} + \alpha u_{xxx} = 0, \tag{2.1}$$

where $\alpha$ is an arbitrary real constant. This equation is linked to the third-order Burgers equation [3]

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$$\tilde{u}_t = \tilde{u}_{xxx} + \frac{3}{2}\tilde{u}\tilde{u}_{xx} + \frac{3}{2}\tilde{u}_x^2 + \frac{3}{4}\tilde{u}^2\tilde{u}_x,$$

under $\tilde{u}(x,t) = 2\sqrt[3]{\alpha}u(\sqrt[3]{\alpha}x, -t)$. Its symmetries and exact solutions and many travelling wave solutions to its generalization have also been presented in [4–6], respectively. For simplicity, we introduce the potential variable $v$ (i.e., $u = v_x$), and then substitute $u = v_x$ into the STO equation (2.1) and integrate the resulting equation with respect to $x$ once, to obtain the potential Sharma–Tasso–Olver (PSTO) equation

$$v_t + \alpha v_x^3 + 3\alpha v_x v_{xx} + \alpha v_{xxx} = 0, \tag{2.2}$$

or equivalently,

$$\begin{cases} v_x - u = 0, \\ v_t + \alpha u^3 + 3\alpha u u_x + \alpha u_{xx} = 0. \end{cases} \tag{2.3}$$

The original STO equation (2.1) is exactly the compatibility condition, $v_{xt} = v_{tx}$, of the PSTO system (2.3). It is obvious that if $(u,v)^{\mathrm{T}}$ solves the PSTO system (2.3), then $u$ and $v$ solve the STO equation (2.1) and the PSTO equation (2.2), respectively. The PSTO system (2.3) is our example, of which we will analyze symmetry constraints.

A symmetry of the PSTO system (2.3),

$$\sigma := \begin{bmatrix} \sigma^u \\ \sigma^v \end{bmatrix} = \begin{bmatrix} \sigma^u(x,t,u,v,u_x,u_t,v_x,v_t,\ldots) \\ \sigma^v(x,t,u,v,u_x,u_t,v_x,v_t,\ldots) \end{bmatrix}, \tag{2.4}$$

is a solution of its linearized system:

$$\begin{cases} D_x\sigma^v - \sigma^u = 0, \\ D_t\sigma^v + 3\alpha(u^2+u_x)\sigma^u + 3\alpha u D_x\sigma^u + \alpha D_x^2\sigma^u = 0, \end{cases} \tag{2.5}$$

where $D_t$ and $D_x$ are the total derivatives with respect to $t$ and $x$, respectively. That is to say, the PSTO system (2.3) is form invariant under the infinitesimal transformation

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \end{bmatrix} + \varepsilon \begin{bmatrix} \sigma^u \\ \sigma^v \end{bmatrix},$$

where $\varepsilon$ is an infinitesimal parameter. We look for Lie point symmetries of the form

$$\begin{cases} \sigma^u = \xi u_x + \tau u_t - p, \\ \sigma^v = \xi v_x + \tau v_t - q, \end{cases}$$

where $\xi$, $\tau$, $p$ and $q$ are functions of $x$, $t$, $u$ and $v$. A straightforward computation leads to the following Lie point symmetries of the PSTO system (2.3) [4]:

$$\begin{cases} \sigma^u = (c_0x + c_2)u_x + (3c_0t + c_1)u_t - (\mathrm{e}^{v_0-v} - c_0)u + \mathrm{e}^{v_0-v}u_0, \\ \sigma^v = (c_0x + c_2)v_x + (3c_0t + c_1)v_t + \mathrm{e}^{v_0-v} - c_3, \end{cases} \tag{2.6}$$

where $(u_0, v_0)^{\mathrm{T}}$ is a given solution of the PSTO system (2.3). Of these symmetries defined by (2.6), the $c_1$, $c_2$ and $c_3$ parts correspond to translational invariances in $t$, $x$ and $v$, respectively, the $c_0$ part corresponds to the scaling invariance of the model (2.3), and the remaining part is the potential symmetry of the model. Note that the potential symmetry depends on a seed solution $(u_0, v_0)^{\mathrm{T}}$ of the model (2.3).

In what follows, we would like to consider two special cases of symmetries defined by (2.6), to exhibit the phenomenon that not all symmetry constraints yield exact solutions.

