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Dispersion-Managed Lump Waves in a Spatial Symmetric KP Model

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Abstract. This paper aims to explore dispersion-managed lump waves in a spatial symmetric KP model. Negative second-order linear dispersive terms play an important role in creating lump waves with the nonlinearity in the model. The starting point is a Hirota bilinear form with an ansatz on quadratic function solutions to the corresponding Hirota bilinear equation. Symbolic computation with `Maple` is conducted to determine lump waves, and characteristic behaviors are analyzed for the resulting lump wave solutions.

AMS subject classifications: 35Q51, 35Q53, 37K40

Key words: Soliton, lump wave, Hirota bilinear derivative, symbolic computation, dispersion.

1. Introduction

Generally speaking, the amplitudes and widths of waves change during propagation in nonlinear media. Under certain circumstances, however, the effects of nonlinearity and dispersion can cancel each other to create permanent and localized waves called solitons. Such a kind of phenomenon was first observed in water waves [50, 51] and then in optical fibers [47].

In mathematical physics, there are a few methods to determine solitons in nonlinear dispersive models, two of which are the inverse scattering transform [2] and the Hirota bilinear method [8]. The inverse scattering transform can be used to solve Cauchy problems

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of integrable models [63] and obtain long-time asymptotics of solitonless waves [1]. We will apply the Hirota bilinear method to our analysis on lump waves in (2+1)-dimensions below.

Suppose that P is a polynomial in two space variables x, y and time t. A Hirota bilinear differential equation in (2+1)-dimensions is defined as follows:

$$P(D_x, D_y, D_t) f \cdot f = 0,$$

where D_x, D_y and D_t are Hirota's bilinear derivatives [8]

$$\begin{aligned} & D_x^p D_y^q D_t^r f \cdot f \\ = & \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^p \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^q \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^r f(x,y,t) f(x',y',t')\Big|_{x'=x,y'=y,t'=t} \end{aligned}$$

for nonnegative integers p, q, r. An associated partial differential equation (PDE) with a dependent variable u is often determined by some logarithmic derivative transformation of

$$u = 2(\ln f)_x, \quad u = 2(\ln f)_{xx}, \quad u = 2(\ln f)_{xy}.$$

Within the Hirota bilinear theory, an N-soliton solution (please refer to, e.g., [7,25,26,39]) is presented through

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\right),$$

where $\sum_{\mu=0,1}$ stands for the sum over all possibilities for $\mu_1, \mu_2, \cdots, \mu_N$ taking either 0 or 1, and the wave variables and the phase shifts are defined by

$$\begin{aligned} & \xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, && 1 \le i \le N, \\ & \mathrm{e}^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, && 1 \le i < j \le N. \end{aligned}$$

In the above N-soliton solution, the wave numbers k_i, l_i and the frequencies ω_i, $1 \le i \le N$, are required to satisfy the associated dispersion relations

$$P(k_i, l_i, -\omega_i) = 0, \quad 1 \le i \le N,$$

but the phase shifts $\xi_{i,0}$, $1 \le i \le N$, are arbitrary constants. There are abundant applications of the Hirota bilinear method to nonlinear dispersive wave equations [10, 34].

Lump waves (or rogue waves) in integrable models are remarkably varied, and they can describe diverse nonlinear phenomena [43]. Such waves are determined by means of rational functions, and localized in all directions in space [43, 44, 54]. Computing long wave limits of solitons can also lead to lump wave solutions [52]. It is known that the KPI equation has diverse lump wave solutions [19], and its special lump waves can be generated from its solitons, indeed [45]. Other integrable models which possess lump waves include the three-wave resonant interaction [12], the Davey-Stewartson II equation [52], the BKP equation [6, 59], and the KP equation with a self-consistent source [61]. Moreover, nonintegrable models can possess lump waves, among which are several generalized KP, BKP, KP-Boussinesq and Sawada-Kotera equations in (2+1)-dimensions [23,37,40,64], and there are lump waves in linear models [22,24] and with dispersion relations of higher-order [42]. An essential step in generating lump waves is to construct positive quadratic function solutions to Hirota bilinear equations, and then by means of quadratic function solutions, being positive, the logarithmic derivative transformations generate lump waves for nonlinear PDE models [43,44].

