PROCEEDINGS OF THE CONFERENCE ON

DIFFERENTIAL & DIFFERENCE EQUATIONS AND APPLICATIONS

EDITED BY
RAVI P. AGARWAL
AND KANISHKA PERERA

Hindawi Publishing Corporation
410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA
Nasr City Free Zone, Cairo 11816, Egypt
Fax: +1-866-HINDAWI (USA toll-free)

ISBN 977-5945-380

ferential equations with
, 633–647.
plications to population

al equations with state-
, 271–288.
ılsive functional differen-
ıputational and Applied

.terloo, ON,

'aterloo, ON,

MIXED RATIONAL-SOLITON SOLUTIONS TO THE TODA LATTICE EQUATION

WEN-XIU MA

We present a way to solve the Toda lattice equation using the Casoratian technique, and construct its mixed rational-soliton solutions. Examples of the resulting exact solutions are computed and plotted.

1. Introduction

Differential and/or difference equations serve as mathematical models for various phenomena in the real world. The study of differential and/or difference equations enhances our understanding of the phenomena they describe. One of important fundamental questions in the subject is how to solve differential and/or difference equations.

There are mathematical theories on existence and representations of solutions of linear differential and/or difference equations, especially constant-coefficient ones. Soliton theory opens the way to studies of nonlinear differential and/or difference equations. There are different solution methods for different situations in soliton theory, for example, the inverse scattering transforms for the Cauchy problems, Bäcklund transformations for geometrical equations, Darboux transformations for compatibility equations of spectral problems, Hirota direct method for bilinear equations, and truncated series expansion methods (including Painlevé series, and sech and tanh function expansion methods) for Riccati-type equations.

Among the existing methods in soliton theory, Hirota bilinear forms are one of the most powerful tools for solving soliton equations, a kind of nonlinear differential and/or difference equations. In this paper, we would like to construct mixed rational-soliton solutions to the Toda lattice equation:

$$\dot{a}_n = a_n(b_{n-1} - b_n), \qquad \dot{b}_n = a_n - a_{n+1}, \tag{1.1}$$

where $\dot{a}_n = da_n/dt$ and $\dot{b}_n = db_n/dt$. The approach we will adopt to solve this equation is the Casoratian technique. Its key is to transform bilinear forms into linear systems

Hindawi Publishing Corporation
Proceedings of the Conference on Differential & Difference Equations and Applications, pp. 711–720

of solvable differential-difference equations. For the Toda lattice equation (1.1), we will present the general solutions to the corresponding linear systems and further generate mixed rational-soliton solutions.

2. Constructing solutions using the Casoratian technique

Let us start from the Toda bilinear form. Under the transformation

$$a_n = 1 + \frac{d^2}{dt^2}\log\tau_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad b_n = \frac{d}{dt}\log\frac{\tau_n}{\tau_{n+1}} = \frac{\dot{\tau}_n\tau_{n+1} - \tau_n\dot{\tau}_{n+1}}{\tau_n\tau_{n+1}}, \tag{2.1}$$

the Toda lattice equation (1.1) becomes

$$\left[D_t^2 - 4\sinh^2\left(\frac{D_n}{2}\right)\right]\tau_n\cdot\tau_n = 0, \tag{2.2}$$

where D_t and D_n are Hirota's operators. That is,

$$\ddot{\tau}_n\tau_n - \left(\dot{\tau}_n\right)^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2 = 0. \tag{2.3}$$

In the Casoratian formulation, we use the Casorati determinant

$$\tau_n = \operatorname{Cas}(\phi_1,\phi_2,\dots,\phi_N) = \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix} \tag{2.4}$$

to construct exact solutions, and we call such solutions Casoratian solutions. It is known [7] that such a τ-function τ_n solves the bilinear Toda lattice equation (2.3) if

$$\phi_i(n+1) + \phi_i(n-1) = 2\varepsilon_i(\cosh\alpha_i)\phi_i(n), \qquad (\phi_i(n))_t = \phi_i(n+1), \tag{2.5}$$

where $\varepsilon_i = \pm 1$, α_i are nonzero constants and $(\phi_i(n))_t = \partial_t\phi_i(n) = \partial_t\phi_i(n,t)$. The resulting solutions are negatons, that is, a kind of solutions only involving exponential functions of the space variable n.

