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Letter

Matrix extension of the Kuralay-II Equation and its associated Darboux transformation

Wen-Xiu Ma *

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
Research Center of Astrophysics and Cosmology, Khazar University, 41 Mehseti Street, Baku 1096, Azerbaijan
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
School of Mathematics, South China University of Technology, Guangzhou 510640, China
Material Science Innovation and Modelling, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

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ABSTRACT

Starting from the matrix AKNS spectral problem, we construct a Lax pair featuring a first-order non-zero pole in the spectral parameter and derive a matrix generalization of the Kuralay-II equation. The associated Darboux transformation is developed within the AKNS framework. By applying this transformation to a non-zero seed solution, we obtain a class of exact and explicit solutions.

1. Introduction

In soliton theory, numerous powerful methods have been developed to construct solutions of integrable models. Among them, the Darboux transformation stands out as an effective algebraic technique for generating new solutions from known ones, typically through the use of Lax pairs [1].

Fundamentally, the Darboux transformation modifies the potential of a given integrable model by applying a specific transformation to its associated Lax pair. This process preserves the integrability of the original model and enables the iterative construction of multi-soliton solutions, rational solutions, rogue waves, and other intricate structures. The Lax pair formalism lies at the core of this framework and has been extended to encompass matrix-valued systems, higher-dimensional models, noncommutative geometries, and supersymmetric cases (see, e.g., [2,3]).

Let us consider the zero-curvature formulation and the associated Darboux transformations for integrable models [2,4]. We begin with the matrix spectral problems

$$\phi_x = U\phi = U(u,\lambda)\phi,\ \ \phi_t = V\phi = V(u,\lambda)\phi, \tag{1}$$

where λ is the spectral parameter and ϕ is the vector eigenfunction. The compatibility condition of this overdetermined linear system leads to the zero-curvature equation:

$$U_t - V_x + [U,V] = 0, \tag{2}$$

which often yields an integrable model of the form:

$$u_t = K(u) = K(x,t,u,u_x,\ldots). \tag{3}$$

* Corresponding author at: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA.
E-mail address: mawx@cas.usf.edu.

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In recent developments, Lax matrices V featuring negative powers of the spectral parameter have been employed to study associated integrable models, referred to as negative-order flows.

A key objective in the theory of integrable systems is to construct Darboux transformations for such models from the zero-curvature representation. A Darboux transformation consists of a gauge transformation $\phi' = D\phi$ along with a new potential $u' = u'(u)$, where $D = D(u, \lambda)$ is a matrix function of the potential and the spectral parameter. The transformed eigenfunction ϕ' is required to satisfy the same type of spectral problems:

$$\phi'_x = U'\phi' = U(u', \lambda)\phi', \ \phi'_t = V'\phi' = V(u', \lambda)\phi', \tag{4}$$

with U' and V'preserving the structure of the original Lax pair (see, e.g., [4,5]). The matrix D is called a Darboux matrix, and it must satisfy the compatibility conditions:

$$U'D = DU + D_x, \ V'D = DV + D_t. \tag{5}$$

Assume the matrices U and V are of order N. A commonly used first-order Darboux matrix takes the form:

$$D(\lambda) = \lambda I_N - S, \tag{6}$$

where I_N is the $N \times N$ identity matrix, and S is an $N \times N$ matrix independent of λ. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be distinct eigenvalues, with corresponding eigenfunctions $\phi^{[j]}$ satisfying:

$$\phi^{[j]}_x U(u, \lambda_j)\phi^{[j]}, \ \phi^{[j]}_t = V(u, \lambda_j)\phi^{[j]}, \ 1 \leq j \leq N, \tag{7}$$

where u is a known solution to (3). Define the matrix

$$H = (\phi^{[1]}, \ldots, \phi^{[N]}), \ \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N), \tag{8}$$

and then the matrix S is given by [2,4]:

$$S = H\Lambda H^{-1}. \tag{9}$$

The transformed potential $u' = u'(u)$ defines a new solution to the integrable model (3). This Darboux framework applies effectively to general AKNS-type flows in both lower- and higher-dimensional cases [4].