**Case 1.** $c_0 = c_1 = c_3 = 0$

Let the only non-zero constant be $c_2 = -a$ for convenience, where $a$ is an arbitrary non-zero constant. For each non-zero $a$, we make a symmetry constraint between the corresponding special symmetry and the zero symmetry:

$$\sigma^u = au_x + (u - u_0)\mathrm{e}^{v_0-v} = 0, \tag{2.7}$$

$$\sigma^v = av_x - \mathrm{e}^{v_0-v} = 0. \tag{2.8}$$

The first equation (2.7) is linear in $u$, and the second equation (2.8) can be transformed into the linear equation

$$w_x = a^{-1} - v_{0,x}w,$$

upon introducing $w = \mathrm{e}^{v-v_0}$. Therefore, we can obtain the solution set of (2.7) and (2.8):

$$u = a^{-1}e^{-v}(e^{v_0} + C_1(t)), \tag{2.9}$$

$$v = \ln(a^{-1}\int_0^x e^{v_0(x',t)}dx' + C_2(t)), \tag{2.10}$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of $t$.

A proper selection of the functions $C_1$ and $C_2$ in the above solution set can engender solutions to the PSTO system (2.3). Actually, a direct computation verifies that if

$$C_1 = 0, \qquad C_2' = -\alpha a^{-1}(e^{v_0})_{xx}|_{x=0}, \tag{2.11}$$

then (2.9) and (2.10) present a solution to the PSTO system (2.3). Therefore, starting from the zero solution $u_0 = v_0 = 0$ and choosing $C_1 = C_2 = 0$, we obtain the solution

$$u = \frac{1}{x}, \qquad v = \ln(a^{-1}x) \tag{2.12}$$

and starting from the solution $u_0 = k$, $v_0 = kx - \alpha k^3 t + \eta$, and choosing $a = k^{-1}$, $C_1 = 0$ and $C_2 = b + e^{-\alpha k^3 t + \eta}$, we obtain the solution

$$u = \frac{ke^{kx-\alpha k^3 t+\eta}}{b + e^{kx-\alpha k^3 t+\eta}}, \qquad v = \ln(b + e^{kx-\alpha k^3 t+\eta}), \tag{2.13}$$

where $k$, $\eta$ and $b$ are arbitrary constants. Further, two hierarchies of rational solutions and negaton type solutions to the STO equation (2.1) can recursively be computed.

However, the symmetry constraint defined by (2.7) and (2.8) does not always present solutions $(u, v)$ to the PSTO system (2.3). To observe this, it will be good enough only to start from the zero solution $u_0 = 0$ and $v_0 = 0$. Then, the selection of $C_1 = 1$ and $C_2 = 0$ leads to the first counter-example:

$$u = \frac{2}{x}, \qquad v = \ln(a^{-1}x). \tag{2.14}$$

This solution of (2.7) and (2.8) does not satisfy $u = v_x$, and thus not a solution to the PSTO system (2.3). The selection of $C_1 = 0$ and $C_2 = t$ leads to the second counter-example:

$$u = \frac{1}{x + at}, \qquad v = \ln(a^{-1}x + t). \tag{2.15}$$

This solution of (2.7) and (2.8) satisfies $u = v_x$, but it is not a solution to the PSTO system (2.3) since $v$ does not satisfy the PSTO equation (2.2).

On the other hand, the symmetry constraint defined by (2.7) and (2.8) presents a Bäcklund transformation of the PSTO system (2.3) from $(u, v)^T$ to $(u_0, v_0)^T$. This can be shown as follows. Assume that $(u, v)^T$ is a solution to the PSTO system (2.3). Solving (2.8) for $v_0$ and then (2.7) for $u_0$ algebraically, we can have

$$u_0 = (\ln u)_x + u, \qquad v_0 = \ln\left(a\frac{\partial}{\partial x}e^{v(x,t)}\right). \tag{2.16}$$

It is now direct to check that this offers a solution to the PSTO system (2.3).

**Case 2.** $c_0 = c_2 = c_3 = 0$

Let the only non-zero constant be $c_1 = -a$ for convenience, where $a$ is an arbitrary non-zero constant. Similarly, for each non-zero $a$, we make a symmetry constraint between the corresponding special symmetry and the zero symmetry:

$$\sigma^u = au_t + (u - u_0)e^{v_0 - v} = 0, \tag{2.17}$$

$$\sigma^v = av_t - e^{v_0 - v} = 0. \tag{2.18}$$

Using the same idea as before, we can solve this system for $u$ and $v$:

$$u = a^{-1}e^{-v}(e^{v_0} + C_1(x)), \tag{2.19}$$

$$v = \ln\left(a^{-1}\int_0^t e^{v_0(x,t')}dt' + C_2(x)\right), \tag{2.20}$$

where $C_1(x)$ and $C_2(x)$ are arbitrary functions of $x$.