There are other type solutions such as rational solutions [3], positons [5, 8], and complexitons [2]. Similar to [2, 3], we can prove that τ_n is a solution to the bilinear Toda lattice equation (2.3) if

$$\phi_i(n+1) + \phi_i(n-1) = \sum_{j=1}^{N}\lambda_{ij}\phi_j(n), \qquad (\phi_i(n))_t = \zeta\phi_i(n+\delta), \tag{2.6}$$

where $\zeta = \pm 1$, $\delta = \pm 1$ (i.e., $|\zeta| = |\delta| = 1$, $\zeta,\delta\in\mathbb{R}$), and λ_{ij} are arbitrary constants. Under the transformation $t \to -t$, the bilinear Toda lattice equation (2.3) is invariant, and $(\phi_i(n))_t = \phi_i(n+\delta)$ becomes $(\phi_i(n))_t = -\phi_i(n+\delta)$. Therefore, we only need to consider one of the cases $\zeta = \pm 1$ while constructing solutions, since the replacement of t with $-t$ generates solutions from one case to the other. We will only consider the case of $\zeta = 1$ below.

Let us now begin to analyze the system of differential-difference equations (2.6) with $\zeta = 1$. The corresponding system can be compactly written as

$$\Phi_N(n+1,t)+\Phi_N(n-1,t)=\Lambda\Phi_N(n,t), \qquad \left(\Phi_N(n,t)\right)_t=\Phi_N(n+\delta,t), \tag{2.7}$$

where $\Phi_N=\Phi_N(n,t):=(\phi_1(n,t),\dots,\phi_N(n,t))^T$ and $\Lambda:=(\lambda_{ij})_{N\times N}$. Note that a constant similar transformation for the coefficient matrix Λ does not change the resulting Casoratian solution. Actually, if we have $M=P^{-1}\Lambda P$ for an invertible constant matrix P, then $\widetilde{\Phi}_N=P\Phi_N$ satisfies

$$\widetilde{\Phi}_N(n+1,t)+\widetilde{\Phi}_N(n-1,t)=M\widetilde{\Phi}_N(n,t), \qquad \left(\widetilde{\Phi}_N(n,t)\right)_t=\widetilde{\Phi}_N(n+\delta,t). \tag{2.8}$$

Obviously, the Casorati determinants generated from Φ_N and $\widetilde{\Phi}_N$ have just a constant-factor difference, and thus the transformation (2.1) leads to the same Casoratian solutions from Φ_N and $\widetilde{\Phi}_N$. Therefore, as in the KdV case [4], we can focus on the following two types of Jordan blocks of Λ:

$$\begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 1 & \lambda_i \end{bmatrix}_{k_i\times k_i}, \qquad \begin{bmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & I_2 & A_i \end{bmatrix}_{l_i\times l_i}, \quad A_i=\begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}, \tag{2.9}$$

where λ_i, α_i and $\beta_i>0$ are all real constants, I_2 is the identity matrix of order 2, and k_i and l_i are positive integers. A Jordan block of the first type has the real eigenvalue λ_i with algebraic multiplicity k_i, and a Jordan block of the second type has the pair of complex eigenvalues $\lambda_{i,\pm}=\alpha_i\pm\beta_i\sqrt{-1}$ with algebraic multiplicity l_i.

Case $k_i = 1$ of type 1. The representative systems read as follows:

$$\phi_i(n+1)+\phi_i(n-1)=\lambda_i\phi_i(n), \qquad \left(\phi_i(n)\right)_t=\phi_i(n+\delta), \tag{2.10}$$

where $\delta=\pm 1$ and $\lambda_i=$ consts. Their eigenfunctions are classified as

$$\begin{aligned} &\phi_i=C_{1i}\varepsilon_i^n e^{\varepsilon_i t}+C_{2i}(n+\varepsilon_i\delta t)\varepsilon_i^n e^{\varepsilon_i t}, \quad \lambda_i=2\varepsilon_i, \quad \varepsilon_i=\pm 1,\\ &\phi_i=C_{1i}e^{t\cos\alpha_i}\cos(\alpha_i n+\delta t\sin\alpha_i)\\ &\qquad +C_{2i}e^{t\cos\alpha_i}\sin(\alpha_i n+\delta t\sin\alpha_i), \quad \lambda_i=2\cos\alpha_i, \quad \alpha_i\neq m\pi,\ m\in\mathbb{Z},\\ &\phi_i=C_{1i}\varepsilon_i^n e^{\alpha_i n+\varepsilon_i t e^{\delta\alpha_i}}+C_{2i}\varepsilon_i^n e^{-\alpha_i n+\varepsilon_i t e^{-\delta\alpha_i}}, \quad \lambda_i=2\varepsilon_i\cosh\alpha_i, \quad \alpha_i\neq 0, \end{aligned} \tag{2.11}$$

where C_{1i} and C_{2i} are arbitrary constants. The above three sets of eigenfunctions generate rational solutions, positon solutions, and negaton solutions, respectively.

