A METHOD OF ZERO CURVATURE REPRESENTATION FOR CONSTRUCTING SYMMETRY ALGEBRAS OF INTEGRABLE SYSTEMS

WEN-XIU MA
Institute of Mathematics, Fudan University
Shanghai 200433, P. R. of China

ABSTRACT

By establishing Lax operator algebras related to zero curvature (zc) representation, a general skeleton of constructing symmetry algebras of integrable systems is proposed along with an application to coupled KdV hierarchy. Moreover it is clearly explained why nonisospectral vector fields can become the first order common master symmetries of the hierarchy of isospectral flows for a given spectral problem.

1. Introduction

In the last two decades, a lot of progress in the investigation of symmetry algebras of integrable systems has been achieved.[1] The recursion operator, which was first proposed by Olver[2] to provide a mechanism for generating an infinite family of symmetries, plays an important role in the theory. Moreover this kind of operators usually possesses a fundamental algebraic-geometrical property called hereditary[3] or Nijenhuis[4,5] property, and has a close relation[4-10] to the multi-Hamiltonian formulation. However, by the recursion operator it can not be exposed why an integrable system possesses a symmetry algebra, particularly a τ-algebra[11] of symmetries which is a semi-product Lie algebra of a Kac-Moody algebra and a Virasoro algebra. Recently for integrable systems with Lax representations, through introducing Lax operator algebras we have given a clear explanation for the existence of τ-algebras of symmetries.[12] In the present paper for integrable systems with zero curvature (zc) representations, we want to display tersely a general skeleton for constructing symmetry algebras by establishing analogous Lax operator algebras, and thus also explain the same problem, that is, why nonisospectral vector fields can become the first order master symmetries of isospectral flows.

2. The Algebraic Structure Related to ZC Representation

Assume that the spectral operator $U = U(u,\lambda)$ has an injective Gateaux derivative operator U'. Let us discuss the spectral problem

$$\begin{cases} \phi_x = U\phi = U(u,\lambda)\phi, \ \lambda_t = f(\lambda), \ f \in C^\infty, \\ \phi_t = V\phi = V(u,\lambda)\phi, \ u = (u_1, u_2, \cdots, u_q)^T. \end{cases} \tag{2.1}$$

Noticing that $U_t = U'[u_t] + \lambda_t U_\lambda$, we can see that the equation $u_t = K = K(x,t,u)$ has a zc representation $U_t - V_x + [U,V] = 0$, which is the compatibility condition of (2.1), iff

$$U'[K] + f(\lambda)U_\lambda - V_x + [U,V] = 0. \tag{2.2}$$

This equality provides a connection between the equation $u_t = K$ and the spectral problem (2.1) and plays a crucial role in our deduction. For two groups of (V, K, f), (W, S, g) satisfying (2.2), we introduce

$$[\![f,g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda), \tag{2.3}$$

$$[\![V,W]\!] = V'[S] - W'[K] + [V,W] + gV_\lambda - fW_\lambda. \tag{2.4}$$

We may verify the following by a direct computation.

Theorem 2.1. *We generally have the equality*

$$U'[[K,S]] + [\![f,g]\!](\lambda)U_\lambda - [\![V,W]\!]_x + [U,[\![V,W]\!]] = 0, \tag{2.5}$$

namely, the equation $u_t = [K,S]$ is the compatibility condition of the spectral problem $\phi_x = U\phi$, $\lambda_t = [\![f,g]\!](\lambda)$, and the auxiliary problem $\phi_t = [\![V,W]\!]\phi$ provided that (2.2) holds for two groups of (V,K,f), (W,S,g).

By the above theorem, we easily show an important fact that the eigenvector fields (K) corresponding to a given evolution function (f) are certain to possess exactly the same algebraic structure as their Lax operators (V) because of the injective property of U'. This may give rise to a way to construct symmetry algebras, which will be discussed in the following sects. 3 and 4.

3. ZC Representations of Hierarchies

Let Φ be a hereditary symmetry associated with the spectral problem $\phi_x = U\phi$.[13] For a given vector field X, we construct an operator equation of Ω:

$$[\Omega, U] + \Omega_x = U'[\Phi X] - \lambda U'[X] \tag{3.1}$$

and call it the characteristic equation of U at X. Assume that (3.1) has solution for any X and $\Omega = \Omega(X)$ is a solution at X.