In this paper, we would like to look for lump waves in a spatial symmetric KP model. We will apply the Hirota bilinear method in the solution process [17,43,44]. The proposed spatial symmetric KP model contains two sets of second-order linear dispersion terms and nonlinear terms. The dispersion terms balance the nonlinear terms to yield lump wave solutions, which will be computed through symbolic computation with `Maple`. Characteristic properties will be explored for the resulting lump waves. A conclusion and a few concluding remarks will be provided in the final section.

2. A Spatial Symmetric KP Model

Motivated by recent studies on lump waves with symbolic computation [19, 43], we introduce a spatial symmetric KP model equation

$$\begin{aligned} P(u,v,w) = {} & u_{xt} + 6u_x v_x + 6u_{xx} v + u_{xxxx} - u_{yy} \\ & + u_{yt} + 6u_y w_y + 6u_{yy} w + u_{yyyy} - u_{xx} = 0, \end{aligned} \tag{2.1}$$

where $v_y = u_x$ and $w_x = u_y$, to explore dispersion-managed lump waves.

It is straightforward to observe that through the logarithmic derivative transformations

$$u = 2(\ln f)_{xy}, \quad v = 2(\ln f)_{xx}, \quad w = 2(\ln f)_{yy}, \tag{2.2}$$

the above spatial symmetric KP model equation (2.1) is changed into the following Hirota bilinear equation

$$\begin{aligned} B(f) &= \left(D_x^4 + D_y^4 + D_x D_t + D_y D_t - D_x^2 - D_y^2\right) f \cdot f \\ &= 2\Big[\left(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2\right) + \left(f_{yyyy} f - 4 f_{yyy} f_y + 3 f_{yy}^2\right) \\ &\quad + (f_{xt} f - f_x f_t) + (f_{yt} f - f_y f_t) - \left(f_{xx} f - f_x^2\right) - \left(f_{yy} f - f_y^2\right)\Big] = 0, \end{aligned} \tag{2.3}$$

where D_x, D_y, D_t are three Hirota bilinear derivative operators. Actually, the connection between the nonlinear model equation and the bilinear equation reads

$$P(u,v,w) = \left(\frac{B(f)}{f^2}\right)_{xy}, \tag{2.4}$$

where u, v, w are defined through the use of f in (2.2). Therefore, when f solves the bilinear equation (2.3), u, v, w determined by (2.2) solve the spatial symmetric KP model equation (2.1). We will show in the next section that there exist various lump wave solutions to our model equation (2.1).

3. Lump Wave Solutions

In this section, we would like to compute lump wave solutions to the spatial symmetric KP model equation (2.1) via symbolic computation with Maple, though the equation itself does not pass the three-soliton test [18, 25].

Applying a genetic ansatz on lump wave solutions in (2+1)-dimensions [19], we start to determine positive quadratic function solutions

$$f = \xi_1^2 + \xi_2^2 + a_9, \quad \xi_1 = a_1 x + a_2 y + a_3 t + a_4, \quad \xi_2 = a_5 x + a_6 y + a_7 t + a_8 \tag{3.1}$$

to the corresponding Hirota bilinear equation (2.3), where $a_i,\ 1 \le i \le 9$, are real constants to be determined. It is recognized that this is a general form for lump wave solutions of lower order in (2+1)-dimensions [43]. The crucial task is then to conduct symbolic computation to determine those constant parameters $a_i,\ 1 \le i \le 9$.

A direct computation with a Maple code determines a set of solutions for the parameters

$$\begin{aligned} a_3 &= \frac{(a_1+a_2)\big(a_1^2+a_2^2+2a_5a_6\big)+(a_1-a_2)\big(a_5^2-a_6^2\big)}{(a_1+a_2)^2+(a_5+a_6)^2}, \\ a_7 &= \frac{(a_5+a_6)\big(2a_1a_2+a_5^2+a_6^2\big)+(a_5-a_6)\big(a_1^2-a_2^2\big)}{(a_1+a_2)^2+(a_5+a_6)^2}, \\ a_9 &= \frac{3\big[(a_1+a_2)^2+(a_5+a_6)^2\big]\Big[\big(a_1^2+a_5^2\big)^2+\big(a_2^2+a_6^2\big)^2\Big]}{2(a_1a_6-a_2a_5)^2}, \end{aligned} \tag{3.2}$$

and all other a_i are arbitrary. The above solutions for a_3 and a_7 tell a kind of dispersion relations in (2+1)-dimensional dispersive waves, and the solution for a_9 exhibits a complicated coefficient in quadratic function solutions to Hirota bilinear equations. Lump waves with dispersion relations of higher-order have also been explored for the second equation in the integrable KP hierarchy [42].