Generally, the following two results provide ways to solve the linear system (2.6). The detailed proof will be published elsewhere.

Theorem 2.1 ($\lambda_i = \pm 2$). *Let $\varepsilon = \pm 1$ and $\delta = \pm 1$ (i.e., $|\varepsilon| = |\delta| = 1$, $\varepsilon, \delta \in \mathbb{R}$). If $(f(n,t))_t = f(n+\delta, t)$, then the nonhomogeneous system*

$$\phi(n+1,t) + \phi(n-1,t) = 2\varepsilon\phi(n,t) + f(n,t), \qquad (\phi(n,t))_t = \phi(n+\delta,t), \tag{2.12}$$

has the general solution

$$\phi(n,t) = \left[\alpha(n)t + \beta(n) + \int_0^t \int_0^s f(n+\delta, r)e^{-\varepsilon r}\,dr\,ds\right]e^{\varepsilon t}, \tag{2.13}$$

where $\alpha(n)$ and $\beta(n)$ are determined by

$$\alpha(n+\delta) - \varepsilon\alpha(n) = f(n+\delta, 0), \qquad \beta(n+\delta) - \varepsilon\beta(n) = \alpha(n). \tag{2.14}$$

Theorem 2.2 ($\lambda_i \neq \pm 2$). *Let $\lambda \neq \pm 2$ and $\delta = \pm 1$.*

(a) *The homogeneous system*

$$\phi(n+1,t) + \phi(n-1,t) = \lambda\phi(n,t), \qquad (\phi(n,t))_t = \phi(n+\delta,t), \tag{2.15}$$

has its general solution

$$\phi(\lambda; c, d)(n) = c\omega^n e^{t\omega^\delta} + d\omega^{-n}e^{t\omega^{-\delta}}, \tag{2.16}$$

where c and d are arbitrary constants, and

$$\lambda = \omega + \omega^{-1}, \tag{2.17}$$

that is,

$$\omega^2 - \lambda\omega + 1 = 0. \tag{2.18}$$

(b) *Define $f_k = \phi(\lambda_i; c_k, d_k)$, $1 \le k \le k_i$, where c_k and d_k are arbitrary constants. The nonhomogeneous system*

$$\phi_k(n+1,t) + \phi_k(n-1,t) = \lambda_i\phi_k(n,t) + \phi_{k-1}(n,t), \qquad (\phi_k(n,t))_t = \phi_k(n+\delta,t), \tag{2.19}$$

where $1 \le k \le k_i$ and $\phi_0 = 0$, has its general solution

$$\phi_k = \sum_{p=0}^{k-1} \frac{1}{p!}\frac{\partial^p f_{k-p}}{\partial\lambda_i^p} = \sum_{p=0}^{k-1} \frac{1}{p!}\frac{\partial^p \phi(\lambda_i; c_{k-p}, d_{k-p})}{\partial\lambda_i^p}, \quad 1 \le k \le k_i. \tag{2.20}$$

Remark 2.3. The soliton case of $\lambda_i = 2\cosh\alpha_i$ ($\alpha_i \neq 0$) corresponds to $\omega_i = e^{\alpha_i}$ in (a). The nonhomogeneous system in (b) is associated with one Jordan block of type 1.

Begin with

$$\Lambda = \begin{bmatrix} 2\varepsilon & & & 0 \\ * & 2\varepsilon & & \\ \vdots & \ddots & \ddots & \\ * & \cdots & * & 2\varepsilon \end{bmatrix}_{k_i \times k_i}, \quad *\text{-arbitrary consts.}, \tag{2.21}$$

where $\varepsilon = \pm 1$. By the general solution formula in Theorem 2.1, we can have

$$\phi_i(n,t) = \varepsilon^n e^{\varepsilon t} \psi_i(n,t), \quad 1 \le i \le N, \tag{2.22}$$

where $\psi_i(n,t)$ are polynomials in n and t. Thus,

$$\tau_n = \operatorname{Cas}(\psi_1, \ldots, \psi_N) \tag{2.23}$$

presents polynomial solutions to the bilinear Toda lattice equation (2.3) and thus rational solutions to the Toda lattice equation (1.1) through (2.1).

