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A binary Darboux transformation for multicomponent NLS equations and their reductions

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Abstract
We present a binary Darboux transformation for multicomponent NLS equations and their reduced integrable counterparts. The starting point is to apply two pairs of eigenfunctions and adjoint eigenfunctions, and the resulting binary Darboux transformation can be decomposed into an N-fold Darboux transformation. By taking the zero potential as a seed solution, soliton solutions are generated from the binary Darboux transformation for multicomponent NLS equations and their reductions.

Keywords Matrix spectral problem · Binary Darboux transformation · Soliton solution

Mathematics Subject Classification 37K15 · 35Q55 · 37K40

1 Introduction

Soliton theory provides various analytical methods to generate exact solutions to nonlinear partial differential equations [1–3]. One of the efficient approaches to soliton solutions is the Darboux transformation (DT). The key in establishing DTs is to use a pair of spatial and temporal matrix spectral problems (see, e.g., [4–6]). A binary DT begins with two pairs of matrix spectral problems and adjoint matrix spectral problems. We would like to present a binary DT for multicomponent NLS equations and their reduced integrable counterparts.

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Let $u = u(t, x)$ be a potential vector, with t and x being the independent variables. We start from a pair of matrix spectral problems:

$$-i\phi_x = U\phi = U(u, \lambda)\phi, \; -i\phi_t = V\phi = V(u, \lambda)\phi, \tag{1.1}$$

where i is the unit imaginary number, λ denotes a spectral parameter, and ϕ is an m-dimensional column eigenfunction. Usually, an integrable equation is associated with the zero curvature equation, i.e., the compatibility condition of the above two matrix spectral problems,

$$U_t - V_x + i[U, V] = 0, \tag{1.2}$$

where $[\cdot, \cdot]$ is the matrix commutator [1–3]. The adjoint spectral problems of (1.1) are defined by

$$i\tilde{\phi}_x = \tilde{\phi}U, \; i\tilde{\phi}_t = \tilde{\phi}V. \tag{1.3}$$

Their compatibility condition yields the same zero curvature equation as above. Moreover, we can reduce matrix spectral problems (or Lax pairs) to generate reduced integrable equations (see, for example, [7]).

A binary DT consists of

$$\phi' = T^+\phi, \; \tilde{\phi}' = \tilde{\phi}T^-, \; u' = f(u), \tag{1.4}$$

provided that a new Lax pair is presented by

$$U' = -iT^+_x(T^+)^{-1} + T^+U(T^+)^{-1}, \; V' = -iT^+_t(T^+)^{-1} + T^+V(T^+)^{-1}, \tag{1.5}$$

where $(T^+)^{-1} = T^-$ and $U' = U|_{u=u'}$ and $V' = V|_{u=u'}$. This implies that ϕ' and $\tilde{\phi}'$ satisfy

$$- i\phi'_x = U'\phi', \; -i\phi'_t = V'\phi', \tag{1.6}$$

and

$$i\tilde{\phi}'_x = \tilde{\phi}'U', \; i\tilde{\phi}'_t = \tilde{\phi}'V', \tag{1.7}$$

respectively. Either (1.6) or (1.7) ensures that the new Lax pair, U' and V', generates the same zero curvature equation with u replaced with u', and hence u' gives a new solution to the corresponding integrable equation. There are plenty of examples of binary DTs for integrable equations of NLS type in the literature (see, for example, [4,8–11]).

In this paper, we would like to present a binary DT for multicomponent NLS equations and their reduced integrable counterparts, starting from an arbitrary-order matrix spectral problem. Upon taking the zero potential as a seed solution, applications of the resulting binary DT present N-soliton solutions. A few concluding remarks are finally given in the last section.

2 Multicomponent NLS equations

2.1 Unreduced case

Let n be an arbitrarily given natural number and I_n denote the identity matrix of size n. We consider a pair of matrix spectral problems (see, for example, [12,13]):

$$\begin{cases} -i\phi_x = U\phi = U(u,\lambda)\phi, \\ -i\phi_t = V\phi = V(u,\lambda)\phi, \end{cases} \tag{2.1}$$

where $u = (p, q^T)^T$ with $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)^T$, and the Lax pair, U and V, is defined by

$$U = \lambda\Lambda + P, \ \ V = \lambda^2\Omega + Q. \tag{2.2}$$

The involved four square matrices, Λ, Ω, P and Q, are given as follows:

$$\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_n), \ \ P = P(u) = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{2.3}$$

$$\Omega = \mathrm{diag}(\beta_1, \beta_2 I_n), \ \ Q = Q(u,\lambda) = \frac{\beta}{\alpha}\lambda \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2} \begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}, \tag{2.4}$$

where α_1, α_2 and β_1, β_2 are two pairs of different numbers, $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$. Obviously, the matrix Q can be expressed in terms of the potential matrix P as follows:

$$Q = Q(P, P_x) = \frac{\beta}{\alpha}\lambda P - \frac{\beta}{\alpha^2} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix} P^2 + \begin{bmatrix} i & 0 \\ 0 & -iI_n \end{bmatrix} P_x \right\}. \tag{2.5}$$

A simplest example of the spatial spectral problem in (2.1) with $p_j = q_j = 0, \ 2 \leq j \leq n$, gives the standard AKNS spectral problem [14]. Owing to the existence of a multiple eigenvalue of Λ, the multicomponent spatial matrix spectral problem in (2.1) is degenerate.

The zero curvature equation associated with the matrix spectral problems in (2.1) leads to the following multicomponent NLS equations:

$$\begin{cases} p_{j,t} = -\dfrac{\beta}{\alpha^2} i[p_{j,xx} + 2(\sum_{l=1}^{n} p_l q_l)p_j], \ 1 \leq j \leq n, \\ q_{j,t} = \dfrac{\beta}{\alpha^2} i[q_{j,xx} + 2(\sum_{l=1}^{n} p_l q_l)q_j], \ 1 \leq j \leq n. \end{cases} \tag{2.6}$$

When $n = 1$, we can have

$$ip_{1,t} = p_{1,xx} + 2p_1^2 q_1, \ -iq_{1,t} = q_{1,xx} + 2p_1 q_1^2. \tag{2.7}$$

When $n = 2$, we can get

$$ip_{j,t} = p_{j,xx} + 2(p_1q_1 + p_2q_2)p_j, \ -iq_{j,t} = q_{j,xx} + 2(p_1q_1 + p_2q_2)q_j, \quad 1 \le j \le 2. \tag{2.8}$$

Under a special kind of symmetric reductions, the above multicomponent NLS equations (2.8) can be reduced to the Manokov system [15], and an integrable decomposition into finite-dimensional Hamiltonian systems was presented for that reduced system in [16].

2.2 Reduced case

Let us now conduct reductions (see also [7] for the basic idea). We make a specific kind of group reductions for the spectral matrix U:

$$U^\dagger(x, t, \lambda^*) = CU(x, t, \lambda)C^{-1}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix}, \ \Sigma^\dagger = \Sigma. \tag{2.9}$$

This equivalently requires that

$$P^\dagger(x, t) = CP(x, t)C^{-1}. \tag{2.10}$$

Henceforth, Σ is a constant invertible Hermitian matrix, $\dagger$ stands for the Hermitian transpose, and $*$ denotes the complex conjugate.

Corresponding to the reductions in (2.10), we have the reductions for the potential vector:

$$q(x, t) = \Sigma^{-1} p^\dagger(x, t), \tag{2.11}$$

where Σ is an arbitrary invertible Hermitian matrix. These reductions imply that

$$V^\dagger(x, t, \lambda^*) = CV(x, t, \lambda)C^{-1}, \ Q^\dagger(x, t, \lambda^*) = CQ(x, t, \lambda)C^{-1}, \tag{2.12}$$

where V and Q are defined in (2.2) and (2.4), respectively.

It is now direct to see that each reduction in (2.10) (or (2.11)) is compatible with the zero curvature equation of the reduced spatial and temporal matrix spectral problems of (2.1). Therefore, under (2.10), the multicomponent NLS equations (2.6) generate the following reduced multicomponent NLS equations:

$$ip_t = \frac{\beta}{\alpha^2}[p_{xx} + 2p\Sigma^{-1}p^\dagger p], \tag{2.13}$$

where $p = (p_1, p_2, \cdots, p_n)$ and Σ is an arbitrary invertible Hermitian matrix.

