Wronskian and Grammian solutions to a (3 + 1)-dimensional generalized KP equation

Wen-Xiu Ma *, Alrazi Abdeljabbar, Magdy Gamil Asaad

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

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A B S T R A C T

Wronskian and Grammian formulations are established for a (3 + 1)-dimensional generalized KP equation, based on the Plücker relation and the Jacobi identity for determinants. Generating functions for matrix entries satisfy a linear system of partial differential equations involving a free parameter. Examples of Wronskian and Grammian solutions are computed and a few particular solutions are plotted.

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1. Introduction

It is known that soliton equations such as the KdV equation, the Boussinesq equation and the KP equation possess multisoliton solutions generated from a combination of exponential waves. Hirota bilinear form plays a crucial role in constructing soliton solutions [1,2]. The multiple exp-function method, oriented towards the ease of use and capability of computer algebra systems, provides a direct and efficient way to search for generic multi-exponential wave solutions to nonlinear equations including bilinear equations [3]. Besides soliton solutions, another class of interesting multi-exponential wave solutions is a linear combination of exponential waves. It is shown that a kind of nonlinear equations can possess such a linear superposition principle, and a sufficient criterion for its existence was furnished for Hirota bilinear equations in [4].

Solitons, and positons (a kind of periodic solutions), can be expressed as Wronskian determinants [5,6]. Particular solutions combining exponential functions and trigonometrical functions are presented and called complexiton (or briefly complexitons) [7], and lattice soliton equations have a similar situation [8]. Complexions are also shown to exist for source solution equations [9] and soliton equations with sources [10]. For higher-dimensional soliton equations, there exist Grammian solutions and Pfaffian solutions [11]. Grammian solutions to the KP equation were constructed by Nakamura [11] and Pfaffian solutions to the BKP equation were presented by Hirota [12].

In this paper, we would like to study a (3 + 1)-dimensional generalized KP equation

\[ u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{tz} = 0, \]

which can be written in Hirota bilinear form but does not belong to a class of generalized KP and Boussinesq equations [13]

\[ (u_{x_1 x_1})_1 - 6u u_{x_1} + \sum_{i=1}^M a_i a_j = 0, \quad a_i = \text{constant}, \quad M \in \mathbb{N}. \]

We will show this generalized KP equation has a class of Wronskian solutions and a class of Grammian solutions, with all generating functions for matrix entries satisfying a linear system of partial differential equations involving a free parameter. The Plücker relation and the Jacobi identity for determinants are the key to establish the Wronskian and Grammian
formulations [1]. Examples of Wronskian solutions and Grammian solutions are computed, and a few plots of particular solutions are made.

2. Wronskian formulation

Let us introduce the following helpful notation

\[
|N - j - 1, i_1, \ldots, i_j| = \det(\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-j-1)}, \phi^{(i_1)}, \ldots, \phi^{(i_j)}) = \det(\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-j-1)}, \phi^{(i_1)}, \ldots, \phi^{(i_j)}), \quad 1 \leq j \leq N - 1,
\]

(2.1)

where \(i_1, \ldots, i_j\) are non-negative integers, and the vectors of functions \(\phi^{(i)}\) are defined by

\[
\phi^{(i)} = \left(\phi^{(i)}_1, \phi^{(i)}_2, \ldots, \phi^{(i)}_N\right)^T, \quad \phi^{(i)}_j = \frac{\partial}{\partial x^j} \phi^{(i)}.
\]

(2.2)

A Wronskian determinant is given by

\[
W(\phi_1, \phi_2, \ldots, \phi_N) = \det(\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-1)}) = \det(\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-1)}),
\]

(2.3)

We also use the assumption for convenience that if \(i < 0\), the column vector \(\phi^{(i)}\) does not appear in the determinant \(\det(\ldots, \phi^{(i)}), \ldots)\).

We consider the following \((3 + 1)\)-dimensional nonlinear equation:

\[
u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0.
\]

(2.4)

When \(y = x\), this equation reduces the KP equation, and so we call it a generalized KP equation. The KP equation was also generalized by constructing decomposition of \((2 + 1)\)-dimensional equations into \((1 + 1)\)-dimensional equations [14].