We emphasize that all the above expressions for the wave frequencies and the constant term in (3.2) were simplified with the help of Maple. Note that

$$a_1 + a_2 = a_5 + a_6 = 0$$

leads to

$$\Delta = a_1a_6 - a_2a_5 = 0.$$

Therefore, to generate lump wave solutions from the logarithmic derivative transformations, we need only one basic condition

$$\Delta = a_1a_6 - a_2a_5 \neq 0, \tag{3.3}$$

which is necessary and sufficient to guarantee the characteristic properties of the lump waves: both the analyticity of the rational solutions and the localization of the solutions in all spatial directions.

4. Characteristic Behaviors

Let us consider the system

$$f_x(x(t),y(t),t)=0,\quad f_y(x(t),y(t),t)=0$$

to determine critical points of the function f. Since f is quadratic, it gives

$$a_1\xi_1+a_5\xi_2=0,\quad a_2\xi_1+a_6\xi_2=0,$$

which is equivalent to

$$\xi_1=a_1x+a_2y+a_3t+a_4=0,\quad \xi_2=a_5x+a_6y+a_7t+a_8=0 \tag{4.1}$$

under the condition (3.3). Solving this system (4.1) for both x and y, we obtain all critical points of f

$$\begin{aligned} x=x(t)&=-\frac{a_1^2+2a_1a_2-a_2^2+a_5^2+2a_5a_6-a_6^2}{(a_1+a_2)^2+(a_5+a_6)^2}t+\frac{a_2a_8-a_4a_6}{a_1a_6-a_2a_5},\\ y=y(t)&=\frac{a_1^2-2a_1a_2-a_2^2+a_5^2-2a_5a_6-a_6^2}{(a_1+a_2)^2+(a_5+a_6)^2}t-\frac{a_1a_8-a_4a_5}{a_1a_6-a_2a_5}, \end{aligned} \tag{4.2}$$

where t is an arbitrary time parameter. All these critical points form two characteristic lines traveling with fixed velocities. Because the sum of two squares, namely, the function $f-a_9$, vanishes at all critical points, it follows that f is positive if and only if $a_9>0$. This implies that u,v,w determined by (2.2) are analytical in $\mathbb{R}^3$ if and only if $a_9>0$. It further follows that our rational solutions u,v,w are all analytical in $\mathbb{R}^3$, since we have $a_9>0$ in our solutions by (3.2).

For any fixed time t, it is direct to see that each point $(x(t),y(t))$ by (4.2) is also a critical point of the functions u,v and w determined by (2.2). Then by the second derivative test, we see that the solutions v and w have a peak at the point $(x(t),y(t))$, upon noting that

$$\begin{aligned} v_{xx}&=-\frac{32\big(a_1^2+a_5^2\big)^2(a_1a_6-a_2a_5)^4}{3\big[(a_1+a_2)^2+(a_5+a_6)^2\big]^2\big[\big(a_1^2+a_5^2\big)^2+\big(a_2^2+a_6^2\big)^2\big]^2}<0,\\ v_{xx}v_{yy}-v_{xy}^2&=\frac{1024\big(a_1^2+a_5^2\big)^2(a_1a_6-a_2a_5)^{10}}{27\big[(a_1+a_2)^2+(a_5+a_6)^2\big]^4\big[\big(a_1^2+a_5^2\big)^2+\big(a_2^2+a_6^2\big)^2\big]^4}>0, \end{aligned}$$

and

$$\begin{aligned} w_{xx}&=-\frac{32(a_1a_6-a_2a_5)^4\big[3(a_1a_2+a_5a_6)^2+(a_1a_6-a_2a_5)^2\big]}{9\big[(a_1+a_2)^2+(a_5+a_6)^2\big]^2\big[\big(a_1^2+a_5^2\big)^2+\big(a_2^2+a_6^2\big)^2\big]^2}<0,\\ w_{xx}w_{yy}-w_{xy}^2&=\frac{1024\big(a_2^2+a_6^2\big)^2(a_1a_6-a_2a_5)^{10}}{27\big[(a_1+a_2)^2+(a_5+a_6)^2\big]^4\big[\big(a_1^2+a_5^2\big)^2+\big(a_2^2+a_6^2\big)^2\big]^4}>0. \end{aligned}$$

