Applied Mathematics Letters 25 (2012) 1500–1504

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A bilinear Bäcklund transformation of a $(3+1)$-dimensional generalized KP equation

Wen-Xiu Ma *, Alrazi Abdeljabbar

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

ARTICLE INFO

Article history:
Received 24 May 2011
Accepted 1 January 2012

Keywords:
Hirota bilinear form
Bäcklund transformation
Solitary wave solution

ABSTRACT

A bilinear Bäcklund transformation is presented for a $(3+1)$-dimensional generalized KP equation, which consists of six bilinear equations and involves nine arbitrary parameters. Two classes of exponential and rational traveling wave solutions with arbitrary wave numbers are computed, based on the proposed bilinear Bäcklund transformation.

Published by Elsevier Ltd

1. Introduction

It is significantly important to search for exact solutions to nonlinear equations of mathematical physics [1,2]. The transformed rational function method [3] and the multiple exp-function method [4] provide two generic approaches for constructing traveling wave solutions and multiple wave solutions, respectively. If an equation possesses a Hirota bilinear form, then a perturbation expansion often generates a specific class of multiple wave solutions including N-soliton solutions [5]. Moreover, the linear superposition principle may apply to Hirota bilinear equations, and in particular, this presents linear subspaces of solutions for nonlinear equations [6].

Bäcklund transformations are another powerful approach to solutions of nonlinear equations, and they can be written in the Hirota bilinear form when an equation under consideration has a bilinear form [7,8]. For example, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1}$$

which can be written as

$$D_x(D_t + D_x^3)f \cdot f = 0, \tag{1.2}$$

under $u = 2(\ln f)_{xx}$, D_r being Hirota's bilinear operator [2], has the bilinear Bäcklund transformation [7]:

$$\begin{cases} (D_x^2 - \lambda)f' \cdot f = 0, \\ (D_t + 3\lambda D_x + D_x^3)f' \cdot f = 0. \end{cases} \tag{1.3}$$

This means that f solves the bilinear KdV equation (1.2) if and only if f' solves the bilinear KdV equation (1.2). The $(2+1)$-dimensional generalized KdV equation, i.e., the KP equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0, \tag{1.4}$$

which can be written as

$$(-4D_xD_t + 3D_y^2 + D_x^4)f \cdot f = 0, \tag{1.5}$$

* Corresponding author. Tel.: +1 813 9749563; fax: +1 813 9742700.
E-mail addresses: mawx@cas.usf.edu, wma3@usf.edu (W.X. Ma).

0893-9659/$ – see front matter. Published by Elsevier Ltd
doi:10.1016/j.aml.2012.01.003

under $u = 2(\ln f)_{xx}$, has the bilinear Bäcklund transformation [2,9]:

$$\begin{cases}(D_y - D_x^2)f' \cdot f = 0,\\ (3D_yD_x - 4D_t + D_x^3)f' \cdot f = 0.\end{cases} \tag{1.6}$$

Such bilinear Bäcklund transformations also connect with Lax pairs and generate the modified soliton equations [9,10].

In this paper, we would like to study a $(3+1)$-dimensional generalized KP equation

$$u_{xxxy} + 3(u_xu_y)_x + u_{tx} + u_{ty} - u_{zz} = 0,$$

which can be written in the Hirota bilinear form

$$(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f \cdot f = 0,$$

under $u = 2(\ln f)_x$. This equation was presented for the first time in a study on the linear superposition principle for exponential waves [6], and it is similar to a generalized $(3+1)$-dimensional BKP equation [3,11]

$$(3D_xD_z - 2D_yD_t - D_yD_x^3)f \cdot f = 0,$$

whose bilinear Bäcklund transformation was presented in [12].

We would like to construct a bilinear Bäcklund transformation for the above $(3+1)$-dimensional generalized KP equation, which consists of six equations and contains nine arbitrary parameters. The exchange formula for Hirota's bilinear operators are the basis for carrying out the necessary interchanges in deriving the bilinear Bäcklund transformation. Exponential and rational traveling wave solutions with arbitrary wave numbers are computed by applying the proposed bilinear Bäcklund transformation.