When $n = 1$, taking $\alpha = \beta = 1$ and $\Sigma = \frac{1}{\sigma}$, we obtain the focusing and defocusing NLS equations:

$$ip_{1,t} = p_{1,xx} + 2\sigma p_1^2 p_1^*,\ \sigma = \mp 1, \tag{2.14}$$

respectively. When $n = 2$, we can obtain a new system of integrable two-component NLS equations:

$$\begin{cases} ip_{1,t} = p_{1,x,x} + (c_1|p_1|^2 + c_2|p_2|^2)p_1, \\ ip_{2,t} = p_{2,x,x} + (c_1|p_1|^2 + c_2|p_2|^2)p_2, \end{cases} \tag{2.15}$$

where c_1 and c_2 are arbitrary nonzero real constants.

3 Binary Darboux transformation

3.1 General skeleton of *M*-matrices and Darboux matrices

Let N be annother arbitrarily given natural number. We start from two sets of eigenfunctions and adjoint eigenfunctions:

$$-iv_{k,x} = U(u,\lambda_k)v_k,\ -iv_{k,t} = V(u,\lambda_k)v_k,\ 1 \le k \le N, \tag{3.1}$$

and

$$i\hat{v}_{k,x} = \hat{v}_k U(u,\hat{\lambda}_k),\ i\hat{v}_{k,t} = \hat{v}_k V(u,\hat{\lambda}_k),\ 1 \le k \le N, \tag{3.2}$$

where λ_k and $\hat{\lambda}_k$, $1 \le k \le N$, are arbitrary eigenvalues and adjoint eigenvalues, respectively. Let us set

$$v = (v_1, \cdots, v_N),\ \hat{v} = (\hat{v}_1^T, \cdots, \hat{v}_N^T)^T, \tag{3.3}$$

and then we can compactly write the equations for the eigenfunctions as follows:

$$-iv_x = \Lambda v A + Pv,\ i\hat{v}_x = \hat{A}\hat{v}\Lambda + \hat{v}P, \tag{3.4}$$

and

$$\begin{aligned} -iv_t &= \Omega v A^2 + (Q(\lambda_1)v_1, \cdots, Q(\lambda_N)v_N), \\ i\hat{v}_t &= \hat{A}^2\hat{v}\Omega + (\hat{v}_1 Q(\hat{\lambda}_1), \cdots, \hat{v}_N Q(\hat{\lambda}_N)), \end{aligned} \tag{3.5}$$

where

$$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_N),\ \hat{A} = \mathrm{diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_N). \tag{3.6}$$

Now introduce a square M-matrix:

$$M=(m_{kl})_{N\times N},\ m_{kl}=\begin{cases}\dfrac{\hat{v}_k v_l}{\lambda_l-\hat{\lambda}_k}, & \text{if } \lambda_l\neq\hat{\lambda}_k,\\ 0, & \text{if } \lambda_l=\hat{\lambda}_k,\end{cases}\quad \text{where } 1\le k,l\le N. \tag{3.7}$$

This M-matrix incorporates zero entries, if $\lambda_l=\hat{\lambda}_k$, where $1\le k,l\le N$, and so it generalizes the traditional soliton case without zero entries (see, for example, [3,17]) and particularly presents soliton solutions to nonlocal integrable equations (see, e.g., [18]). Further, if M is invertible, we can introduce two Darboux matrices:

$$\begin{cases} T^+=T^+(\lambda)=I_{n+1}-\displaystyle\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\hat{\lambda}_l},\\ T^-=T^-(\lambda)=I_{n+1}+\displaystyle\sum_{k,l=1}^{N}\frac{v_k(M^{-1})_{kl}\hat{v}_l}{\lambda-\lambda_k},\end{cases} \tag{3.8}$$

and define

$$T_1^{\pm}(\lambda)=\lim_{\lambda\to\infty}[\lambda(T^{\pm}(\lambda)-I_{n+1})]. \tag{3.9}$$

We can easily rewrite

$$T^+=I_{n+1}-vM^{-1}\hat{R}\hat{v},\ T^-=I_{n+1}+vRM^{-1}\hat{v}, \tag{3.10}$$

where

$$R=\mathrm{diag}(\frac{1}{\lambda-\lambda_1},\cdots,\frac{1}{\lambda-\lambda_N}),\ \hat{R}=\mathrm{diag}(\frac{1}{\lambda-\hat{\lambda}_1},\cdots,\frac{1}{\lambda-\hat{\lambda}_N}); \tag{3.11}$$

and obtain

$$T_1^+=-vM^{-1}\hat{v},\ T_1^-=vM^{-1}\hat{v}, \tag{3.12}$$

which also implies that

$$T_1^+=-T_1^-. \tag{3.13}$$

These two Darboux matrices possess the following properties.
(a) Spectral property:

$$\Big(\prod_{l=1}^{N}(\lambda-\hat{\lambda}_l)T^+\Big)(\lambda_k)v_k=0,\ \hat{v}_k\Big(\prod_{l=1}^{N}(\lambda-\lambda_l)T^-\Big)(\hat{\lambda}_k)=0,\ 1\le k\le N. \tag{3.14}$$