But the solution u will have a peak or valley at the point $(x(t),y(t))$, depending on $a_1a_2+a_5a_6>0$ or $a_1a_2+a_5a_6<0$, when $3(a_1a_2+a_5a_6)^2>(a_1a_6-a_2a_5)^2$. The point $(x(t),y(t))$ is a saddle point of u, when $3(a_1a_2+a_5a_6)^2<(a_1a_6-a_2a_5)^2$. The second derivative test will be inconclusive, when $3(a_1a_2+a_5a_6)^2=(a_1a_6-a_2a_5)^2$. All this is because we have

$$u_{xx}=-\frac{32(a_1a_6-a_2a_5)^4\left(a_1^2+a_5^2\right)(a_1a_2+a_5a_6)}{3\left[(a_1+a_2)^2+(a_5+a_6)^2\right]\left[\left(a_1^2+a_5^2\right)^2+\left(a_2^2+a_6^2\right)^2\right]^2},$$

$$u_{xx}u_{yy}-u_{xy}^2=\frac{1024(a_1a_6-a_2a_5)^{10}\left[3(a_1a_2+a_5a_6)^2-(a_1a_6-a_2a_5)^2\right]}{81\left[(a_1+a_2)^2+(a_5+a_6)^2\right]^4\left[\left(a_1^2+a_5^2\right)^2+\left(a_2^2+a_6^2\right)^2\right]^4}.$$

The extreme values of u,v and w at the critical points $(x(t),y(t))$ read

$$u_{extremum}=\frac{8(a_1a_6-a_2a_5)^2(a_1a_2+a_5a_6)}{3\left[(a_1+a_2)^2+(a_5+a_6)^2\right]\left[\left(a_1^2+a_5^2\right)^2+\left(a_2^2+a_6^2\right)^2\right]},$$

$$v_{maximum}=\frac{8\left(a_1^2+a_5^2\right)(a_1a_6-a_2a_5)^2}{3\left[(a_1+a_2)^2+(a_5+a_6)^2\right]\left[\left(a_1^2+a_5^2\right)^2+\left(a_2^2+a_6^2\right)^2\right]},$$

$$w_{maximum}=\frac{8\left(a_2^2+a_6^2\right)(a_1a_6-a_2a_5)^2}{3\left[(a_1+a_2)^2+(a_5+a_6)^2\right]\left[\left(a_1^2+a_5^2\right)^2+\left(a_2^2+a_6^2\right)^2\right]}.$$

Those three extreme values do not depend on time t, and they can tend to either zero or nonzero, when $\Delta=a_1a_6-a_2a_5$ goes to zero.

5. Dispersion Effect

The two negative second-order linear dispersion terms in the spatial symmetric KP model (2.1) are essential to guarantee the existence of lump waves.

Taking two second-order linear dispersion terms with the positive sign, we have another spatial symmetric KP model

$$\begin{aligned}P(u,v,w)=&\,u_{xt}+6u_xv_x+6u_{xx}v+u_{xxxx}+u_{yy}\\&+u_{yt}+6u_yw_y+6u_{yy}w+u_{yyyy}+u_{xx}=0,\end{aligned}\tag{5.1}$$

where $v_y=u_x$ and $w_x=u_y$. This spatial symmetric model equation has the following Hirota bilinear form:

$$\begin{aligned}B(f)=&\left(D_x^4+D_y^4+D_xD_t+D_yD_t+D_x^2+D_y^2\right)f\cdot f\\=&\,2\Big[\left(f_{xxxx}f-4f_{xxx}f_x+3f_{xx}^2\right)+\left(f_{yyyy}f-4f_{yyy}f_y+3f_{yy}^2\right)\\&+(f_{xt}f-f_xf_t)+(f_{yt}f-f_yf_t)+\left(f_{xx}f-f_x^2\right)+\left(f_{yy}f-f_y^2\right)\Big]=0.\end{aligned}\tag{5.2}$$