Theorem 2.4. *The Jordan block of type 1 with $\lambda_i = \pm 2$ leads to rational solutions to the Toda lattice equation (1.1).*

This adds one case of $\lambda = -2$ to the result in [3]. A few examples of rational solutions associated with one Jordan block case with $\lambda = 2$ were presented in [3].

3. Mixed rational-soliton solutions

Let us now show a way to construct mixed rational-soliton solutions. We use the following procedure.

Step 1. Solve the triangular systems whose coefficient matrices possess Jordan blocks of type 1 with $\lambda_i = \pm 2$ or $\lambda_i = 2\cosh \alpha_i$ ($\alpha_i \ne 0$) to form a set of eigenfunctions $(\phi_1, \ldots, \phi_N)$.

Step 2. Evaluate the τ-function $\tau_n = \operatorname{Cas}(\phi_1, \ldots, \phi_N)$.

Step 3. Evaluate a_n and b_n by the transformation (2.1), to obtain mixed rational-soliton solutions to the Toda lattice equation (1.1).

The τ-functions generated above are quite general. In what follows, we would like to present two sets of special eigenfunctions required in forming such τ-functions.

Special eigenfunctions yielding rational solutions. We take two specific Taylor expansions as in [9]:

$$\begin{aligned} \phi_+(n,t) &= e^{kn+te^k} + e^{-kn+te^{-k}} = \sum_{i=0}^{\infty} a_{i+1}(n,t) k^{2i}, \\ \phi_-(n,t) &= e^{kn+te^k} - e^{-kn+te^{-k}} = \sum_{i=0}^{\infty} a_{i+1}(n,t) k^{2i+1}, \end{aligned} \tag{3.1}$$

the coefficients of which satisfy

$$a_i(n+1,t) + a_i(n-1,t) = \sum_{j=0}^{i} \frac{2}{(2j)!} a_{i-j+1}(n,t), \qquad (a_i(n,t))_t = a_i(n+1,t), \tag{3.2}$$

where $i \ge 1$. These two sets of functions are given by

$$a_{i+1}(n,t) = e^t \sum_{j=0}^{2i} \frac{2n^{2i-j}}{(2i-j)!} \beta_j(t), \qquad a_{i+1}(n,t) = e^t \sum_{j=0}^{2i+1} \frac{2n^{2i+1-j}}{(2i+1-j)!} \beta_j(t), \tag{3.3}$$

respectively. The functions $\beta_j(t)$ above are defined by

$$\sum_{j=0}^{\infty}\beta_j(t)k^j = \sum_{p=0}^{\infty}\frac{t^p}{p!}\left(\sum_{q=1}^{\infty}\frac{1}{q!}k^q\right)^p. \tag{3.4}$$

This is a rational solution case, since $\lambda_{ii} = 2$, $1 \le i \le N$.

Special eigenfunctions yielding solitons. We start from the same eigenfunctions

$$\begin{aligned}\phi_+(n,t) &= \mathrm{e}^{kn+te^k} + \mathrm{e}^{-kn+te^{-k}} = 2\mathrm{e}^{t\cosh k}\cosh(kn+t\sinh k),\\ \phi_-(n,t) &= \mathrm{e}^{kn+te^k} - \mathrm{e}^{-kn+te^{-k}} = 2\mathrm{e}^{t\sinh k}\cosh(kn+t\sinh k),\end{aligned} \tag{3.5}$$

which solve

$$\phi(n+1,t)+\phi(n-1,t) = 2(\cosh k)\phi(n,t), \qquad (\phi(n,t))_t = \phi(n+\delta,t). \tag{3.6}$$

Computing derivatives of the above system with the parameter k leads to a set of eigenfunctions as follows:

$$b_i(n,t) = \partial_k^{i-1}\phi(n,t), \quad 1 \le i \le N, \tag{3.7}$$

which satisfies

$$b_i(n+1,t)+b_i(n-1,t) = \sum_{j=1}^{i}\lambda_{ij}b_j(n,t), \qquad (b_i(n,t))_t = b_i(n+\delta,t), \tag{3.8}$$

where $1 \le i \le N$ and $\lambda_{ij} = 2\binom{i-1}{j-1}(\partial_k^{i-j}\cosh k)$. This is a soliton case if $k \ne 0$, since $\lambda_{ii} = 2\cosh k$, $1 \le i \le N$.