Under the dependent variable transformation

\[
u = 2(\ln f)_y,
\]

(2.5)

the above \((3 + 1)\)-dimensional generalized KP equation is mapped into a Hirota bilinear equation

\[
\left(D_x^2 D_y + D_x D_y + D_x D_y - D_z^2\right)f \cdot f = 0,
\]

(2.6)

where \(D_x, D_y, D_z\) and \(D_t\) are Hirota bilinear differential operators [1,15]. Equivalently, we have

\[
(f_{xxxy} + f_{xy} - f_{zz})f - 3f_{xyxf} + 3f_{xpdf} - f_{fxf} + f_{ffy} + f_{ff} = 0.
\]

(2.7)

The equation was presented for the first time in a study on the linear superposition principle for exponential waves [4], and it is similar to a generalized \((3 + 1)\)-dimensional BKP equation

\[
(3D_x D_z - 2D_y D_t - D_x D_y)^2 f \cdot f = 0,
\]

which has Wronskian and Grammian solutions [16,17]. We would like to present a sufficient condition under which the Wronskian determinant solves the generalized Hirota bilinear KP Eq. (2.6).

**Theorem 2.1.** Let a set of functions \(\phi = \phi(x,y,z,t)\) satisfy the following condition:

\[
\phi_{iy} = \gamma_1 \phi_{ix}, \quad \phi_{iz} = \gamma_2 \phi_{ixx}, \quad \phi_{iz} = \gamma_3 \phi_{ixxx}, \quad 1 \leq i \leq N
\]

with

\[
\gamma_1 = \frac{a^2}{3}, \quad \gamma_2 = a, \quad \gamma_3 = \frac{4a^2}{3 - a^2},
\]

(2.8)

(2.9)

where \(a \neq \pm \sqrt{3}\) is a free parameter. Then the Wronskian determinant \(f_N = |N - 1|\) defined by (2.3) solves the \((3 + 1)\)-dimensional generalized bilinear KP Eq. (2.6).

**Proof.** Under the condition (2.8), we can compute various derivatives of the Wronskian determinant \(f_N = |N - 1|\) with respect to the variables \(x, y, z, t\). It is not difficult to obtain, using (2.8), that

\[
\begin{align*}
f_{N,x} &= |N - 2, N|, \\
f_{N,xx} &= |N - 3, N - 1, N| + |N - 2, N + 1|, \\
f_{N,xxx} &= |N - 4, N - 2, N - 1, N| + 2|N - 3, N - 1, N + 1| + |N - 2, N + 2|, \\
f_{N,y} &= \gamma_1 |N - 2, N|,
\end{align*}
\]
\[ f_{N,xy} = \gamma_1(|N-3, N-1, N| + |N-2, N+1|), \]
\[ f_{N,xz} = \gamma_1(|N-4, N-2, N-1, N| + 2|N-3, N-1, N| + |N-2, N+2|), \]
\[ f_{N,xyz} = \gamma_1(|N-5, N-3, N-2, N-1, N| + 3|N-4, N-2, N-1, N+1| + 2|N-3, N-1, N+1| + 3|N-3, N-1, N+2| + |N-2, N+3|), \]
\[ f_{N,x} = \gamma_2(|N-2, N+1| - |N-3, N-1, N|), \]
\[ f_{N,xz} = \gamma_2^2(-|N-4, N-2, N-1, N+1| + 2|N-3, N, N+1| + |N-2, N+3| + |N-5, N-3, N-2, N-1, N| - |N-3, N-1, N+1| + |N-2, N+2|), \]
\[ f_{N,tx} = \gamma_3(|N-5, N-3, N-2, N-1, N| - |N-3, N-1, N+1| + |N-2, N+3|), \]
\[ f_{N,ty} = \gamma_3^2(|N-5, N-3, N-2, N-1, N| - |N-3, N, N+1| + |N-2, N+2|, \]

In the above expressions, the column \( \phi^{(N-5)} \) does not appear if \( N < 5 \), as we assumed before since \( N - 5 < 0 \). Therefore, under the selection (2.9) of \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), we can now compute that

\[ f_{N,xyz} + f_{N,tx} + f_{N,ty} - f_{N,xz} = -4a^2|N-3, N, N+1|, \]
\[ -3f_{N,xy}f_{N,xx} - f_{N,y}f_{N,yy} - f_{N,x}f_{N,xy} = 4a^2|N-2, N||N-3, N-1, N+1|, \]
\[ 3f_{N,xy}f_{N,xx} + (f_{N,x})^2 = -4a^2|N-3, N-1, N||N-2, N+1|, \]

and further obtain that

\[
\left( D_x^2 D_y + D_x D_y + D_x D_y - D_y^2 \right) f_{N,x} = \left( f_{N,xyz} + f_{N,tx} + f_{N,ty} - f_{N,xz} \right) f_{N} - 3f_{N,xy}f_{N,xx} + 3f_{N,xy}f_{N,xx} - f_{N,xy}f_{N,xx} - f_{N,xy}f_{N,xx} - f_{N,xy}f_{N,xx} + (f_{N,x})^2 \\
= -4a^2(|N-1||N-3, N, N+1| - |N-2, N||N-3, N-1, N+1| + |N-3, N-1, N||N-2, N+1|) = 0.
\]