Theorem 3.1. *Suppose $k \geq 0$ and there exist starting Lax operators A_0, B_0 and vector fields f_0, g_0 such that the equalities*

$$U'[f_0] - A_{0x} + [U, A_0] = 0,\ U'[g_0] + \lambda^k U_\lambda - B_{0x} + [U, B_0] = 0 \tag{3.2}$$

hold. Let $K_m = \Phi^m f_0$, $\sigma_n = \Phi^n g_0$, $m, n \geq 0$, and further let

$$V_m = \sum_{i=0}^{m} \lambda^{m-i} A_i = \lambda^m A_0 + \sum_{i=1}^{m} \lambda^{m-i}\Omega(K_{i-1}), \tag{3.3a}$$

$$W_n = \sum_{j=0}^{n} \lambda^{n-j} B_j = \lambda^n B_0 + \sum_{j=1}^{n} \lambda^{n-j}\Omega(\sigma_{j-1}). \tag{3.3b}$$

Then V_m, W_n are respectively Lax operators associated with K_m, σ_n, i.e. we have

$$U'[K_m] - V_{mx} + [U, V_m] = 0,\ m \geq 0, \tag{3.4a}$$

$$U'[\sigma_n] + \lambda^{n+k}U_\lambda - W_{nx} + [U, W_n] = 0,\ n \geq 0. \tag{3.4b}$$

This theorem implies that two hierarchies of equations $u_t = K_m$, $m \geq 0$, and $u_t = \sigma_n$, $n \geq 0$, possess zc representations $U_t - V_{mx} + [U, V_m] = 0$, $m \geq 0$, and $U_t - W_{nx} + [U, W_n] = 0$, $n \geq 0$, respectively. It also tells us that a starting master symmetry g_0 may be worked out by solving an equation of g_0, B_0 in (3.2).

4. Lax Operator Algebras

The aim of this section is to explain why the vector fields K_m, σ_n, $m, n \geq 0$, defined in the previous section, possess a semi-product Lie algebraic structure.

Theorem 4.1. *Let $k' \geq 0$. Suppose that (1) $f_0|_{u=0} = 0$, $\sigma_{k'}|_{u=0} = 0$; (2) if $U'[T] - \Theta_x + [U, \Theta] = 0$ and $\Theta|_{u=0} = 0$, then $\Theta = 0$; (3) the equality $[V_0|_{u=0}, W_{k'}|_{u=0}] = \gamma V_{k'+k-1}|_{u=0}$, $\gamma =$const. holds if $k' > 0$ or the equality*

$$[V_0|_{u=0}, W_0|_{u=0}] = 0,\ [V_0|_{u=0}, W_1|_{u=0}] = \gamma V_k|_{u=0},\ [V_1|_{u=0}, W_0|_{u=0}] = (1+\gamma)V_k|_{u=0},$$

where γ is a constant, holds if $k' = 0$. Then we have a semi-product Lie algebra of Lax operators

$$[\![V_m, V_n]\!] = 0,\ m, n \geq 0, \tag{4.1a}$$

$$[\![V_m, W_n]\!] = (m+\gamma)V_{m+n+k-1},\ V_{-1} = 0,\ m \geq 0,\ n \geq k', \tag{4.1b}$$

$$[\![W_m, W_n]\!] = (m-n)W_{m+n+k-1},\ W_{-1} = 0,\ m, n \geq k', \tag{4.1c}$$

where $[\![\cdot,\cdot]\!]$ is determined by (2.4).

We refer to the above algebra (4.1) as a Lax operator algebra. From a Lax operator algebra (4.1), we can readily deduce the same semi-product algebraic structure of the vector fields K_m, σ_n, $m, n \geq 0$, by noticing the explanation in Sec. 2 or directly using (2.5), and thus may obtain the following at once.

Theorem 4.2 (τ-algebra). *Under the assumption of Theorem 4.1, every integrable systems $u_t = K_s\,(s \geq 0)$ possess the following symmetry algebra:*

$$[K_m, K_n] = 0,\ m, n \geq 0,$$

$$[K_m, \tau_n^{(s)}] = (m+\gamma)K_{m+n+k-1},\ K_{-1} = 0,\ m \geq 0,\ n \geq k',$$

$$[\tau_m^{(s)}, \tau_n^{(s)}] = (m-n)\tau_{m+n+k-1}^{(s)},\ \tau_{-1}^{(s)} = 0,\ m, n \geq k',$$

where τ-symmetries $\tau_n^{(s)}$ read as $\tau_n^{(s)} = t[K_s, \sigma_n] + \sigma_n$, $n \geq 0$.

