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Nonlinearity-managed lump waves in a spatial symmetric HSI model

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Abstract We aim at seeking nonlinearity-managed lump waves in a spatial symmetric HSI model. Nonlinear terms play an important role in formulating such lump waves with the dispersion terms in the nonlinear model. Based on an associated Hirota bilinear form, an ansatz on quadratic function solutions is adopted for the corresponding Hirota bilinear equation, and symbolic computation with Maple is made to construct the required lump waves. A few of characteristic properties of the obtained lump waves are determined and some concluding remarks are given.

1 Introduction

It is known that nonlinear waves usually change while propagating in physical media. Solitons are a particular kind of nonlinear waves that keep the amplitudes and widths during propagation. Such wave motions were first observed in water waves [1,2] and then in optical fibers [3,4]. The effects of nonlinearity and dispersion play important roles in formulating such permanent and localized waves.

Soliton theory provides us with a few powerful approaches to compute solitons in nonlinear dispersive wave models, among which are the inverse scattering transform [5] and the Hirota bilinear method [6]. The inverse scattering transform is a nonlinear analogue and a generalization of the Fourier transform. It is aimed to solve Cauchy problems of nonlinear integrable models [7] and obtain long-time asymptoticis of solitonless waves [8]. We will focus our analysis on the Hirota bilinear method to seek lump waves in (2+1)-dimensions below.

Let P be a polynomial in two space variables x, y and time t. A Hirota bilinear differential equation in (2+1)-dimensions is defined by

$$P(D_x, D_y, D_t) f \cdot f = 0, \tag{1.1}$$

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where D_x, D_y and D_t are Hirota bilinear derivatives [6]:

$$D_x^p D_y^q D_t^r f \cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^p \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^q \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^r f(x, y, t) f(x', y', t') \,|_{x'=x, y'=y, t'=t},$$

p, q, r being nonnegative integers. An associated partial differential equation with a dependent variable u is often determined by one logarithmic derivative transformation of

$$u = 2(\ln f)_x, \quad u = 2(\ln f)_{xx}, \quad u = 2(\ln f)_{xy}. \tag{1.2}$$

Within the Hirota bilinear formulation, an N-soliton solution (see, e.g., [9,10]) is presented through

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\right), \tag{1.3}$$

where $\sum_{\mu=0,1}$ denotes the sum over all possibilities for $\mu_1, \mu_2, \cdots, \mu_N$ taking either 0 or 1, and the wave variables ξ_i and the phase shifts a_{ij} are defined by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \quad 1 \le i \le N, \tag{1.4}$$

and

$$e^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, \quad 1 \le i < j \le N, \tag{1.5}$$

respectively. In this N-soliton solution, the wave numbers k_i, l_i and the frequencies ω_i need to satisfy the associated dispersion relations

$$P(k_i, l_i, -\omega_i) = 0, \quad 1 \le i \le N, \tag{1.6}$$

but the constant phase shifts $\xi_{i,0}$ are arbitrary.

Recent studies show that lump waves (and rogue waves) in integrable models are remarkably rich, and they describe various nonlinear phenomena [11]. Such waves are determined through rational functions localized in all directions in space (see, e.g., [11,12]). Long wave limits of soliton solutions can also produce lump wave solutions (see, e.g., [13]). The KPI equation has a large class of lump wave solutions (see, e.g., [14]), and its special lump waves can be generated from its soliton solutions, indeed [15]. Other integrable models which possess lump waves include the three-wave resonant interaction [16], the BKP equation [17,18], the Davey–Stewartson II equation [13], the Ishimori-I equation [19], and the KP equation with a self-consistent source [20]. Furthermore, nonintegrable models can possess lump waves, among which are several generalized KP, BKP and KP-Boussinesq equations in (2+1)-dimensions [21–23], and there also exist lump waves in linear models (see, e.g., [24,25]) and even with higher-order dispersion relations [26]. An important step in seeking lump waves is to first compute positive quadratic function solutions to bilinear equations, and then the logarithmic derivative transformations produce lump waves for nonlinear model equations [11].

In this paper, we would like to seek lump waves in a spatial symmetric HSI model. We will use the Hirota bilinear form in the solution process (see, e.g., [11,27]). The introduced spatial symmetric HSI model contains two sets of nonlinear terms and second-order linear terms. The nonlinearity terms balance the linear terms to generate lump waves. Symbolic computation with Maple will be made to determine nonlinearity-managed lump waves and a few of characteristic behaviors will be explored for the presented lump waves. Concluding remarks will be given in the last section.