This last equality is nothing but the Plücker relation for determinants:

\[ |B, A_1, A_2, B, A_3, A_4| - |B, A_1, A_2, B, A_3, A_4| - |B, A_1, A_2, B, A_3, A_4| = 0, \]

where \( B \) denotes an \( N \times (N-2) \) matrix, and \( A_i, 1 \leq i \leq 4 \), are four \( N \)-dimensional column vectors. Therefore, we have shown that \( f = f_N \) solves the \((3+1)\)-dimensional generalized Hirota bilinear KP Eq. (2.6), under the condition (2.8).

The condition (2.8) is a linear system of partial differential equations. It has an exponential-type function solution:

\[ \phi_i = \sum_{j=1}^{p} d_{ij} e^{k_j t}, \quad \eta_{ij} = k_j x + y_j t, \]

where \( d_{ij} \) and \( k_j \) are free parameters and \( p \) is an arbitrary natural number.

In particular, we can have the following Wronskian solutions:

\[ u = 2(\ln f_N)_x, \quad f_N = W(\phi_1, \phi_2, \ldots, \phi_N), \]

where

\[ \phi_i = e^{k_x x + y_i t} + e^{k_x x + y_i t}, \]

where \( k_i \) and \( l_i \) are free parameters. To illustrate Wronskian solutions, we set \( N = 3 \). The following three Figs. 2.1, 2.2 and 2.3 of three dimensional plots and two dimensional contour plots show the Wronskian solutions defined by (2.11) on the indicated specific regions, with specific values being chosen for the parameters. In the contour plots, we see the interaction regions and patterns of the involved solitons.

### 3. Grammian formulation

Let us now introduce the following Grammian determinant

\[ f_N = \det(a_{ij})_{1 \leq i, j \leq N}, \quad a_{ij} = c_{ij} + \int_x^1 \phi_i \psi_j dx, \quad c_{ij} = \text{constant}, \]

with \( \phi_i \) and \( \psi_j \) satisfying
Fig. 2.1. $N=3$ with $k_1 = 1$, $k_2 = 2$, $k_3 = -2$, $l_1 = 1$, $l_2 = 3$, $l_3 = -1$, $a = 1$, $y = 0$, $t = 1$.

Fig. 2.2. $N=3$ with $k_1 = -1$, $k_2 = 2$, $k_3 = -1$, $l_1 = -2$, $l_2 = 3$, $l_3 = 1$, $a = 2$, $y = 0$, $t = 1$.

Fig. 2.3. $N=3$ with $k_1 = 1$, $k_2 = 3$, $k_3 = -2$, $l_1 = 2$, $l_2 = 4$, $l_3 = -3$, $a = -1$, $y = 0$, $t = 0$. 
where the constants $\gamma_i$ are defined by (2.9).

**Theorem 3.1.** Let $\phi_i$ and $\psi_j$ satisfy (3.2) and (3.3), respectively. Then the Grammian determinant $f_N = \det(a_{ij})_{1 \leq i, j \leq N}$ defined by (3.1) solves the $(3 + 1)$-dimensional generalized bilinear KP Eq. (2.6).

**Proof.** Let us express the Grammian determinant $f_N$ by means of a Pfaffian as

$$f_N = \det(\phi_{ij})_{1 \leq i, j \leq N},$$

where $(i,j) = 0$ and $(i,j) = 0$. To compute derivatives of the entries $a_{ij}$ and the Grammian $f_N$, we introduce new Pfaffian entries

$$\frac{\partial}{\partial x}a_{ij} = \phi_{ij}, \quad \frac{\partial}{\partial y}a_{ij} = \frac{\partial}{\partial t}a_{ij} = \phi_{ij}, \quad (d_m, d_n) = (d_m, n) = (d_m, i, j) = 0, \quad m, n \geq 0,$

as usual. In terms of these new entries, derivatives of the entries $a_{ij} = (i,j)$ are given, upon using (3.2) and (3.3), by