(b) Partial fractional decomposition:

$$T^{+}=I_{n+1}-\sum_{k=1}^{N}\frac{v_k^M\hat{v}_k}{\lambda-\hat{\lambda}_k},\ \ T^{-}=I_{n+1}+\sum_{k=1}^{N}\frac{v_k\hat{v}_k^M}{\lambda-\lambda_k},\tag{3.15}$$

where

$$\begin{cases}(v_1^M,\cdots,v_N^M)=(v_1,\cdots,v_N)M^{-1},\\ ((\hat{v}_1^M)^T,\cdots,(\hat{v}_N^M)^T)^T=M^{-1}(\hat{v}_1^T,\cdots,\hat{v}_N^T)^T.\end{cases}$$

(c) Binary Darboux characteristic: If an orthogonal condition

$$\hat{v}_k v_l=0,\ \text{if}\ \lambda_l=\hat{\lambda}_k,\ \text{where}\ 1\leq k,l\leq N,\tag{3.16}$$

is satisfied, then we have

$$\hat{R}\hat{v}vR=MR-\hat{R}M,\ T^{+}(\lambda)T^{-}(\lambda)=I_{n+1}.\tag{3.17}$$

3.2 Binary DT in the unreduced case

To formulate a DT, we need to compute the derivatives of the M-matrix with respect to x and t. It is direct to see that if

$$\hat{v}_k\Lambda v_l=0,\ \text{if}\ \lambda_l=\hat{\lambda}_k,\ \text{where}\ 1\leq k,l\leq N,\tag{3.18}$$

then we have

$$M_x=i\hat{v}\Lambda v;\tag{3.19}$$

and if

$$\hat{v}_k\Omega_{[k,l]}v_l=0,\ \text{if}\ \lambda_l=\hat{\lambda}_k,\ \text{where}\ 1\leq k,l\leq N,\tag{3.20}$$

with

$$\Omega_{[k,l]}=(\hat{\lambda}_k+\lambda_l)\Omega+\frac{\beta}{\alpha}P,\ 1\leq k,l\leq N,\tag{3.21}$$

then we have

$$M_t=i(\hat{A}\hat{v}\Omega v+\hat{v}\Omega vA+\frac{\beta}{\alpha}\hat{v}Pv).\tag{3.22}$$

Now, a general binary DT can be formulated (see also [19] for details) as follows.

Theorem 3.1 (General structure) *Let $\Omega_{[k,l]}$ be defined by (3.21). Then, when the conditions:*

$$\hat{v}_k v_l = \hat{v}_k \Lambda v_l = \hat{v}_k \Omega_{[k,l]} v_l = 0, \ \textit{if}\ \lambda_l = \hat{\lambda}_k, \ \textit{where}\ 1 \le k, l \le N, \tag{3.23}$$

are satisfied, we have a binary DT:

$$\phi' = T^+\phi, \ \tilde{\phi}' = \tilde{\phi}T^-, \ P' = P + [T_1^+, \Lambda], \tag{3.24}$$

for the multicomponent NLS equations (2.6).

Moreover, if $\{\lambda_k | \, 1 \le k \le N\} \cap \{\hat{\lambda}_k | \, 1 \le k \le N\} = \emptyset$, which is the standard case, we can decompose the above general binary DT into an N-fold binary DT:

$$T^+ = T^+[\![N]\!]T^+[\![N-1]\!] \cdots T^+[\![1]\!], \ T^- = T^-[\![1]\!] \cdots T^-[\![N-1]\!]T^-[\![N]\!], \tag{3.25}$$

by introducing new eigenfunctions and adjoint eigenfunctions. In the above formulas, $T^+[\![k]\!]$ and $T^-[\![k]\!]$, $1 \le k \le N$, are recursively defined as single binary Darboux matrices:

$$\begin{cases} T^+[\![k]\!] = T^+[\![k]\!](\lambda) = I_{n+1} - \dfrac{\lambda_k - \hat{\lambda}_k}{\lambda - \hat{\lambda}_k} \dfrac{v_k' \hat{v}_k'}{\hat{v}_k' v_k'}, \ 1 \le k \le N, \\ T^-[\![k]\!] = T^-[\![k]\!](\lambda) = I_{n+1} + \dfrac{\lambda_k - \hat{\lambda}_k}{\lambda - \lambda_k} \dfrac{v_k' \hat{v}_k'}{\hat{v}_k' v_k'}, \ 1 \le k \le N, \end{cases} \tag{3.26}$$

where new eigenfunctions and adjoint eigenfunctions read

$$v_k' = T^+\{k-1\}(\lambda_k) v_k, \ \hat{v}_k' = \hat{v}_k T^-\{k-1\}(\hat{\lambda}_k), \ 1 \le k \le N, \tag{3.27}$$

in which we have $T^+\{0\} = T^-\{0\} = I_{n+1}$ and

$$T^+\{k\} = T^+[\![k]\!] \cdots T^+[\![2]\!]T^+[\![1]\!], \ T^-\{k\} = T^-[\![1]\!]T^-[\![2]\!] \cdots T^-[\![k]\!], \ 1 \le k \le N. \tag{3.28}$$

3.3 Binary DT in the reduced case

To ensure the reduction property for U', determined by (2.9), let us take

$$\hat{\lambda}_k = \lambda_k^*, \ 1 \le k \le N. \tag{3.29}$$

At this moment, we can find that

$$(T_1^+(x,t))^\dagger = -C T_1^+(x,t) C^{-1} \tag{3.30}$$

will guarantee the reduction property (2.9) for U'. To satisfy this condition (3.30) , we then take

$$\hat{v}_k(x,t,\hat{\lambda}_k) = v_k^{\dagger}(x,t,\lambda_k)C,\ 1 \le k \le N, \tag{3.31}$$

and impose the following three conditions:

$$v_k^{\dagger}Cv_l = v_k^{\dagger}C\Lambda v_l = v_k^{\dagger}C\Omega_{[k,l]}v_l = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k, \tag{3.32}$$

where $1 \le k, l \le N$.

Finally, the binary DT (3.24) is reduced to a binary DT for the reduced multicomponent NLS equations (2.13). We state the result in the following theorem.

Theorem 3.2 *Let $\{\hat{\lambda}_k | \, 1 \le k \le N\}$ be determined by (3.29) and $\{\hat{v}_k | \, 1 \le k \le N\}$ be taken as in (3.31) with the orthogonal properties for $\{v_k | \, 1 \le k \le N\}$ in (3.32). Then the binary Darboux transformation (3.24) is reduced to a binary Darboux transformation for the reduced multicomponent NLS equations (2.13).*

4 Soliton solutions

4.1 Unreduced case

We begin with two arbitrary sets of eigenvalues and adjoint eigenvalues: $\{\lambda_k \in \mathbb{C} | \, 1 \le k \le N\}$ and $\{\hat{\lambda}_k \in \mathbb{C} | \, 1 \le k \le N\}$, respectively. Upon taking $P = 0$ as a seed solution, we can work out the corresponding eigenfunctions and adjoint eigenfunctions

$$v_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k,\ 1 \le k \le N, \tag{4.1}$$

$$\hat{v}_k(x,t) = \hat{w}_k \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t},\ 1 \le k \le N, \tag{4.2}$$

where w_k and $\hat{w}_k$, $1 \le k \le N$, are arbitrary constant column and row vectors, respectively, but need to satisfy three orthogonal conditions:

$$\hat{w}_k w_l = \hat{w}_k \Lambda w_l = (\hat{\lambda}_k + \lambda_l)\hat{w}_k \Omega w_l = 0,\ \text{if}\ \lambda_l = \hat{\lambda}_k,\ \text{where}\ 1 \le k, l \le N, \tag{4.3}$$

where Λ is defined as in (2.3).

Now based on the binary DT (3.24), we obtain a new potential matrix:

$$P' = [T_1^+, \Lambda],\ T_1^+ = -vM^{-1}\hat{v} = -\sum_{k,l=1}^{N} v_k (M^{-1})_{kl} \hat{v}_l. \tag{4.4}$$

Consequently, this yields a kind of N-soliton solutions to the multicomponent NLS equations (2.6):

$$\begin{cases} p_j = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,j+1}, \ 1 \leq j \leq n, \\ q_j = -\alpha \sum_{k,l=1}^{N} v_{k,j+1}(M^{-1})_{kl}\hat{v}_{l,1}, \ 1 \leq j \leq n, \end{cases} \tag{4.5}$$

where we set $v_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \cdots, \hat{v}_{k,n+1})$, $1 \leq k \leq N$.

