Encyclopedia of
Nonlinear Science

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In the path integral approach to quantum field theory, physical values are derived by certain weighted integrations over all possible field configurations. These path integrals are often intractable, and it is common to expand the integration about the stationary points of the action. This works because of the way the action appears in the integrand of the path integral. Calculations of this type are known as semi-classical calculations because they are effectively an expansion in Planck's constant, $\hbar$. This expansion must include a sum over all possible stationary points, and so, in Yang–Mills theory, it includes a sum over instanton configurations. By studying the symmetry properties of the measure in quantum chromodynamics (QCD), the Yang–Mills theory describing the strong nuclear force, it can be shown that terms in this expansion break the axial U(1) symmetry. This symmetry is unbroken if the instanton terms are omitted. The symmetry breaking allows processes that violate baryon and lepton conservation; however, the amplitudes for these effects are highly suppressed. One useful approach is to consider these processes as tunneling events between different vacua, in fact, instanton calculations in quantum field theory are very similar to Wentzel–Kramers–Brillouin (WKB) calculations in quantum mechanics.

While instantons provide a qualitative explanation for a host of phenomena in quantum chromodynamics, useful quantitative results are not available within the semi-classical approach, and even qualitatively, instanton calculations are not rigorous since they relate only to the semi-classical approximation, a truncation of the full quantum theory. The modern approach to quantum chromodynamics is lattice QCD. It is possible within lattice QCD to verify the original ideas about the role of instantons in the physics of the strong force; however, there are limits to the precision with which the lattice and semi-classical approaches can be compared (Negele, 1998).

Finite action soliton solutions in other equation systems are sometimes referred to as instantons. Examples include the finite action solutions to the forced Burgers equation, which arises in the study of turbulence, and the vortex-like solutions in the abelian Higgs model, which is related to condensed matter physics. Instantons have many similarities to the lump solitons found in certain sigma models.

CONOR HOUGHTON

See also **Burgers equation; Higgs boson; Particles and antiparticles; Quantum field theory; Solitons; Yang–Mills theory**

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INTEGRABILITY

Central problems in the integrability theory for nonlinear ordinary differential equations (ODEs) or partial differential equations (PDEs) are knowing which systems can be solved analytically and developing appropriate solution techniques. Although in some contexts the term *integrability* is not well defined, let us begin by considering two physically motivated nonlinear ODE systems that can be analytically solved.

Nonlinear Pendulum

Here, the nonlinear system is

$$\ddot{x} = ax + bx^2 + cx^3, \quad a, b, c = \text{constants}. \qquad (1)$$

Multiplying both sides of this equation by $\dot{x}$ and integrating yields an intermediate integral $I = \dot{x}^2/2 - ax^2/2 - bx^3/3 - cx^4/4$, which is determined by the initial conditions: $x(t_0)$ and $\dot{x}(t_0)$. In terms of I, the solution $x(t)$ can be written implicitly as

$$t - t_0 = \int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{2I + ax^2 + 2bx^3/3 + cx^4/2}}, \qquad (2)$$

where $x(t_0) = x_0$. The inversion of this formula is expressed in terms of Jacobi elliptic functions (Byrd & Friedman, 1954). Note that this system has one degree of freedom, and one intermediate integral was sufficient to obtain a solution. Because it is related to an area, the intermediate integral (I) is sometimes called a quadrature, and by extension, Equation (2) is said to be obtained by the method of quadratures.

Calogero–Moser N-Body Problem

In this example, the nonlinear system is

$$\ddot{q}_n = \sum_{m=1, m\neq n}^{N} 2g^2(q_n - q_m)^{-3}, \quad g = \text{constant}, \quad (3)$$

generalizations of which were studied extensively in the last three decades of the 20th century. Using a Lax isospectral deformation technique, the solution matrix $\text{diag}[q_1(t), \ldots, q_N(t)]$ is found to be similar to the matrix $\tilde{Q} = [\tilde{Q}_{nm}(t)]$ defined by

$$\tilde{Q}_{nm}(t) = \delta_{nm}\big[q_n(0) + \dot{q}_n(0)t\big] + i(1 - \delta_{nm})g\big[q_n(0) - q_m(0)\big]^{-1}t. \quad (4)$$

Thus, the solution to the initial value (Cauchy) problem is reduced to the algebraic task of finding the N eigenvalues of the matrix $\tilde{Q}$, and the solution formula for initial data, $q_i(0)$ and $\dot{q}_i(0)$, can be constructed from linear functions, through a finite number of algebraic operations and compositions of functions (Calogero, 2001). Equations (1) and (3) are completely integrable.