2. Bilinear Bäcklund transformation and traveling wave solutions

We consider the following $(3+1)$-dimensional nonlinear equation:

$$u_{xxxy} + 3(u_xu_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \tag{2.1}$$

This equation is different from the $(3+1)$-dimensional KP equation [13]; but when $y = x$, the equation is reduced to the KP equation, and so it is called a generalized KP equation [6]. The KP equation was also generalized by constructing decomposition of $(2+1)$-dimensional equations into $(1+1)$-dimensional equations [14].

Under the dependent variable transformation

$$u = 2(\ln f)_x, \tag{2.2}$$

the above $(3+1)$-dimensional generalized KP equation is put into a Hirota bilinear equation

$$(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f \cdot f = 0, \tag{2.3}$$

where D_x, D_y, D_z and D_t are Hirota's bilinear differential operators [2,7]. This is equivalent to

$$(f_{xxxy} + f_{tx} + f_{ty} - f_{zz})f - 3f_{xxy}f_x + 3f_{xy}f_{xx} - f_yf_{xxx} - f_tf_x - f_tf_y + (f_z)^2 = 0.$$

Its Wronskian and Grammian solutions, Pfaffianized generalized KP system, and non-singular and singular soliton solutions were presented in [15–17], respectively.

2.1. Bilinear Bäcklund transformation

We would like to present a bilinear Bäcklund transformation for the $(3+1)$-dimensional generalized bilinear KP equation (2.3).

Let us assume that we have another solution f' to the generalized bilinear KP equation (2.3):

$$(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f' \cdot f' = 0, \tag{2.4}$$

and introduce a key function

$$P = [(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f' \cdot f']f^2 - [(D_x^3D_y + D_tD_x + D_tD_y - D_z^2)f \cdot f]f'^2. \tag{2.5}$$

If $P = 0$, then f solves the generalized bilinear KP equation (2.3) if and only if f' solves the generalized bilinear KP equation (2.3). Therefore, if we can obtain, from $P = 0$ by interchanging the dependent variables f and f', a system of bilinear equations that guarantees $P = 0$:

$$B_i(D_t, D_x, D_y, D_z)f' \cdot f = 0, \quad 1 \le i \le M,$$

where the B_i's are polynomials in the indicated variables and M is a natural number depending on the complexity of the equation, then this system presents a bilinear Bäcklund transformation for the generalize bilinear KP equation (2.3).

Let us now start to explore what those bilinear equations could be. First we want to list three exchange identities for Hirota's bilinear operators:

$$(D_tD_xa\cdot a)b^2-(D_tD_xb\cdot b)a^2=2D_x(D_ta\cdot b)\cdot ba, \tag{2.6}$$

$$(D_tD_ya\cdot a)b^2-(D_tD_yb\cdot b)a^2=2D_y(D_ta\cdot b)\cdot ba, \tag{2.7}$$

$$\begin{aligned}&2(D_x^3D_ya\cdot a)b^2-2(D_x^3D_yb\cdot b)a^2\\&\quad=D_x[(3D_x^2D_ya\cdot b)\cdot ba+(3D_x^2a\cdot b)\cdot(D_yb\cdot a)+(6D_xD_ya\cdot b)\cdot(D_xb\cdot a)]\\&\qquad+D_y[(D_x^3a\cdot b)\cdot ba+(3D_x^3a\cdot b)\cdot(D_xb\cdot a)].\end{aligned} \tag{2.8}$$

The first and second identities can be found in [2], and the third one can be obtained from the coefficient of ε^1, while taking the independent variable transformation $D_x\to D_x+\varepsilon D_y$ for

$$(D_x^4a\cdot a)b^2-(D_x^4b\cdot b)a^2=2D_x[(D_x^3a\cdot b)\cdot ba+(3D_x^2a\cdot b)\cdot(D_xb\cdot a)],$$

which is the known identity in [2]. All these identities come from the general exchange formula (see [2] for details). Now from the first identity (2.6) or the second identity (2.7), we can easily obtain

$$(D_z^2a\cdot a)b^2-(D_z^2b\cdot b)a^2=2D_z(D_za\cdot b)\cdot ba, \tag{2.9}$$

$$D_r(D_sa\cdot b)\cdot ba=D_s(D_ra\cdot b)\cdot ba, \tag{2.10}$$

by taking $x=t=z$ and noting $D_rD_sg\cdot g=D_sD_rg\cdot g$.