$$\frac{\partial}{\partial x}a_{ij} = \phi_{ij},$$

$$\frac{\partial}{\partial y}a_{ij} = \int (\phi_{i,j} + \phi_{j,i})dx = \gamma_1(\phi_{i,j} + \phi_{j,i})dx = \gamma_1(\phi_{i,j} + \phi_{j,i}),$$

$$\frac{\partial}{\partial z}a_{ij} = \int (\phi_{i,j} + \phi_{j,i})dx = \gamma_2(\phi_{i,j} + \phi_{j,i})dx = \gamma_2(\phi_{i,j} + \phi_{j,i}),$$

$$\frac{\partial}{\partial t}a_{ij} = \int (\phi_{i,j} + \phi_{j,i})dx = \gamma_3(\phi_{i,j} + \phi_{j,i})dx = \gamma_3(\phi_{i,j} + \phi_{j,i}).$$

Then we can develop differential rules for Pfaffians as in [1], and compute various derivatives of the Grammian determinant $f_N = \det(a_{ij})$ with respect to the variables $x, y, z, t$ as follows:

$$f_{N,x} = (d_0, d_0, \bullet),$$

$$f_{N,xx} = (d_1, d_0, \bullet) + (d_0, d_1, \bullet),$$

$$f_{N,xxx} = (d_2, d_0, \bullet) + 2(d_1, d_1, \bullet) + (d_0, d_2, \bullet),$$

$$f_{N,y} = \gamma_1(d_0, d_0, \bullet),$$

$$f_{N,xy} = \gamma_1[(d_1, d_0, \bullet) + (d_0, d_1, \bullet)],$$

$$f_{N,xxy} = \gamma_1[(d_2, d_0, \bullet) + 2(d_1, d_1, \bullet) + (d_0, d_2, \bullet)],$$

$$f_{N,xyz} = \gamma_1[(d_3, d_0, \bullet) + 3(d_2, d_1, \bullet) + 2(d_0, d_0, d_1, d_1, \bullet) + 3(d_1, d_2, d_1, \bullet) + (d_0, d_3, \bullet)],$$

$$f_{N,z} = \gamma_2[(d_1, d_0, \bullet) + (d_0, d_1, \bullet)],$$

$$f_{N,zz} = \gamma_2[(d_2, d_0, \bullet) + 2(d_1, d_1, \bullet) + (d_0, d_2, \bullet)],$$

$$f_{N,xt} = \gamma_3[(d_1, d_0, \bullet) + (d_0, d_1, \bullet) + (d_0, d_2, \bullet)],$$

$$f_{N,tx} = \gamma_3[(d_2, d_0, \bullet) + (d_0, d_0, d_1, d_1, \bullet) + (d_0, d_3, \bullet)],$$

$$f_{N,yt} = \gamma_1[(d_3, d_0, \bullet) + 2(d_0, d_0, d_1, d_1, \bullet) + (d_0, d_3, \bullet)],$$

where the abbreviated notation $\bullet$ denotes the list of indices $1, 2, \ldots, N, N', \ldots, 2', 1'$ common to each Pfaffian.

Noting the selection (2.9) of $\gamma_1$, $\gamma_2$ and $\gamma_3$, we can now compute that

$$f_{N,xxx} + f_{N,xx} + f_{N,yy} + f_{N,zz} = -4a^2(d_0, d_0, d_1, d_1, \bullet),$$

$$-3f_{N,xyy}f_{N,x} - f_{N,yxy}f_{N,x} - f_{N,xxy}f_{N,x} - f_{N,xyy}f_{N,x} = 4a^2(d_0, d_0, \bullet)(d_1, d_1, \bullet),$$

$$3f_{N,x}f_{N,xx} + (f_{N,xx})^2 = -4a^2(d_0, d_0, \bullet)(d_1, d_1, \bullet),$$