2 A spatial symmetric HSI Model

Let α be a non-zero constant. We consider a spatial symmetric HSI model equation:

$$\begin{aligned} P(u) =& \alpha(3u_{xx}p_t + 3u_x p_{tx} + 3u_{tx}v + 3u_t v_x + u_{txxx}) + u_{xx} + u_{ty} \\ &+ \alpha(3u_{yy}q_t + 3u_y q_{ty} + 3u_{ty}w + 3u_t w_y + u_{tyyy}) + u_{yy} + u_{tx} = 0, \end{aligned} \quad (2.1)$$

where $v_y = u_x, w_x = u_y, p_x = v, q_y = w$, to explore nonlinearity-managed lump waves. The first half or the second half of this nonlinear model gives the HSI model equation in (2+1)-dimensions exactly [28].

We can directly check that under the logarithmic derivative transformations

$$u = 2(\ln f)_{xy},\ v = 2(\ln f)_{xx},\ w = 2(\ln f)_{yy},\ p = 2(\ln f)_x,\ q = 2(\ln f)_y, \quad (2.2)$$

the above spatial symmetric HSI model equation (2.1) is put into the Hirota bilinear equation:

$$\begin{aligned} B(f) =& (\alpha D_x^3 D_t + D_y D_t + D_x^2 + \alpha D_y^3 D_t + D_x D_t + D_y^2) f \cdot f \\ =& 2[\alpha(f_{txxx}f - 3f_{txx}f_x + 3f_{tx}f_{xx} - f_t f_{xxx}) + (f_{ty}f - f_t f_y) + (f_{xx}f - f_x^2) \\ &+ \alpha(f_{tyyy}f - 3f_{tyy}f_y + 3f_{ty}f_{yy} - f_t f_{yyy}) + (f_{tx}f - f_t f_x) + (f_{yy}f - f_y^2)] = 0, \end{aligned} \quad (2.3)$$

where D_x, D_y and D_t are three Hirota bilinear derivatives. In fact, the connection between the nonlinear model equation and the bilinear equation is given by

$$P(u) = \left(\frac{B(f)}{f^2}\right)_{xy},$$

where u, v, w, p, q are determined by f in (2.2). It is now clear that if f solves the bilinear equation (2.3), then u, v, w, p, q defined by (2.2) solve the spatial symmetric HSI model equation (2.1). We will exploret abundant lump waves in our model equation (2.1) in the next section.

3 Nonlinearity-managed lump waves

We would now like to construct lump wave solutions to the spatial symmetric HSI model equation (2.1), with the help of symbolic computation by Maple. It is easy to check that the above nonlinear model equation does not pass the three-soliton test (see, e.g., [10] for the three-soliton test).

Using a general ansatz on lump waves in (2+1)-dimensions [14], we begin with looking for positive quadratic function solutions

$$f = \xi_1^2 + \xi_2^2 + a_9,\ \xi_1 = a_1 x + a_2 y + a_3 t + a_4,\ \xi_2 = a_5 x + a_6 y + a_7 t + a_8, \quad (3.1)$$

to the corresponding Hirota bilinear equation (2.3), where the parameters a_i are real constants to be determined. It is known that this is a general form for lump wave solutions of lower order in (2+1)-dimensions [11]. A crucial task is now to make symbolic computation to determine the involved constant parameters a_i.

A standard computation with Maple tells a set of solutions for the parameters:

$$\begin{cases} a_3 = -\dfrac{(a_1+a_2)(a_1^2+a_2^2)+a_1(a_5^2+2a_5a_6-a_6^2)-a_2(a_5^2-2a_5a_6-a_6^2)}{(a_1+a_2)^2+(a_5+a_6)^2}, \\ a_7 = -\dfrac{(a_5+a_6)(a_5^2+a_6^2)+a_5(a_1^2+2a_1a_2-a_2^2)-a_6(a_1^2-2a_1a_2-a_2^2)}{(a_1+a_2)^2+(a_5+a_6)^2}, \\ a_9 = \dfrac{3(a_1^2+a_2^2+a_5^2+a_6^2)}{2(a_1a_6-a_2a_5)^2}\alpha b, \end{cases} \tag{3.2}$$