In the 19th century, Evariste Galois, Niels Henrik Abel, Joseph Liouville, and Sophus Lie tried to rationalize the process of solving differential equations by quadratures, or failing that, massaging them in such a way that useful information might be extracted. Mainly based on geometry and algebra, these efforts initiated much of the mathematics—such as classifications of differential equations in term of symmetries and conservation laws—that has dominated the 20th century. Attempts by Karl Weierstrass and Henri Poincaré to create a systematic theory of integrability are based on complex function theory.

An achievement of 19th-century mathematics was the elaboration of the theory of elliptic and Abelian functions, particularly the introduction of theta functions. A solution comprising such rapidly convergent power series is quite efficient computationally; thus, a natural question for 19th-century mathematicians was this: Which differential equations admit solutions as quotients of power series that converge in large regions, independent of initial or boundary conditions? The Cauchy–Kovalevsky theorem gives local existence for such expressions. This theme traces through the work of Sophia Kovalevsky and Paul Painlevé for nonlinear ODEs and through Bernhard Riemann and Immanuel Fuchs and their successors for linear ODEs (Hermann, 1984).

There are several results about local and global solutions of differential equations that establish existence, uniqueness, smoothness, stability, approximations, and so on; yet attempts to find exact solution formulas in the 19th century were largely unsuccessful. This is because solutions to nonlinear systems may exhibit complex behaviors—such as blow-up, shocks, chaos, fractals, and bifurcation—which are obstacles to integrability. Poincaré put an end to the search for new integrable equations by showing that among all dynamical systems integrable ones are exceptional. Indeed, a small structural perturbation of an integrable system often destroys integrability.

Although Poincaré's results dampened interest in the search for new integrable equations during the first half of the 20th century, the situation changed dramatically with the discovery of the inverse scattering transform as a method of solving the initial value (Cauchy) problem for the Korteweg–de Vries (KdV) equation (Gardner et al., 1967). Rapidly extended to several other nonlinear systems of scientific interest (nonlinear Schrödinger equation, sine-Gordon equation, Toda lattice, and so on), this discovery also led to the emergence of "soliton factories" during the 1980s (Zakharov, 1991).

More generally, integrability theory aims to find global information on solutions and, if possible, to solve differential equations analytically. Integrability itself is an intrinsic characteristic of differential equations, imposing constraints on the way solutions evolve in phase space and suggesting the following working definition.

Definition A differential equation is completely integrable if all solutions to well-posed initial or boundary value problems can be presented beginning with elementary functions, using finitely many algebraic operations and compositions of functions, and evaluating limits.

Algebraic operations include inverting or diagonalizing matrices; compositions of functions include inverses; and the limits can be integrals, infinite series, or asymptotes. Thus, this definition holds for cases in which solutions can be constructed explicitly. It also generalizes the notion of integrability by quadratures (Liouville integrability) that only requires computation of intermediate integrals in addition to algebraic operations and compositions of functions.

For ODEs, the Liouville–Arnol'd theorem on finite-dimensional Hamiltonian systems gives sufficient conditions, called the Liouville conditions, for guaranteeing the integrability by quadratures (Arnol'd, 1989). For a Hamiltonian system of N degrees of freedom, these conditions are:

- the existence of N integrals of motion $\{F_1, F_2, \ldots, F_N\}$,
- that are functionally independent on the level surface $\{F_i = a_i\}$ containing the initial data, and
- that commute with each other under the associated Poisson bracket.