Summarizing, we can state that symmetry algebras may be generated by computing Lax operator algebras, which is a kind of new trick.

5. Application: An Illustrative Example

In terms of the above theory, we consider the coupled KdV case[14] as a particular example. The spectral problem in this case can be rewritten as

$$\phi_x = U(u,\lambda)\phi = \begin{bmatrix} 0 & 1 \\ -Q & 0 \end{bmatrix}\phi,\ Q = Q(u,\lambda) = \lambda^{-l}\sum_{i=0}^{q-1} v_i\lambda^i - \lambda^{q-l},\ 0 \leq l \leq q-1$$

with $u = (v_0, v_1, \cdots, v_{q-1})^T$. The corresponding isospectral hierarchy is $u_t = K_m = \Phi^m u_x$, $m \geq 0$, in which Φ is the common hereditary recursion operator

$$\Phi = \begin{bmatrix} 0 & 0 & \cdots & 0 & R_0^* \\ 1 & 0 & \cdots & 0 & R_1^* \\ 0 & 1 & \cdots & 0 & R_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & R_{q-1}^* \end{bmatrix}, \quad R_i^* = \frac{1}{4}\delta_{il}\partial^2 + v_i + \frac{1}{2}v_{ix}\partial^{-1}, \; 0 \leq i \leq q-1.$$

The characteristic eq. (3.1) has a special solution for any $X = (X_1, \cdots, X_q)^T$:

$$\Omega = \Omega(X) = \begin{bmatrix} -\frac{1}{4}X_q & \frac{1}{2}\partial^{-1}X_q \\ -\frac{1}{2}Q\partial^{-1}X_q - \frac{1}{4}X_{qx} & \frac{1}{4}X_q \end{bmatrix}$$

and Eq. (3.2) with $k = 1$ has two pairs of solutions: $f_0 = u_x$, $A_0 = U$ and

$$g_0 = (\, qv_0 + \frac{1}{2}(q-l)xv_{0x}, \cdots, v_{q-1} + \frac{1}{2}(q-l)xv_{q-1,x}\,)^T,$$

$$B_0 = \begin{bmatrix} -\frac{1}{4}(q-l) & \frac{1}{2}(q-l)x \\ -\frac{1}{2}(q-l)xQ & \frac{1}{4}(q-l) \end{bmatrix}.$$

According to Theorem 4.1 with $k' = 0$, for coupled KdV hierarchy we can present a Lax operator algebra determined by (4.1) with $k = 1$ and $\gamma = \frac{1}{2}(q-l)$, and thus the same structure admitted by the symmetry algebra of the hierarchy can be formulated by Theorem 4.2. The effectiveness of our method may be also demonstrated by applications to the other typical integrable hierarchies, which will be reported elsewhere.

Acknowledgement: The author would like to express his sincere thanks to Profs. C. H. Gu and H. S. Hu for their guidance and support.

References

1. C. H. Gu *et al.*, *Soliton Theory and Its Application*, (Zhejiang Publishing House of Science and Technology, Hangzhou, China, 1990).
2. P. J. Olver, *J. Math. Phys.* **18** (1977), 1212–1215.
3. B. Fuchssteiner, *Nonlinear Anal. Theor. Math. Appl.* **3** (1979), 849–862.
4. F. Magri, *J. Math. Phys.* **19** (1978), 1156–1162.
5. ——, *in: Nonlinear Evolution Equations and Dynamical Systems edited by M. Boiti, F. Pempinelli and G. Soliani*, (Springer-Verlag, New York, 1980), pp. 233–263.
6. A. S. Fokas and B. Fuchssteiner, *Lett. Nuovo Cimento* **28** (1980), 299–303.
7. B. Fuchssteiner and A. S. Fokas, *Physica* **4D** (1981), 47–66.
8. I. M. Gel'fand and I. Y. Dorfman, *Funct. Anal. Appl.* **13** (1979), 248–262.
9. ——, *Funct. Anal. Appl.* **14** (1980), 223–226.
10. G. Magnano and F. Magri, *Rev. Math. Phys.* **3** (1991), 403–466.
11. W. X. Ma, *J. Phys. A: Math. Gen.* **25** (1992), to appear.
12. ——, *J. Math. Phys.* **33** no. 7 (1992).
13. A. S. Fokas and R. L. Anderson, *J. Math. Phys.* **23** (1982), 1066–1073.
14. M. Boiti, P. J. Caudrey and F. Pempinelli, *Il Nuovo Cimento* **83B** (1984), 71–87.