Then, it can be proved that $P=0$ if we take

$$\begin{cases}B_1f'\cdot f\equiv(3D_x^2D_y+4D_t+\lambda_1D_y+4\lambda_8D_z+\lambda_2)f'\cdot f=0,\\B_2f'\cdot f\equiv(D_x^3+4D_t-\lambda_1D_x+4\lambda_9D_z+\lambda_3)f'\cdot f=0,\\B_3f'\cdot f\equiv(3D_x^2+\lambda_4D_y+\lambda_6)f'\cdot f=0,\\B_4f'\cdot f\equiv(3D_x^2+\lambda_5D_x-\lambda_6)f'\cdot f=0,\\B_5f'\cdot f\equiv(D_xD_y+\lambda_7D_x)f'\cdot f=0,\\B_6f'\cdot f\equiv(D_z+\lambda_8D_x+\lambda_9D_y)f'\cdot f=0,\end{cases} \tag{2.11}$$

where nine arbitrary parameters have been introduced. This system provides a bilinear Bäcklund transformation for the (3 + 1)-dimensional generalized KP equation (2.3).

Actually, by using the exchange identities (2.6)–(2.9), we can make the following conversion:

$$\begin{aligned}2P&=[2(D_x^3D_yf'\cdot f')f^2-2(D_x^3D_yf\cdot f)f'^2]+[2(D_tD_xf'\cdot f')f^2-2(D_tD_xf\cdot f)f'^2]\\&\quad+[2(D_tD_yf'\cdot f')f^2-2(D_tD_yf\cdot f)f'^2]-[2(D_z^2f'\cdot f')f^2-2(D_z^2f\cdot f)f'^2]\\&=\{D_x[(3D_x^2D_yf'\cdot f)\cdot ff'+(3D_x^2f'\cdot f)\cdot(D_yf\cdot f')+(6D_xD_yf'\cdot f)\cdot(D_xf\cdot f')]\\&\quad+D_y[(D_x^3f'\cdot f)\cdot ff'+(3D_x^2f'\cdot f)\cdot(D_xf\cdot f')]\}\\&\quad+4D_x(D_tf'\cdot f)\cdot ff'+4D_y(D_tf'\cdot f)\cdot ff'-4D_z(D_zf'\cdot f)\cdot ff'\\&=D_x(3D_x^2D_yf'\cdot f+\lambda_1D_yf'\cdot f+\lambda_2f'f)\cdot ff'\\&\quad+D_x(3D_x^2f'\cdot f+\lambda_4D_yf'\cdot f+\lambda_6f'f)\cdot(D_yf\cdot f')\\&\quad+D_x(6D_xD_yf'\cdot f+6\lambda_7D_xf'\cdot f)\cdot(D_xf\cdot f')\\&\quad+D_y(D_x^3f'\cdot f-\lambda_1D_xf'\cdot f+\lambda_3f'f)\cdot ff'\\&\quad+D_y(3D_x^2f'\cdot f+\lambda_5D_xf'\cdot f-\lambda_6f'f)\cdot(D_xf\cdot f')\\&\quad+4D_x(D_tf'\cdot f)\cdot ff'+4D_y(D_tf'\cdot f)\cdot ff'\\&\quad-4D_z(D_zf'\cdot f+\lambda_8D_xf'\cdot f+\lambda_9D_yf'\cdot f)\cdot ff'\\&\quad+4D_x(\lambda_8D_zf'\cdot f)\cdot ff'+4D_y(\lambda_9D_zf'\cdot f)\cdot ff'\\&=D_x(B_1f'\cdot f)\cdot ff'+D_y(B_2f'\cdot f)\cdot ff'+D_x(B_3f'\cdot f)\cdot(D_yf\cdot f')\\&\quad+D_y(B_4f'\cdot f)\cdot(D_xf\cdot f')+6D_x(B_5f'\cdot f)\cdot(D_xf\cdot f')-4D_z(B_6f'\cdot f)\cdot ff'.\end{aligned}$$

In the above deduction, the coefficients of λ_2, λ_3, λ_4, λ_5 and λ_7 are zero because of $D_rg\cdot g=0$, and the coefficients of λ_1, λ_6, λ_8 and λ_9 are zero because of (2.10). This shows that (2.11) presents a Bäcklund transformation for the (3 + 1)-dimensional generalized bilinear KP equation (2.3).