If the level surface $\{F_i = a_i\}$ is compact and connected, then the Liouville–Arnol'd theorem says that the underlying Hamiltonian system can be expressed as

$$\dot{I}_i = 0, \; \dot{\varphi}_i = \omega_i(I_1, \ldots, I_N), \quad (5)$$

in the action-angle coordinate system (I_i, φ_i) of a neighborhood of the level surface. Thus, the motion is conditionally periodic along an N-dimensional torus with the frequencies ω_i, and the solution to the Cauchy problem can be completely determined by the method of quadratures. The Euler top, the Lagrange top, Kowalewski's top, the Stäckel systems, and geodesic flows on a surface are Liouville-integrable systems (Perelomov, 1990).

For PDEs, there also exist some results on integrability properties from a Hamiltonian perspective, including the bi-Hamiltonian theory (Magri, 1978) from which infinitely many symmetries and conservation laws for PDEs can be deduced. Motivated by the Liouville–Arnol'd theorem, a notion of integrability for PDEs is the existence of infinitely many conservation laws. The KdV equation is integrable in this sense, and it can also be put into a Hamiltonian form in action-angle coordinates like integrable ODEs (Zakharov & Faddeev, 1971).

The existence of infinitely many conservation laws can lead to infinitely many finite-dimensional solution varieties that can be determined analytically (Fuchssteiner, 1992), but it is not clear whether these subvarieties generate the whole infinite-dimensional solution variety. Indeed, the infinitely many degrees of freedom of PDEs introduce large diversity, obscuring their solvability and integrability properties.

The notion of partial integrability arises when systems of differential equations (both ODEs and PDEs) possess more degrees of freedom than conservation laws (Conte & Boccara, 1990). Symmetry constraints developed in soliton theory (Ma & Zhou, 2001) suggest a way to show partial integrability for PDEs through relating PDEs to integrable ODEs. The method was motivated by Moser's work on Hill's equation (Moser, 1980) and nonlinearization of Lax pairs (Cao, 1990). The results generalize the theory of finite-dimensional integrable stationary equations, suggesting the possibility of establishing a Liouville–Arnol'd theorem for infinite-dimensional Hamiltonian systems.

Since the 1970s, several criteria have arisen for testing the integrability of nonlinear PDEs, and corresponding theories include infinitely many symmetries, infinitely many conservation laws, Lax structure, bi-Hamiltonian structure, the Bäcklund transform, Hirota's bilinear form, the inverse scattering transform, and the Painlevé property (Degasperis, 1998; Gu, 1995).

An ODE or a PDE is said to possess the Painlevé property, respectively, if the movable singularities of solutions of the ODE are only poles (Ablowitz & Clarkson, 1991) or if solutions of the PDE are "single-valued" in the neighborhood of noncharacteristic, movable singularity manifolds (Weiss et al., 1983), suggesting that the Painlevé property is an indication of complete integrability (Conte, 1999). One advantage of the Painlevé property is that it can be a tool for computing Lax pairs and conservation laws. Lax pairs, as we have seen in Equation (3), provide linear objects for solving Cauchy problems of nonlinear equations. A thorough understanding of the relationship between Painlevé singularities and integrability may suggest alternatives to the KAM notion of near-integrability (Zakharov, 1991).

The term S-integrability implies integrability of Cauchy problems by the inverse spectral transform technique (Calogero & Degasperis, 1982) or the inverse scattering transform (Ablowitz & Clarkson, 1991; Fokas, 1997). C-integrability, on the other hand, means that a differential equation can be transformed into a linear one by an appropriate change of variables. Examples of C-integrable equations are the Burgers equation and the Calogero–Dagasperis–Ibragimov–Shabat equation:

$$u_t = u_{xx} - 2uu_x, \qquad \text{(B)}$$
$$u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x. \qquad \text{(CDIS)}$$

These two equations are, respectively, cast into the heat equation ($v_t = v_{xx}$) and the linear KdV equation ($v_t = v_{xxx}$) under appropriate dependent variable transformations. Calogero and colleagues have made a systematic study of the construction and classification of C-integrable equations, both in $1 + 1$ dimensions and $N + 1$ dimensions (Zakharov, 1991). Although the property of C-integrability often leads to an infinity of conservation laws, the Burgers equation has only one local conservation law of differential polynomial type.