and all other parameters are arbitrary. In the above solution set, the polynomial b is given by

$$\begin{aligned} b = b(a_1,a_2,a_5,a_6) &= a_1^4+a_2^4+a_1a_2(a_1^2+a_2^2)+2(a_1^2-a_2^2)(a_5^2-a_6^2) \\ &\quad + a_1a_2(a_5+a_6)^2+a_5a_6(a_1+a_2)^2+(a_5^2-a_5a_6+a_6^2)(a_5+a_6)^2, \end{aligned} \tag{3.3}$$

which satisfies a symmetric property

$$b(a_1,a_2,a_5,a_6) = b(a_2,a_1,a_6,a_5). \tag{3.4}$$

The expressions for a_3, a_7 and a_9 by (3.2) also satisfy this symmetric property, which reflects the spatial symmetric character of the nonlinear model equation (2.1). The above solutions for a_3 and a_7 represent a kind of dispersion relations in (2+1)-dimensional dispersive waves, and the solution for a_9 determines a complicated coefficient in quadratic function solutions to Hirota bilinear equations. Higher-order dispersion relations in lump waves have also been exhibited for the second member in the integrable KP hierarchy [26].

All the above expressions for the wave frequencies and the constant term in (3.2) with (3.3) were simplified through applying Maple. Obviously,

$$a_1 + a_2 = a_5 + a_6 = 0 \tag{3.5}$$

yields

$$\Delta = a_1a_6 - a_2a_5 = 0. \tag{3.6}$$

Therefore, to formulate lump wave solutions by means of the logarithmic derivative transformations, we require two basic conditions:

$$\Delta = a_1a_6 - a_2a_5 \neq 0, \ \alpha b > 0, \tag{3.7}$$

where b is the polynomial determined by (3.3). Those two conditions guarantee the characteristic properties of lump waves: the analyticity of the rational solutions and the localization of the solutions in all spatial directions.

The condition $\alpha b > 0$ involves the coefficient α of the nonlinear terms. When $b > 0$, we can have lump waves for the spatial symmetric HSI model with $\alpha > 0$, and when $b < 0$, we can have lump waves for the spatial symmetric HSI model with $\alpha < 0$. In other words, the nonlinear terms with different signs determine different lump waves for the model equation (2.1).

4 Characteristic behaviors

In order to obtain critical points of the function f, let us solve the system

$$f_x(x(t), y(t), t) = 0, \quad f_y(x(t), y(t), t) = 0. \tag{4.1}$$

Since f is quadratic, it equivalently gives

$$a_1\xi_1 + a_5\xi_2 = 0, \quad a_2\xi_1 + a_6\xi_2 = 0, \tag{4.2}$$

which exactly requires

$$\xi_1 = a_1x + a_2y + a_3t + a_4 = 0, \quad \xi_2 = a_5x + a_6y + a_7t + a_8 = 0, \tag{4.3}$$

under the first condition in (3.7). Solving this system for x and y, we get all critical points of f:

$$\begin{cases} x = x(t) = \dfrac{a_1^2 + 2a_1a_2 - a_2^2 + a_5^2 + 2a_5a_6 - a_6^2}{(a_1 + a_2)^2 + (a_5 + a_6)^2}t + \dfrac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \\ y = y(t) = -\dfrac{a_1^2 - 2a_1a_2 - a_2^2 + a_5^2 - 2a_5a_6 - a_6^2}{(a_1 + a_2)^2 + (a_5 + a_6)^2}t - \dfrac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \end{cases} \tag{4.4}$$

where t is an arbitrary time parameter. Those critical points represent two characteristic lines traveling with constant velocities. Since the sum of two squares, namely, the function $f - a_9$, vanishes at all those critical points, we can see that $f > 0$ if and only if $a_9 > 0$. It then follows that u, v, w defined by (2.2) are analytical in $\mathbb{R}^3$ if and only if $a_9 > 0$, i.e., $\alpha b > 0$, which is the second condition in (3.7).