Examples of mixed rational-soliton solutions. Mixed rational-soliton solutions can now be computed, for example, by

$$\tau_n = \mathrm{Cas}\,(\mathrm{e}^{\pm t}a_1,\dots,\mathrm{e}^{\pm t}a_p,b_1,\dots,b_{N-p}). \tag{3.9}$$

In particular (in the case of ϕ_+), we have

$$\begin{aligned}\tau_n &= \mathrm{Cas}\,(\mathrm{e}^{-t}a_1,b_1) = 4\mathrm{e}^{t\cosh k}\{\cosh[k(n+1)+t\sinh k]-\cosh(kn+t\sinh k)\},\\ \tau_n &= \mathrm{Cas}\,(\mathrm{e}^{-t}a_1,\mathrm{e}^{-t}a_2,b_1)\\ &= 4\mathrm{e}^{t\cosh k}\{2(n+t+1)\cosh[k(n+2)+t\sinh k]\\ &\qquad -4(2n+2t+1)\cosh[k(n+1)+t\sinh k]\\ &\qquad +(2n+2t+3)\cosh(kn+t\sinh k)\}.\end{aligned} \tag{3.10}$$

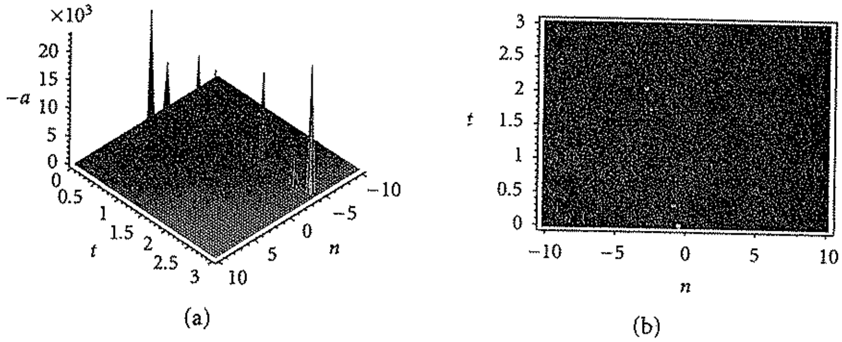

Figure 3.1. 3D and density plots of a_n.

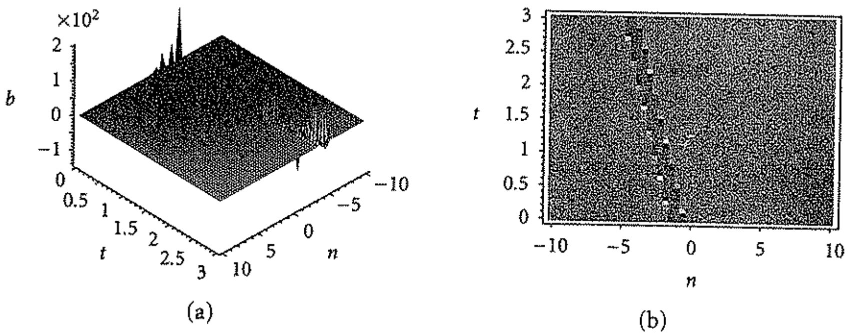

Figure 3.2. 3D and density plots of b_n.

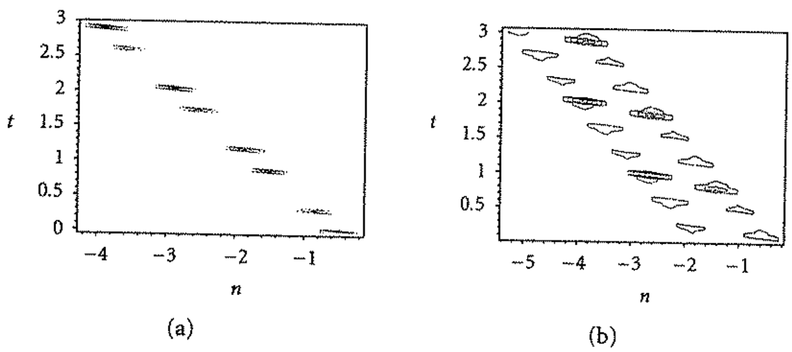

Figure 3.3. Contour plots of (a) a_n and (b) b_n.