In this paper, we start from the matrix AKNS spectral problem, formulate a Lax matrix V featuring a first-order non-zero pole in the spectral parameter, and derive a matrix generalization of the Kuralay-II equation. Based on the proposed Lax pair, we construct the corresponding Darboux transformation within the AKNS framework. Using a non-zero constant seed solution, we apply the transformation to obtain a class of exact and explicit solutions to the new matrix model.

2. Lax pair and matrix Kuralay-II equation

Let m and n be natural numbers, and let α_1 and α_2 be two distinct constants. Consider the AKNS spectral matrix [6] given by

$$U = i\lambda\Lambda + Q, \ \Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \ Q = \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}, \tag{10}$$

where I_k denotes the $k \times k$ identity matrix, and q and r are the potential matrices of sizes $m \times n$ and $n \times m$, respectively. We define the associated Lax matrix as:

$$V = \frac{1}{1 - \alpha\lambda} W, \tag{11}$$

where

$$\alpha = \alpha_1 - \alpha_2, \ W = W_1 + W_2, \ W_1 = -i \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}, \ W_2 = i \begin{bmatrix} 0 & -q_t \\ r_t & 0 \end{bmatrix}, \tag{12}$$

with v and w being two square potential matrices of orders m and n, respectively.

The zero-curvature equation

$$U_t - V_x + [U, V] = 0 \tag{13}$$

is equivalent to

$$(1 - \alpha\lambda)U_t - W_x + [U, W] = 0. \tag{14}$$

By equating powers of λ, this gives:

$$\begin{cases} -\alpha Q_t + i[\Lambda, W] = 0, \\ Q_t - W_x + [Q, W] = 0. \end{cases}$$

The first equation is satisfied automatically, since

$$[\Lambda, W_1] = 0, \ [\Lambda, W_2] = -i\alpha Q_t.$$

To verify the second equation, we compute

$$[Q, W_1] = -i \begin{bmatrix} 0 & qw - vq \\ rv - wr & 0 \end{bmatrix}, \ [Q, W_2] = i \begin{bmatrix} (qr)_t & 0 \\ 0 & -(rq)_t \end{bmatrix}.$$

Hence, we obtain the following system:

$$\begin{cases} iq_t - q_{tx} + qw - vq = 0, \\ ir_t + r_{tx} + rv - wr = 0, \\ v_x + (qr)_t = 0, \\ w_x - (rq)_t = 0. \end{cases} \tag{15}$$

This is the matrix form of the Kuralay-II equation, which also extends integrable models arising from dual group reductions (see, e.g., [7–10]). Remarkably, the resulting equation is independent of the specific values of α_1 and α_2, and it remains valid even when $\alpha = 0$.

In the scalar case $m = n = 1$, setting $w = -v$, we recover

$$\begin{cases} iq_t - q_{tx} - 2vq = 0, \\ ir_t + r_{tx} + 2vr = 0, \\ v_x + (qr)_t = 0. \end{cases} \tag{16}$$

This reduces, via the scaling $v \to \frac{1}{2}v, q \to dq, r \to dr$, to the standard Kuralay-II equation [11]:

$$\begin{cases} iq_t - q_{tx} - vq = 0, \\ ir_t + r_{tx} + vr = 0, \\ v_x + 2d^2(qr)_t = 0, \end{cases} \tag{17}$$

where d is a non-zero constant. Many other negative-order AKNS flows have been studied in the scalar case (see, e.g., [12–16]).