Similarly, a direct computation verifies that if

$$C_1 = \int_0^t \frac{\partial}{\partial x} e^{v_0(x,t')} dt' - e^{v_0} + aC_2', \qquad C_2''' = -\alpha^{-1}a^{-1}e^{v_0}|_{t=0}, \tag{2.21}$$

then (2.19) and (2.20) present a solution to the PSTO system (2.3). Therefore, starting from the zero solution $u_0 = v_0 = 0$, we can obtain the solution

$$u = v_x, \qquad v = \ln\left(a^{-1}t - \frac{1}{6}\alpha^{-1}a^{-1}x^3 + bx^2 + cx + d\right), \tag{2.22}$$

where $b$, $c$ and $d$ are arbitrary constants; and starting from the solution $u_0 = k$, $v_0 = kx - \alpha k^3 t + \eta$, we can obtain the solution

$$u = v_x, \qquad v = \ln(-\alpha^{-1}a^{-1}k^{-3}e^{kx-\alpha k^3 t+\eta} + bx^2 + cx + d), \tag{2.23}$$

where $k$, $\eta$, $b$, $c$ and $d$ are arbitrary constants. Further, two hierarchies of rational solutions and rational-negaton type solutions to the STO equation (2.1) can recursively be generated.

However, starting from the zero solution of (2.3): $u_0 = 0$ and $v_0 = 0$, we can present two counter-examples:

$$u = 0, \qquad v = \ln(a^{-1}t + 1), \tag{2.24}$$

under the selection of $C_1 = -1$ and $C_2 = 1$, and

$$u = \frac{1+x}{t}, \qquad v = \ln(a^{-1}t), \tag{2.25}$$

under the selection of $C_1 = x$ and $C_2 = 0$. The first solution (2.24) of (2.17) and (2.18) satisfies $u = v_x$, but the second solution (2.25) of (2.17) and (2.18) does not satisfy $u = v_x$. Either of both functions for $v$ does not solve the PSTO equation (2.2), and thus both pairs of functions defined by (2.24) and (2.25) do not offer solutions to the PSTO system (2.3).

On the other hand, the symmetry constraint defined by (2.17) and (2.18) also presents a Bäcklund transformation of the PSTO system (2.3) from $(u,v)^T$ to $(u_0,v_0)^T$. The solutions obtained so are

$$u_0 = \frac{u_t}{v_t} + u, \qquad v_0 = \ln\left(a\frac{\partial}{\partial t}e^{v(x,t)}\right), \tag{2.26}$$

which can also directly be checked.

## 3. Discussion and conclusion

The phenomenon exhibited above can be explained by the general symmetry theory. Observe that associated with a given symmetry $(\sigma^u, \sigma^v)^T$ of the PSTO system (2.3), the Cauchy problem

$$\begin{cases} u_\tau = \sigma^u, & v_\tau = \sigma^v, \\ u|_{\tau=0} = u_0, & v|_{\tau=0} = v_0, \end{cases}$$

where $(u_0,v_0)^T$ is a given solution to (2.3), defines the one-parameter symmetry group: $(u_0,v_0)^T \mapsto (u(\tau,u_0,v_0), v(\tau,u_0,v_0))^T$ of the PSTO system (2.3). The symmetry constraint defined by $\sigma^u = \sigma^v = 0$ means that the corresponding symmetry group does not explicitly depend on a small parameter $\tau$, i.e., there is only one transformation in the symmetry group. This is fine with the solution process. However, the initial conditions at $\tau = 0$ were not required in the above analysis of symmetry constraints. This is only the problem that we have met while generating exact solutions from symmetry constraints. To obtain exact solutions, the initial point $(u_0,v_0)^T$ of the transformations in the symmetry group at $\tau = 0$ should be in the solution space.

To conclude, we have already showed that not all symmetry constraints lead to exact solutions. The reason for that is that the initial status of the transformations in a symmetry group is not specified. Nevertheless, there is a large class of symmetry constraints [1,7] and even adjoint symmetry constraints [8] arising from Lax pairs, which yield exact solutions to soliton equations.

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