Wen-Xiu Ma

See also **Burgers equation; Constants of motion and conservation laws; Hamiltonian systems; Hirota's method; Inverse scattering method or transform; Painlevé analysis; Solitons**

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INTEGRABLE CELLULAR AUTOMATA

Integrable cellular automata (CAs) are discrete dynamical systems that possess attributes of integrability including conserved quantities, symmetries, and localized solutions. They show how to discretize a given integrable system of differential equations, maintaining the integrability property, suggesting that integrability of discrete systems and their coherent structures are inherently connected with the iterated string processing performed by automata.

The propagating solutions (or coherent objects) on certain cellular automata have been found to exhibit nondestructive collisions similar to those observed for soliton systems like the Korteweg–de Vries equation. In such cases, pulse-like disturbances propagating along a uniform nonlinear medium are represented by strings of symbols (zero-one patterns), passing through a one-way structure—the pipeline of identical automata M. The analysis of these moving objects reduces to investigating the repeated automaton action over strings, also called an iterated automaton map (IAM); $M(a^t) = a^{t+1}$, $t = 0, 1, \ldots$.

All known models capable of supporting such discrete localized structures are described by automata (Siwak, 2001, 2002), and a method (called ultradiscretization) leads to discrete systems in which solutions of some familiar soliton equations are preserved as soliton-like patterns (Tokihiro et al., 1999). Thus, an IAM is a fundamental discrete mechanism that supports localized soliton-like periodic structures. This is why a new term *iterons* (Siwak, 2001, 2002) has been proposed for these objects. Note also that fractals owe their existence to the iterating process of some maps.

There are two classes of iterons. The first consists of particles well known in cellular automata models (Delorme & Mazoyer, 1999) where the parallel processing of strings occurs. The second class is not widely known and consists of so-called filtrons. The filtrons are emergent coherent objects occurring in serial processing of strings (Siwak, 2001).

In parallel processing of a string a^t at a given time t all symbols a_i^{t+1} of the next string a^{t+1} are updated simultaneously, for example, by the function $a_i^{t+1} = f(a_{i-r}^t, \ldots, a_i^t, \ldots, a_{i+r}^t)$. When the same function f is used for all positions i, one has a 1-dimensional CA, with f being called its local function or rule. The arguments of f are determined by the so-called neighborhood window $N_i = (-r, \ldots, +r)$; here N_i designates r neighbors on both sides of position i.

The listing of consecutive strings a^t for $t = 0, 1, \ldots$ one under another forms what is called a space-time (ST) diagram. This diagram visualizes the evolution of a given string a^0—the dynamics of a CA system in phase space for initial global state a^0. Occasionally, some moving and periodic patterns of symbols or segments of a string are seen on ST diagrams of CA processing. These are just particles or signals of CAs (Delorme & Mazoyer, 1999).

The serial processing of strings is performed by a computational model called a finite (state) automaton. The Mealy type automaton is described by $M = (S, \Sigma, \Omega, \delta, \beta, s_0)$, where S, Σ, and Ω are nonempty finite sets of—respectively—states, inputs, and outputs; $\delta: S \times \Sigma \rightarrow S$ is called the next state function of M, $\beta: S \times \Sigma \rightarrow \Omega$ is called the output function of M, and s_0 is the initial (extinction) state of the automaton. The automaton converts sequences of symbols, preserving their length. Any input string is read sequentially from left to right, one symbol at each instant of time (pulse of clock). For all $\tau = 1, 2, \ldots$ the automaton:

(i) reads input symbol $\sigma(\tau)$,
(ii) changes its current state $s(\tau)$ onto the next one according to $\delta(s(\tau), \sigma(\tau)) = s(\tau + 1)$, and
(iii) generates the symbol $\beta(s(\tau), \sigma(\tau)) = \omega(\tau)$ of the resulting string.

Thus the complete one step conversion at $a^t \rightarrow a^{t+1}$ requires a series of clock pulses τ. To consider IAMs, one has to assume the unified input-output alphabet $A = \Sigma = \Omega = 0, 1, \ldots, m$. Then the automaton's operation can be described by a