The solution from $\tau_n = \text{Cas}(e^{-t}a_1, b_1)$ with $k = 1$ is depicted in Figures 3.1, 3.2, and 3.3, and the solution from $\tau_n = \text{Cas}(e^{-t}a_1, e^{-t}a_2, b_1)$ with $k = -1$ in Figures 3.4, 3.5, and 3.6.

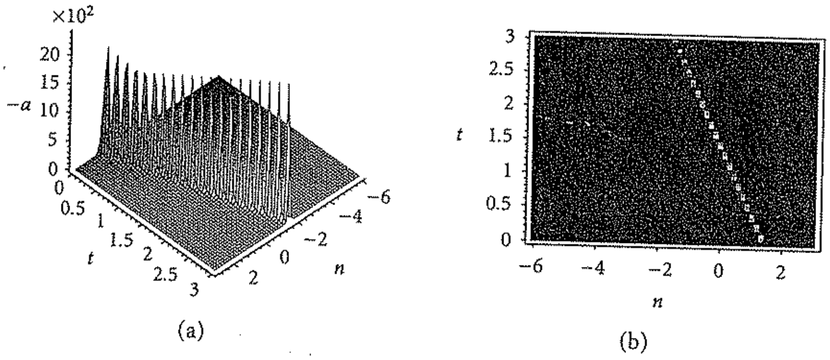

Figure 3.4. 3D and density plots of a_n.

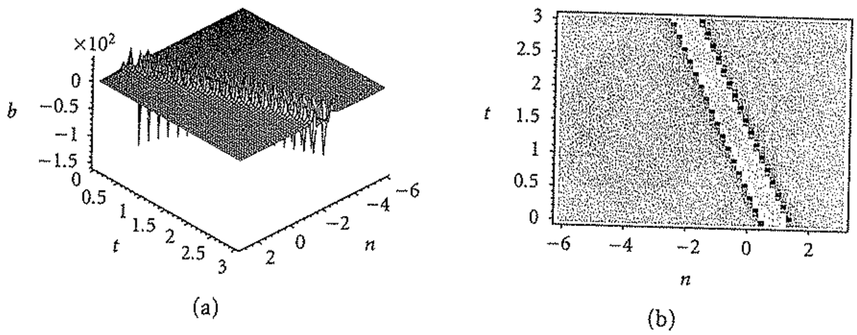

Figure 3.5. 3D and density plots of b_n.

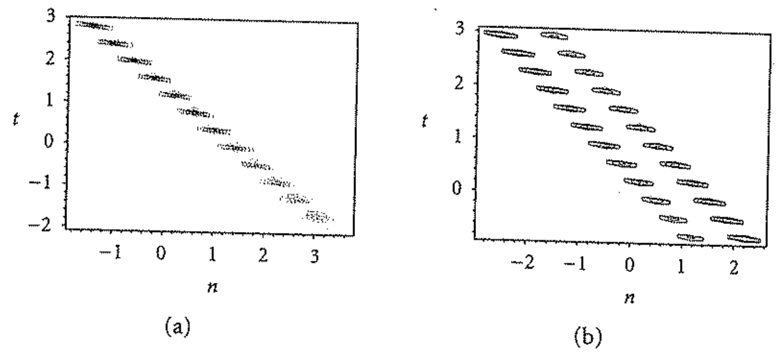

Figure 3.6. Contour plots of (a) a_n and (b) b_n.