In fact, this equation (5.2) connects (5.1) as in (2.4), under (2.2). The model equation (5.1) possesses a singular rational solution through (3.1) with

$$a_3 = -\frac{(a_1+a_2)\big(a_1^2+a_2^2+2a_5a_6\big)+(a_1-a_2)\big(a_5^2-a_6^2\big)}{(a_1+a_2)^2+(a_5+a_6)^2},$$
$$a_7 = -\frac{(a_5+a_6)\big(2a_1a_2+a_5^2+a_6^2\big)+(a_5-a_6)\big(a_1^2-a_2^2\big)}{(a_1+a_2)^2+(a_5+a_6)^2},$$
$$a_9 = -\frac{3\big[(a_1+a_2)^2+(a_5+a_6)^2\big]\big[\big(a_1^2+a_5^2\big)^2+\big(a_2^2+a_6^2\big)^2\big]}{2(a_1a_6-a_2a_5)^2}.$$

The singularity of the resulting solution is caused by the negative value of the constant term a_9. This phenomenon is completely similar to the one in the standard KP model [19].

Moreover, we have the spatial symmetric KP model without second-order linear dispersion terms

$$\begin{aligned} P(u,v,w) = {} & u_{xt}+6u_xv_x+6u_{xx}v+u_{xxxx} \\ & +u_{yt}+6u_yw_y+6u_{yy}w+u_{yyyy}=0, \end{aligned} \tag{5.3}$$

where $v_y=u_x$ and $w_x=u_y$. This model equation possesses a Hirota bilinear form

$$B(f)=\Big(D_x^4+D_y^4+D_xD_t+D_yD_t\Big)f\cdot f=0 \tag{5.4}$$

under (2.2). The link between (5.3) and (5.4) is the same as (2.4). However, we fail to find any lump waves to this dispersionless KP model by symbolic computation with `Maple`. The model equation (5.3) is similar to the spatial symmetric (2+1)-dimensional KdV model

$$P(u,v,w)=u_{yt}+3(uv)_{xy}+u_{xxxy}+u_{xt}+3(uw)_{xy}+u_{yyyx}=0, \tag{5.5}$$

where $v_y=u_x$ and $w_x=u_y$. It is transformed into

$$\begin{aligned} B(f) &= \Big(D_x^3D_y+D_yD_t+D_y^3D_x+D_xD_t\Big)f\cdot f \\ &= 2\Big[(f_{xxxy}f-f_{xxx}f_y-3f_{xxy}f_x+3f_{xx}f_{xy})+(f_{yt}f-f_yf_t) \\ &\quad +(f_{yyyx}f-f_{yyy}f_x-3f_{yyx}f_{yx}+3f_{yy}f_{yx})+(f_{xt}f-f_xf_t)\Big]=0 \end{aligned} \tag{5.6}$$

under (2.2). The link between (5.5) and (5.6) is the same as (2.4), again. There is no lump wave found for this nonlinear (2+1)-dimensional model, either.

6. Concluding Remarks

By conducting `Maple` symbolic computation, we have shown that there are lump waves in a spatial symmetric KP model with negative second-order linear dispersion terms, and the second-order linear dispersion terms are absolutely essential to achieving lump wave solutions. The proposed lump wave solutions were explicitly worked out, through presenting the frequencies a_3, a_7 and the constant term a_9, in terms of the wave numbers in the quadratic function solutions. Characteristic properties were also explored, together with an analysis on the role that the second-order linear dispersion terms play.

We remark that the adopted ansatz on lump waves is increasingly being used in computations of exact and explicit solutions to nonlinear dispersive wave equations [4, 9, 57], and all such presented solutions provide insights about lump wave generation. Links to other solution methods in soliton theory should be interesting, including Darboux transformations [58], the Wronskian technique [41, 56], the multiple-wave expansion approach [15, 22], the generalized bilinear approach [16], auto-Bäcklund transformations [11, 48], the Riemann-Hilbert technique [21, 29–34, 36], symmetry reductions [5, 55], and symmetry constraints — cf. [14, 38] and [35] for the continuous and discrete cases, respectively.

We also point out that various recent studies exhibit the striking richness of lump wave solutions to both linear PDEs [22, 24], and nonlinear PDEs in (2+1)-dimensions [46, 49, 62] and (3+1)-dimensions [20, 53, 65]. With the help of the Hirota bilinear forms and the generalized bilinear forms, some more general formulations have been presented for lump wave solutions [3, 43, 44]. Other classes of homoclinic and heteroclinic interaction solutions between lump wave solutions and other dispersive wave solutions have also been constructed for various integrable models [13, 40, 60].

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