For the case $m = 1$ and $n = 2$, if we set

$$q = (q_1, q_2), \ r = (r_1, r_2)^T, \ v = (v_{11}), \ w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \tag{18}$$

then the system of equations becomes:

$$\begin{cases} iq_{1,t} - q_{1,tx} + q_1 w_{11} + q_2 w_{21} - v_{11} q_1 = 0, \\ iq_{2,t} - q_{2,tx} + q_1 w_{12} + q_2 w_{22} - v_{11} q_2 = 0, \\ ir_{1,t} + r_{1,tx} + r_1 v_{11} - w_{11} r_1 - w_{12} r_2 = 0, \\ ir_{2,t} + r_{2,tx} + r_2 v_{11} - w_{21} r_1 - w_{22} r_2 = 0, \\ v_{11,x} + (q_1 r_1 + q_2 r_2)_t = 0, \\ w_{11,x} - (r_1 q_1)_t = 0, \ w_{12,x} - (r_1 q_2)_t = 0, \\ w_{21,x} - (r_2 q_1)_t = 0, \ w_{22,x} - (r_2 q_2)_t = 0. \end{cases} \tag{19}$$

In general, the resulting system comprises $(m+n)^2$ equations, corresponding to all possible components of the dependent variables. It is also a special case of the system proposed in [4], associated with the following parameter and spectral matrix choices:

$$n = 2, \ p_1 = x, \ x_2 = t, \ A_1 = -U, \ a_2 = (1 - \alpha\lambda), \ A_2 = W.$$

3. The Darboux transformation and its application

3.1. Compatibility conditions

We assume that the Darboux matrix takes a linear form in λ:

$$D(\lambda) = \lambda I_{m+n} - S, \tag{20}$$

where S is an auxiliary matrix to be determined. The spatial compatibility condition

$$U' D = DU + D_x,$$

with U being given by (10) and

$$U' = i\lambda\Lambda + Q', \tag{21}$$

leads to

$$(i\lambda\Lambda + Q')(\lambda I_{m+n} - S) = (\lambda I_{m+n} - S)(i\lambda\Lambda + Q) - S_x.$$

By comparing the powers of λ, we find

$$Q' = Q + i[\Lambda, S], \tag{22}$$

and

$$S_x = Q'S - SQ = [Q + i\Lambda S, S]. \tag{23}$$

The temporal compatibility condition:

$$V'D = DV + D_t,$$

with

$$V' = \frac{1}{1-\alpha\lambda}W', \tag{24}$$

yields

$$W'(\lambda I_{m+n} - S) = (\lambda I_{m+n} - S)W - (1-\alpha\lambda)S_t.$$

It then follows that

$$W' = W + \alpha S_t, \tag{25}$$

and

$$S_t = W'S - SW = (W + \alpha S_t)S - SW = [W, S] + \alpha S_t S. \tag{26}$$

3.2. Construction of the Darboux matrix

Following the general AKNS framework (see, e.g., [4]), the matrix S can be constructed as

$$S = HAH^{-1}, \tag{27}$$

where

$$H = (\phi^{[1]}, \dots, \phi^{[m+n]}),\ \ A = \mathrm{diag}(\lambda_1, \dots, \lambda_{m+n}), \tag{28}$$

and each column $\phi^{[j]}$ satisfies the Lax pair equations:

$$\phi^{[j]}_x = U(u, \lambda_j)\phi^{[j]},\ \ \phi^{[j]}_t = V(u, \lambda_j)\phi^{[j]},\ \ 1 \le j \le m+n. \tag{29}$$

To verify compatibility, we compute the derivatives of S using the identities

$$H_x = i\Lambda HA + QH,\ \ H_t = WHB, \tag{30}$$

where

$$B = \mathrm{diag}(\frac{1}{1-\alpha\lambda_1}, \dots, \frac{1}{1-\alpha\lambda_{m+n}}). \tag{31}$$

Differentiating S with respect to x, we obtain

$$\begin{aligned} S_x &= H_xAH^{-1} - HA(H^{-1}H_xH^{-1}) \\ &= i\Lambda HA^2H^{-1} + QHAH^{-1} - HAH^{-1}(i\Lambda HA + QH)H^{-1} \\ &= i\Lambda HA^2H^{-1} + QHAH^{-1} - iHAH^{-1}\Lambda HAH^{-1} - HAH^{-1}Q \\ &= QS - SQ + i\Lambda S^2 - iS\Lambda S, \end{aligned}$$

which verifies the spatial compatibility condition (23).