2.2. Traveling wave solutions

Let us take a simple solution $f = 1$ to the $(3+1)$-dimensional generalized KP equation (2.3), which is transformed into the original variable u as $u = 2(\ln f)_x = 0$. Noting that

$$D_r^n g \cdot 1 = \frac{\partial^n}{\partial r^n} g, \quad n \geq 1,$$

the bilinear Bäcklund transformation (2.11) associated with $f = 1$ becomes a system of linear partial differential equations

$$\begin{cases} 3f'_{xxy} + 4f'_t + \lambda_1 f'_y + 4\lambda_8 f'_z + \lambda_2 f' = 0, \\ f'_{xxx} + 4f'_t - \lambda_1 f'_x + 4\lambda_9 f'_z + \lambda_3 f' = 0, \\ 3f'_{xx} + \lambda_4 f'_y + \lambda_6 f' = 0, \\ 3f'_{xx} + \lambda_5 f'_x - \lambda_6 f' = 0, \\ f'_{xy} + \lambda_7 f'_x = 0, \\ f'_z + \lambda_8 f'_x + \lambda_9 f'_y = 0. \end{cases} \tag{2.12}$$

Let us first consider a class of exponential wave solutions

$$f' = 1 + \varepsilon e^{kx+ly+mz-\omega t}, \tag{2.13}$$

where ε, k, l, m and ω are constants to be determined. Upon selecting

$$\lambda_2 = 0, \qquad \lambda_3 = 0, \qquad \lambda_6 = 0, \tag{2.14}$$

a direct computation tells

$$m = -(\lambda_8 k + \lambda_9 l), \qquad \omega = \frac{k^3 l - (\lambda_8 k + \lambda_9 l)^2}{k+l}, \tag{2.15}$$

and

$$\lambda_1 = \frac{k^3 - 3k^2 l + 4\lambda_8^2 k - 4\lambda_8\lambda_9(k-l) - 4\lambda_9^2 l}{k+l}, \qquad \lambda_4 = -\frac{3k^2}{l}, \qquad \lambda_5 = -3k, \qquad \lambda_7 = -l. \tag{2.16}$$

Therefore, we obtain a class of exponential wave solutions to the $(3+1)$-dimensional generalized bilinear KP equation (2.3):

$$f' = 1 + \varepsilon \exp\left[kx + ly - (\lambda_8 k + \lambda_9 l)z - \frac{k^3 l - (\lambda_8 k + \lambda_9 l)^2}{k+l} t\right], \tag{2.17}$$

where ε, k, l, λ_8 and λ_9 are arbitrary constants; and $u = 2(\ln f')_x$ solves the $(3+1)$-dimensional generalized KP equation (2.1).

Let us second consider a class of first-order polynomial solutions

$$f' = kx + ly + mz - \omega t, \tag{2.18}$$

where ε, k, l, m and ω are constants to be determined. Similarly upon selecting

$$\lambda_i = 0, \quad 2 \leq i \leq 7, \tag{2.19}$$

a direct computation shows that the system (2.12) becomes

$$\begin{cases} l\lambda_1 + 4m\lambda_8 - 4\omega = 0, \\ -k\lambda_1 + 4m\lambda_9 - 4\omega = 0, \\ k\lambda_8 + l\lambda_9 + m = 0. \end{cases} \tag{2.20}$$

Obviously, this system requires a necessary but not sufficient (see the last section for a counterexample) condition

$$(k+l)\omega + m^2 = 0 \tag{2.21}$$

for the existence of λ_1, λ_8 and λ_9. Under this condition (2.21), it is direct to check that f' defined by (2.18) solves the $(3+1)$-dimensional generalized bilinear KP equation (2.3), and so,

$$u = 2(\ln f')_x = \frac{2k}{kx + ly + mz - \omega t} \tag{2.22}$$

produces a class of rational solutions to the $(3+1)$-dimensional generalized KP equation (2.1).

3. Conclusions and remarks

We have computed a bilinear Bäcklund transformation for the (3 + 1)-dimensional generalized KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0.$$

The facts used in our construction are the exchange identities for Hirota's bilinear operators. The obtained bilinear Bäcklund transformation consists of six bilinear equations and involves nine arbitrary parameters. It is therefore a pretty large system, which in turn implies that the above (3 + 1)-dimensional generalized KP equation should have diverse solutions. Indeed, two classes of exponential and rational traveling wave solutions with arbitrary wave numbers have been generated from the proposed bilinear Bäcklund transformation.