and further obtain that

$$\left(D_1^2D_y + D_1D_x + D_1D_y - D_y^2\right) f_{N,xx} = f_{N,xxx} + f_{N,xx} + f_{N,yy} + f_{N,zz}f_{N,xx} - 3f_{N,xyy}f_{N,xx} - f_{N,yxy}f_{N,xx} - f_{N,xxy}f_{N,xx} - f_{N,xyy}f_{N,xx} - f_{N,yy}f_{N,xx} - f_{N,zz}f_{N,xx} - f_{N,zz}f_{N,xx} = 0.$$
The systems (3.2) and (3.3) have solutions
\[
\phi_i = \sum_{j=1}^{p} d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij} x + \gamma_1 k_{ij} y + \gamma_2 k_{ij}^2 z + \gamma_3 k_{ij}^3 t, \\
(3.6)
\]
\[
\psi_j = \sum_{i=1}^{q} e_{ji} e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji} x + \gamma_1 l_{ji} y - \gamma_2 l_{ji}^2 z + \gamma_3 l_{ji}^3 t, \\
(3.7)
\]
where \(d_{ij}, e_{ji}, l_{ji}\) and \(k_{ij}\) are free parameters and \(p, q\) are two arbitrary natural numbers. In particular, we have the following specific solutions
\[
\phi_i = e^{\eta_{i}}, \quad \eta_{i} = k_{i} x + \gamma_1 k_{i} y + \gamma_2 k_{i}^2 z + \gamma_3 k_{i}^3 t, \\
(3.8)
\]
\[
\psi_j = e^{\zeta_{j}}, \quad \zeta_{j} = l_{j} x + \gamma_1 l_{j} y - \gamma_2 l_{j}^2 z + \gamma_3 l_{j}^3 t, \\
(3.9)
\]
where \(k_{i}\) and \(l_{j}\) are free parameters. To illustrate Grammian solutions, we further choose \((c_{ij})_{N\times N}\) as the identity matrix of order \(N\). The following three Figs. 3.1, 3.2 and 3.3 of three dimensional plots and two dimensional contour plots show the corresponding Grammian solutions \(u = 2(\ln f)_{x}\) on the indicated specific regions with \(N = 2, 3, 4\), respectively, with specific values being chosen for the parameters. In the contour plots, we see the interaction regions and patterns of the involved solitons.

4. Conclusions and remarks

We have established one Wronskian formulation and one Grammian formulation for the \((3 + 1)\)-dimensional generalized KP equation
\[
u_{xxx} + 3(u_x u_y)_{x} + u_{tx} + u_{ty} - u_{zz} = 0.
\]
The facts used in our construction of Wronskian and Grammian solutions are the Plücker relation for determinants and the Jacobi identity for determinants, respectively. Theorems 2.1 and 3.1 present the main results on Wronskian and Grammian solutions, which say that
\[
u = 2(\ln f)_{x}, \quad f = |N-1| = W(\phi_1, \phi_2, \ldots, \phi_N), \\
\]
with \(\phi_i\) satisfying
\[
\phi_{iy} = -\frac{a^2}{3} \phi_{lx}, \quad \phi_{lx} = a \phi_{lxx}, \quad \phi_{lz} = \frac{4a^2}{3 - a^2} \phi_{lxxx}, \\
\]
and
\[
u = 2(\ln f)_{x}, \quad f = \det(a_{ij})_{1\leq i,j\leq N}, \quad a_{ij} = c_{ij} + \int_{x}^{x} \phi_i \psi_j dx, \quad c_{ij} = \text{constant}, \\
\]
Fig. 3.1. \(N = 2\) with \(k_1 = 1, k_2 = 2, l_1 = 1, l_2 = 3, a = 1, y = 0, t = 1\).
with $\phi_i$ and $\psi_j$ satisfying

$$
\phi_{iy} = -\frac{a^2}{3} \phi_{ix}, \quad \phi_{iz} = a \phi_{ixx}, \quad \phi_{i\ell} = \frac{4a^2}{3 - a^2} \phi_{ixxx},
$$

$$
\psi_{jy} = -\frac{a^2}{3} \psi_{jx}, \quad \psi_{jz} = -a \psi_{jxx}, \quad \psi_{j\ell} = \frac{4a^2}{3 - a^2} \psi_{jxxx},
$$

solve the above $(3+1)$-dimensional generalized KP equation. Here $a$ can be any real number, except $\pm \sqrt{3}$. Examples of Wronskian and Grammian solutions were made, along with a few plots of particular solutions.

In Theorems 2.1 and 3.1, we considered only specific sufficient conditions: (2.8), (3.2) and (3.3), though there is a free parameter $a$ in the conditions. There should exist more general conditions involving combined equations for Wronskian and Grammian solutions. Such conditions were presented for Wronskian solutions of the KdV equation [18], the Boussinesq equation [19,20] and the Toda lattices [8,21], and for Grammian solutions of the KP equation [22]. It should be interesting to find such a condition consisting of combined equations for the $(3+1)$-dimensional generalized KP equation.

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