Similarly, differentiating S with respect to t, we compute

$$S_t = H_tAH^{-1} - HA(H^{-1}H_tH^{-1}) = WHBAH^{-1} - HA(H^{-1}WHBH^{-1}).$$

Using the identity

$$B = \alpha BA + I_{m+n}, \tag{32}$$

we obtain

$$\begin{aligned} S_t &= WH(A + \alpha BA^2)H^{-1} - HAH^{-1}WH(\alpha BA + I_{m+n})H^{-1} \\ &= WHAH^{-1} + \alpha(WHBAH^{-1} - HAH^{-1}WHBH^{-1})HAH^{-1} - HAH^{-1}W \\ &= WS + \alpha S_tS - SW, \end{aligned}$$

which confirms the temporal compatibility condition (26).

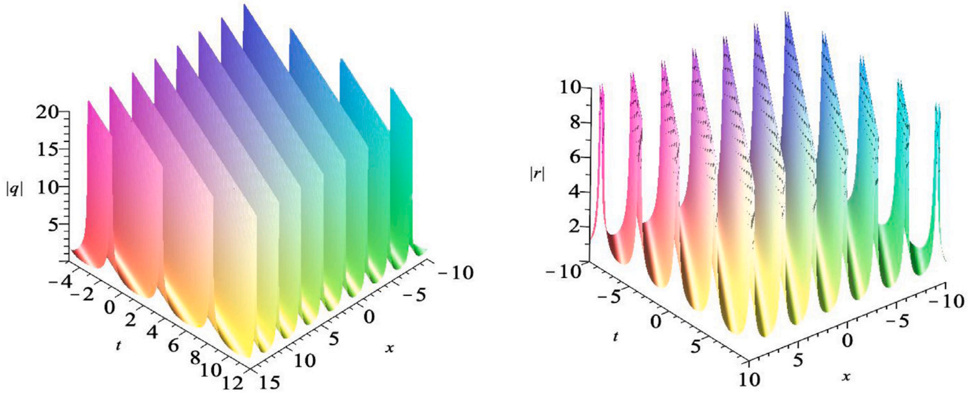

Fig. 1. 3D plots of $|q|$ (left) and $|r|$ (right).

3.3. The Darboux transformation

From the above construction, we obtain the Darboux transformation:

$$\phi' = (\lambda I_{m+n} - S)\phi,\ Q' = Q + i[\Lambda, S],\ W' = W + \alpha S_t. \tag{33}$$

Using the structures of U and W given in (10) and (12), respectively, the transformed potentials are explicitly given by:

$$q' = q + i[\Lambda, S]_{12},\ r' = r + i[\Lambda, S]_{21},\ v' = v + i\alpha S_{11,t},\ w' = w + i\alpha S_{22,t}, \tag{34}$$

where S is defined by (27), and the subscript notation M_{jk} denotes the (j,k)-block component of the matrix M under the partitioning determined by the spectral matrix U.

3.4. An application: explicit solutions

We illustrate the procedure by considering a non-zero constant seed solution:

$$q = r = 0,\ v = v_0,\ w = w_0, \tag{35}$$

where v_0 and w_0 are arbitrary constant matrices. For each spectral parameter λ_j, the associated eigenfunction

$$\phi^{[j]} = (\phi_1^{[j]T}, \phi_2^{[j]T})^T \tag{36}$$

is given explicitly by

$$\begin{cases} \phi_1^{[j]} = \exp(i\alpha_1\lambda_j I_m x - \frac{i}{1-\alpha\lambda_j} v_0 t)\mu_1^{[j]}, \\ \phi_2^{[j]} = \exp(i\alpha_2\lambda_j I_n x - \frac{i}{1-\alpha\lambda_j} w_0 t)\mu_2^{[j]}, \end{cases} \quad 1 \le j \le m+n, \tag{37}$$