4.2 Reduced case

Let us now consider the reduced case. We need to guarantee the involution condition (3.30) to engender N-soliton solutions to the reduced multicomponent NLS equations (2.13). This equivalently needs us to check if the newly obtained potential matrix P' through the binary DT satisfies the reduction property (2.10). When this is true, the N-soliton solution to the multicomponent NLS equations (2.6) is reduced to the N-soliton solution:

$$p_j = \alpha \sum_{k,l=1}^{N} v_{k,1}(M^{-1})_{kl}\hat{v}_{l,j+1}, \ 1 \leq j \leq n, \tag{4.6}$$

for the corresponding reduced multicomponent NLS equations (2.13), where $v_k = (v_{k,1}, v_{k,2}, \cdots, v_{k,n+1})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \cdots, \hat{v}_{k,n+1})$, $1 \leq k \leq N$, as before.

To ensure the involution property (3.30), we then take N eigenvalues $\lambda_k \in \mathbb{C}$, $1 \leq k \leq N$, and define $\{\hat{\lambda}_k | \, 1 \leq k \leq N\}$ as in (3.29). Further, upon taking $P = 0$, we can determine the corresponding eigenfunctions v_k, $1 \leq k \leq N$, by

$$v_k(x,t) = v_k(x,t,\lambda_k) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k, \ 1 \leq k \leq N, \tag{4.7}$$

respectively, where w_k, $1 \leq k \leq N$, are arbitrary column vectors. Now, based on the previous analysis on the reductions, the corresponding adjoint eigenfunctions $\hat{v}_k$, $1 \leq k \leq N$, can be taken as

$$\hat{v}_k(x,t) = \hat{v}_k(x,t,\hat{\lambda}_k) = v_k^\dagger(x,t,\lambda_k)C = w_k^\dagger \mathrm{e}^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t} C, \ 1 \leq k \leq N, \tag{4.8}$$

respectively. The three orthogonal properties in (3.32) become the following three new conditions:

$$w_k^\dagger C w_l = w_k^\dagger C \Lambda w_l = (\hat{\lambda}_k + \lambda_l) w_k^\dagger C \Omega w_l = 0, \ \text{if } \lambda_l = \hat{\lambda}_k, \ \text{where } 1 \leq k, l \leq N, \tag{4.9}$$

on $\{w_k | \, 1 \le k \le N\}$. It is worth noting that the situation of $\lambda_k = \hat{\lambda}_k$ occurs only when taking $\lambda_k \in \mathbb{R}$. Obviously, due to $\alpha_1 \ne \alpha_2$, the two conditions in (4.9) equivalently require that

$$w_{k,1}^* w_{l,1} = 0, \ (w_{k,2}^*, \cdots, w_{k,n+1}^*)\Sigma(w_{l,2}, \cdots, w_{l,n+1})^T = 0, \text{ if } \lambda_l = \hat{\lambda}_k, \text{ where } 1 \le k, l \le N, \tag{4.10}$$

in which we set $w_k = (w_{k,1}, w_{k,2}, \cdots, w_{k,n+1})^T$, $1 \le k \le N$.

Finally, we see that the formula (4.6), together with (3.7), (4.7) and (4.8), gives N-soliton solutions to the reduced multicomponent NLS equations (2.13).

5 Concluding remarks

The paper aims to present a binary Darboux transformation (DT) for a kind of multicomponent NLS equations and their reduced integrable counterparts. The crucial step is to utilize pairs of eigenfunctions and adjoint eigenfunctions. The resulting formulation can be applied to construction of soliton solutions to other multicomponent integrable equations such as the mKdV equations and the Hirota equations.

Our success is to introduce a generalized M-matrix in establishing binary DTs. The motivation is derived from various recent studies on Riemann-Hilbert problems for nonlocal integrable equations (see, for example, [18]) . Our general formulation of binary DTs can be applied to both local and nonlocal integrable equations (see, for example, [18,20–23] for nonlocal theories).

Further interesting questions include how one can determine other kinds of exact solutions, for example, lump solutions [24,25], through DTs; and what binary DTs there exist for integrable couplings, i.e., integrable equations associated with general Lie algebras (see [26] for DTs for integrable couplings).

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

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