It is interesting to note that the condition (2.21) has a solution

$$k = l = m = 0, \qquad \omega \neq 0,$$

but this makes it impossible to solve (2.20). Therefore, the corresponding function

$$f' = -\omega t$$

provides a solution for the generalized bilinear KP equation (2.3), but it is not generated from the bilinear Bäcklund transformation (2.11) associated with $f = 1$. It is actually a limit solution of the presented polynomial solutions.

We remark that the above (3 + 1)-dimensional generalized KP equation possesses linear subspaces of exponential wave solutions [6]. This shows a nice integrability property that nonlinear equations normally do not possess. One can also get some nonlinear superposition formulas of solutions generated from the proposed bilinear Bäcklund transformation [9,18], but it is hard to prove that the resulting functions are solutions due to a large number of different equations involved in the Bäcklund transformation. To overcome this complexity, once should find a bilinear Bäcklund transformation consisting of a small number of bilinear equations. However, it is a very difficult challenge for us to get a bilinear Bäcklund transformation defined by a system of less than six equations, for example, two or three equations for the above (3 + 1)-dimensional generalized KP equation. Some new specific exchange identities must be developed for use in merging terms resulted from $P = 0$. There might also be other equations different from $P = 0$ which one can begin with to formulate bilinear Bäcklund transformations.

Acknowledgments

The work was supported in part by the State Administration of Foreign Experts Affairs of China, the National Natural Science Foundation of China (Nos. 10831003, 61072147 and 11071159), Chunhui Plan of the Ministry of Education of China, Zhejiang Innovation Project (Grant No. T200905), the Natural Science Foundation of Shanghai (No. 09ZR1410800) and the Shanghai Leading Academic Discipline Project (No. J50101).

References

[1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[2] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, 2004.
[3] W.X. Ma, J.-H. Lee, A transformed rational function method and exact solutions to the 3+1 dimensional Jimbo–Miwa equation, Chaos Solitons Fractals 42 (2009) 1356–1363.
[4] W.X. Ma, T.W. Huang, Y. Zhang, A multiple exp-function method for nonlinear differential equations and its application, Phys. Scr. 82 (2010) 065003.
[5] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192–1194.
[6] W.X. Ma, E.G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011) 950–959.
[7] R. Hirota, A new form of Bäcklund transformations and its relation to the inverse scattering problem, Progr. Theoret. Phys. 52 (1974) 1498–1512.
[8] X.B. Hu, A Bäcklund transformation and nonlinear superposition formula of a modified Korteweg-De Vries-type bilinear equation, J. Math. Phys. 35 (1994) 4739–4745.
[9] R. Hirota, J. Satsuma, A simple structure of superposition formula of the Bäcklund transformation, J. Phys. Soc. Japan 45 (1978) 1741–1750.
[10] A. Nakamura, R. Hirota, Second modified KdV equation and its exact multi-soliton solution, J. Phys. Soc. Japan 48 (1980) 1755–1762.
[11] X.G. Geng, Y.L. Ma, N-soliton solution and its Wronskian form of a (3+1)-dimensional nonlinear evolution equation, Phys. Lett. A 369 (2007) 285–289.
[12] J.P. Wu, A bilinear Bäcklund transformation and explicit solutions for a (3 + 1)-dimensional soliton equation, Chin. Phys. Lett. 25 (2008) 4192–4194.
[13] W.X. Ma, Comment on the 3 + 1 dimensional Kadomtsev–Petviashvili equations, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2663–2666.
[14] F.C. You, T.C. Xia, D.Y. Chen, Decomposition of the generalized KP, cKP and mKP and their exact solutions, Phys. Lett. A 372 (2008) 3184–3194.
[15] W.X. Ma, A. Abdeljabbar, M.G. Asaad, Wronskian and Grammian solutions to a (3 + 1)-dimensional generalized KP equation, Appl. Math. Comput. 217 (2011) 8722–8730.
[16] W.X. Ma, T.C. Xia, Pfaffianization of a (3 + 1)-dimensional generalized KP equation, 2011 (submitted for publication).
[17] A.M. Wazwaz, Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 491–495.
[18] Y. Matsuno, Bilinear Transformation Method, Academic Press, Orlando, 1984.