where $\mu_1^{[j]} \in \mathbb{C}^m$ and $\mu_2^{[j]} \in \mathbb{C}^n$ are arbitrary constant vectors. Substituting these into the Darboux framework, the transformed solution is explicitly given by:

$$q' = i[\Lambda, S]_{12},\ r' = i[\Lambda, S]_{21},\ v' = v_0 + i\alpha S_{11,t},\ w' = w_0 + i\alpha S_{22,t}, \tag{38}$$

where $S = HAH^{-1}$ is constructed from the explicit eigenfunctions determined above, as defined in (27) and (28). The block components are taken with respect to the partitioning induced by the spectral matrix U.

For the case $m = n = 1$, the four pictures in Figs. 1 and 2 exhibit 3D plots of the solution (q, r, v, w), derived based on the parameter choices and initial solution given below:

$$\alpha_1 = 1,\ \alpha_2 = -1,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \mu_1^{[1]} = \mu_2^{[1]} = -1,\ \mu_1^{[2]} = \mu_2^{[2]} = -2,$$

and

$$(q, r, v, w) = (q_0, r_0, v_0, w_0) = (0, 0, 1, -1).$$

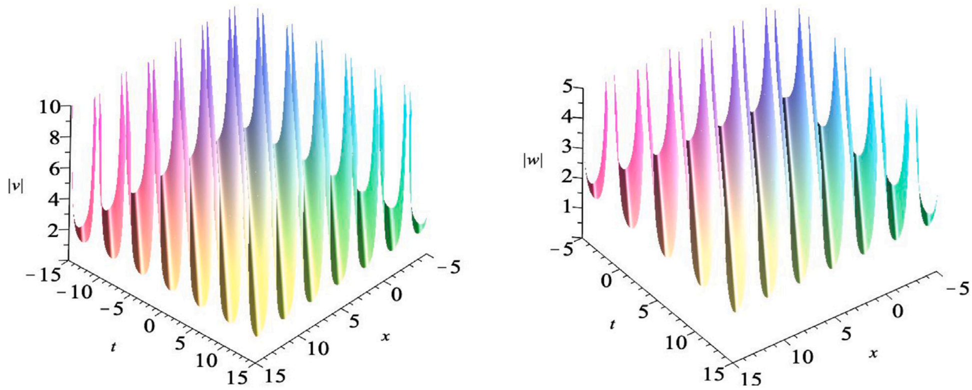

Fig. 2. 3D plots of $|v|$ (left) and $|w|$ (right).

4. Conclusions

Based on the matrix AKNS spectral problem, we have constructed a matrix generalization of the Kuralay-II equation and its associated Darboux transformation, along with a class of exact and explicit solutions. A key feature of the construction is the presence of a non-zero pole in the spectral parameter. Related developments on Darboux transformations for negative-order flows can be found in the literature (see, e.g., [17–21]).

It is well known that reduced Lax pairs often yield novel integrable models, and group reductions play a central role in implementing consistent constraints on the general spectral matrix (see, e.g., [22,23]). These reductions also give rise to nonlocal integrable models [24,25]. A key open question is how to construct Darboux transformations for such constrained matrix systems.

A broader generalization involves Lax operators with higher-order poles in the spectral parameter, although this direction presents considerable difficulty. It introduces new types of constraints and significantly increases the complexity of both the Lax pair structure and the associated Darboux transformations. Determining the form of Darboux transformations in this more singular framework remains a challenging and compelling problem, particularly since the structure given in (27) and (28) may no longer be applicable. Progress in this direction would contribute to the classification of integrable systems and enhance our understanding of their underlying nonlinear dynamics.

CRediT authorship contribution statement

Wen-Xiu Ma: Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The author declares that there are no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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Data availability

No new data were created or analysed in this study.

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