4. Discussions

A careful analysis based on Theorems 2.1 and 2.2 can prove that the Jordan blocks of type 1 with $\lambda_i = \pm 2$, $|\lambda_i| > 2$, and $|\lambda_i| < 2$ generate rational solutions, negatons, and positons,

respectiv
generate
tions:

where ζ =
this case

where (f
rational-s
The ab
than the
The basic
Volterra l

The trans
following

The Caso

where 1 ≤
Casoratian
of constru

Acknowle

The autho
sions.

respectively; and that the Jordan blocks of type 2, which possess complex eigenvalues, generate complexitons. Moreover, we can have another case of conditions on eigenfunctions:

$$\begin{aligned} \phi_i(n+1)+\phi_i(n-1) &= \sum_{j=1}^{N}\lambda_{ij}\phi_j(n), \\ (\phi_i(n,t))_t &= \frac{1}{2}\zeta(\phi_i(n+1,t)-\phi_i(n-1,t)), \end{aligned} \tag{4.1}$$

where $\zeta = \pm 1$ and λ_{ij} are arbitrary constants. An analysis is left for future publication, on this case of conditions and its representative system

$$\begin{aligned} \phi(n+1,t)+\phi(n-1,t) &= \lambda\phi(n,t)+f(n,t), \\ (\phi(n,t))_t &= \frac{1}{2}\phi(n+1,t)-\frac{1}{2}\phi(n-1,t), \end{aligned} \tag{4.2}$$

where $(f(n,t))_t = 1/2f(n+1,t)-(1/2)f(n-1,t)$, which will lead to different mixed rational-soliton solutions to the Toda lattice equation (1.1).

The above construction of mixed rational-soliton solutions is direct and much easier than the existing approach by computing long-wave limits of soliton solutions [1, 6]. The basic idea can also be applied to other integrable lattice equations, for example, the Volterra lattice equation:

$$\dot{u}_n = u_n(u_{n+1}-u_{n-1}). \tag{4.3}$$

The transformation of $u_n = \tau_{n+2}\tau_{n-1}/\tau_{n+1}\tau_n$ puts the Volterra lattice equation into the following bilinear form:

$$\dot{\tau}_{n+1}\tau_n - \tau_{n+1}\dot{\tau}_n - \tau_{n+2}\tau_{n-1} + \tau_{n+1}\tau_n = 0. \tag{4.4}$$

The Casorati determinant $\tau_n = \mathrm{Cas}(\phi_1,\ldots,\phi_N)$ solves this equation if

$$\phi_i(n+1,t)+\phi_i(n-1,t) = \sum_{j=1}^{N}\lambda_{ij}\phi_j(n,t), \qquad (\phi_i(n,t))_t = \phi_i(n+2,t), \tag{4.5}$$

where $1 \le i \le N$ and λ_{ij} are arbitrary constants. Therefore, this allows us to construct Casoratian solutions to the Volterra lattice equation in a simple and direct way. The details of constructing Casoratian solutions will be published elsewhere.

Acknowledgment

The author would like to thank Professors K. Maruno and Y. You for stimulating discussions.

f type
itons,

References

[1] A. S. Cârstea and D. Grecu, *On a class of rational and mixed soliton-rational solutions of the Toda lattice*, Progress of Theoretical Physics **96** (1996), no. 1, 29–36.

[2] W. X. Ma and K.-I. Maruno, *Complexiton solutions of the Toda lattice equation*, Physica A **343** (2004), no. 1–4, 219–237.

[3] W. X. Ma and Y. You, *Rational solutions of the Toda lattice equation in Casoratian form*, Chaos, Solitons & Fractals **22** (2004), no. 2, 395–406.

[4] ———, *Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions*, Transactions of the American Mathematical Society **357** (2005), no. 5, 1753–1778.

[5] K. Maruno, W. X. Ma, and M. Oikawa, *Generalized Casorati determinant and positon-negaton-type solutions of the Toda lattice equation*, Journal of the Physical Society of Japan **73** (2004), 831–837.

[6] K. Narita, *Solutions for the Mikhailov-Shabat-Yamilov difference-differential equations and generalized solutions for the Volterra and the Toda lattice equations*, Progress of Theoretical Physics **99** (1998), no. 3, 337–348.

[7] J. J. C. Nimmo, *Soliton solution of three differential-difference equations in Wronskian form*, Physics Letters. A **99** (1983), no. 6-7, 281–286.

[8] A. A. Stahlhofen and V. B. Matveev, *Positons for the Toda lattice and related spectral problems*, Journal of Physics. A. Mathematical and General **28** (1995), no. 7, 1957–1965.

[9] H. Wu and D.-J. Zhang, *Mixed rational-soliton solutions of two differential-difference equations in Casorati determinant form*, Journal of Physics. A. Mathematical and General **36** (2003), no. 17, 4867–4873.

Wen-Xiu Ma: Department of Mathematics, University of South Florida, Tampa, FL 33620, USA
E-mail address: mawx@cas.usf.edu