For any fixed time t, we can directly show that each point $(x(t), y(t))$ defined by (4.4) is also a critical point of the functions u, v and w defined by (2.2). By the second partial derivative test, we know that the solutions v and w have a peak at the point $(x(t), y(t))$, because

$$\begin{cases} v_{xx} = -\dfrac{32(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^4}{3[\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b]^2} < 0, \\ v_{xx}v_{yy} - v_{xy}^2 = \dfrac{1024(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^{10}}{27[\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b]^4} > 0, \end{cases} \tag{4.5}$$

and

$$\begin{cases} w_{xx} = -\dfrac{32[3(a_1a_2 + a_5a_6)^2 + (a_1a_6 - a_2a_5)^2](a_1a_6 - a_2a_5)^4}{9[\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b]^2} < 0, \\ w_{xx}w_{yy} - w_{xy}^2 = \dfrac{1024(a_2^2 + a_6^2)^2(a_1a_6 - a_2a_5)^{10}}{27[\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b]^4} > 0. \end{cases} \tag{4.6}$$

But the solution u has a peak (or valley) at the point $(x(t), y(t))$, if $a_1a_2 + a_5a_6 > 0$ (or $a_1a_2 + a_5a_6 < 0$) and $c > 0$; u has a saddle point $(x(t), y(t))$, if $c < 0$; and the second partial derivative test is inconclusive, if $c = 0$; where

$$c = 3(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2. \tag{4.7}$$

All this follows from

$$\begin{cases} u_{xx} = -\dfrac{32(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^4(a_1a_2 + a_5a_6)}{3[\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b]^2}, \\ u_{xx}u_{yy} - u_{xy}^2 = \dfrac{1024c(a_1a_6 - a_2a_5)^{10}}{81[\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b]^4}, \end{cases} \tag{4.8}$$

where c is given by (4.7). The extreme values of v, w and u at the critical points $(x(t), y(t))$ are as follows:

$$v_{\text{maximum}} = \frac{8(a_1^2 + a_5^2)(a_1 a_6 - a_2 a_5)^2}{3\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b}, \tag{4.9}$$

$$w_{\text{maximum}} = \frac{8(a_2^2 + a_6^2)(a_1 a_6 - a_2 a_5)^2}{3\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b}, \tag{4.10}$$

$$u_{\text{extremum}} = \frac{8(a_1 a_6 - a_2 a_5)^2(a_1 a_2 + a_5 a_6)}{3\alpha(a_1^2 + a_2^2 + a_5^2 + a_6^2)b}. \tag{4.11}$$

Observing those three extreme values, we can know that the lump waves may not decay, when $\Delta = a_1 a_6 - a_2 a_5$ tends to zero.

5 Concluding remarks

With Maple symbolic computation, we have shown that there exist nonlinearity-managed lump waves in a spatial symmetric HSI model, and the nonlinear terms play an essential role in the formulation of the indicated lump waves. The resulting lump waves were explicitly presented, through computing the frequencies a_3, a_7 and the constant term a_9, by means of the wave numbers in the quadratic function solutions. A few of characteristic behaviors were explored, along with the discussion on the role that the nonlinear terms play.

We remark that extensive studies show the striking richness of lump wave solutions to both linear wave equations [24,25], and nonlinear wave equations in (2+1)-dimensions (see, e.g., [29–32]) and (3+1)-dimensions (see, e.g., [33,34]). Based on the Hirota bilinear forms and the generalized bilinear forms, a few more generic solution formulations have been established for lump waves [11,35]. Other kinds of homoclinic and heteroclinic solutions, including interaction solutions between lump waves and other nonlinear waves (see, e.g., [22,36]), have also been generated for integrable models in (2+1)-dimensions.

We also point out that the adopted ansatz on lump wave solutions is increasingly being adopted in computations of exact solutions to nonlinear wave equations (see, e.g., [37–39]). It will be interesting to explore connections with other solution methods in soliton theory, such as Darboux transformations (see, e.g., [40]), the Wronskian technique (see, e.g., [41]), the multiple-wave expansion approach (see, e.g., [25,42–44]), the Riemann-Hilbert technique (see, e.g., [45]), the generalized bilinear approach (see, e.g., [46]), symmetry reductions (see, e.g., [47]), and symmetry constraints (see, e.g., [48] and [49] for the continuous and discrete cases, respectively). We hope that the studies of lump solutions will enhance our understanding of nonlinear dispersive wave propagation and interaction.

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Compliance with ethical standards

Conflicts of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

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