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NONLOCALITY, INTEGRABILITY, AND SOLITONS

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We explore integrable equations that involve involution points, along with the solution phenomena for Cauchy problems associated with nonlocal differential equations. By applying group reductions to classical Lax pairs, we generate nonlocal integrable equations. Soliton solutions of these models are derived using binary Darboux transformations or reflectionless Riemann–Hilbert problems in the nonlocal context. Further discussion on the well-posedness of nonlocal differential equations is also presented.

Keywords: Lax pair, integrable model, Darboux transformation

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1. Introduction

Differential equations are fundamental in describing dynamical systems in Nature. They are primarily classified into two categories: ordinary differential equations (ODEs) and partial differential equations (PDEs). Integrable equations are among the most elegant examples of differential equations [1]–[3]. One of their hallmark features is the existence of sufficiently many conserved quantities, which always commute with each other under a specific form of Poisson brackets [4]–[6]. Integrable ODEs have a finite number of conserved quantities, whereas integrable PDEs require infinitely many [7], [8].

In modern mathematical physics, nonlocal differential equations have emerged as an important field of study. Examples of such equations include nonlocal ODEs

$$u_t = u(t-a), \qquad u_t = u(\lambda t), \qquad u_t = u(-t), \tag{1.1}$$

where $a > 0$ and $0 < \lambda < 1$, as well as nonlocal PDEs

$$u_t = u_{xx}(-x,-t), \qquad u_t = u(-x,-t)u_x. \tag{1.2}$$

Nonlocal differential equations pose significant challenges in formulating well-posed problems and determining their solutions.

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Motivated by non-Hermitian quantum mechanics, many nonlocal integrable equations have been studied recently (see, e.g., [9]–[12]). We introduced nonlocal integrable equations through group reductions of matrix spectral problems [13], [14]. In particular, novel nonlocal integrable nonlinear Schrödinger-type equations have been derived through pairs of group reductions (see, e.g., [15]–[18]). These nonlocal models give rise to new questions, prompting us to explore innovative ideas and techniques to address them.

In this paper, we discuss the concept of group reduction and its applications to the AKNS integrable nonlinear Schrödinger (NLS) equations and modified Korteweg–de Vries (mKdV) equations, employing the Lax pair formulation as the primary tool. Soliton solutions for these models are derived by formulating binary Darboux transformations, which are equivalent to reflectionless Riemann–Hilbert problems in the nonlocal setting. We present a classification of the resulting nonlocal integrable equations and explore some novel solution phenomena in the nonlocal context. The final section concludes with a summary, including our future work on further applications and soliton structures in the nonlocal setting, as well as a discussion of the well-posedness of nonlocal differential equations.

2. Solution phenomena in the nonlocal setting

In this section, we explore some solutions in the nonlocal setting. They are quite different from those in the local setting.

2.1. A Cauchy problem. We consider the Cauchy (i.e., initial-value) problem for a nonlocal heat equation

$$u_t = u_{xx}(-x,-t), \qquad u(x,0) = f(x). \tag{2.1}$$

The method of separation of variables, along with Fourier series theory, can be used to find a solution of this problem

$$u(x,t) = \sum_{n=0}^{\infty} \left[a_n \sin(nx) \cos\left(n^2 t - \frac{\pi}{4}\right) + b_n \cos(nx) \sin\left(n^2 t - \frac{\pi}{4}\right) \right], \tag{2.2}$$

where the constants a_n and b_n are given by the Fourier coefficients of the initial wave:

$$f(x) = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} [a_n \sin(nx) + b_n \cos(nx)]. \tag{2.3}$$

However, the uniqueness and stability of the problem remain open. This PDE does not possess the maximum principle, unlike the standard local heat equation $u_t = u_{xx}$.

2.2. Well-posedness. We consider a linear nonlocal first-order differential equation

$$x'(t) = -x(t) - x(-t), \qquad t \geqslant 0. \tag{2.4}$$

We can easily see that the general solution is

$$x(t) = c(-2t+1), \tag{2.5}$$

where c is an arbitrary constant [19].

It follows that the Cauchy problem

$$\begin{cases} x'(t) = -x(t) - x(-t), \qquad t \geqslant 0, \\ x\left(\dfrac{1}{2}\right) = 0, \end{cases} \tag{2.6}$$

has infinitely many solutions given by (2.5). Therefore, the uniqueness property of Cauchy problems in the nonlocal setting is not always guaranteed. This feature is different from the local case.

Moreover, the Cauchy problem

$$\begin{cases} x'(t) = -x(t) - x(-t), & t \geqslant 0, \\ x\left(\dfrac{1}{2}\right) = x_0 \neq 0, \end{cases} \tag{2.7}$$

has no solution at all, since all solutions are given by (2.5), for which we always have $x(1/2) = 0$. Therefore, the existence of a solution of the Cauchy problem depends on specific conditions.

2.3. Independence of solutions from coefficients. We consider a specific example of the second order,

$$x''(t) = \lambda x(t) - \lambda x(-t) + \mu x'(t) - \mu x'(-t), \tag{2.8}$$

where λ and μ are real constants. By the same argument as presented in [19], we know that the general solution of this linear differential equation is

$$x(t) = c_1 + c_2 t \tag{2.9}$$

if $\lambda = 0$,

$$x(t) = c_1 + c_2 \sinh(\sqrt{2\lambda} t) \tag{2.10}$$

if $\lambda > 0$, and

$$x(t) = c_1 + c_2 \sin(\sqrt{-2\lambda} t) \tag{2.11}$$

if $\lambda < 0$. Interestingly, we find that the solution is independent of the coefficient μ.

2.4. Stability. We consider a set of restricted Cauchy problems with $t_0 \neq 1/2$:

$$\begin{cases} x'(t) = -x(t) - x(-t), & t \geqslant 0, \\ x(t_0) = x_0. \end{cases} \tag{2.12}$$

The solution of this Cauchy problem is

$$x(t) = x_0 \frac{-2t + 1}{-2t_0 + 1}. \tag{2.13}$$

This is not bounded, and hence the solution of the above nonlocal equation with $t_0 \neq 1/2$ is unstable.

2.5. General solution. An approach using the decomposition of functions into sums of even and odd functions was used in [19] to solve linear nonlocal differential equations. Below are two examples.

Let f be continuous, and let λ and μ be constants. The general solution of

$$x'(t) = \lambda x(t) + \mu x(-t) + f(t) \tag{2.14}$$

is given by four formulas for the four distinct cases of the coefficients [19]. Similarly, the second-order equation

$$x''(t) = \lambda x(t) + \mu x(-t) + f(t) \tag{2.15}$$

has nine solution cases [20].

3. Lax pair formulation

To discuss integrable equations, we begin with introducing a Lax pair of matrix spectral problems

$$-i\phi_x = U\phi, \qquad -i\phi_t = V\phi, \tag{3.1}$$

where i is the imaginary unit, $U = U(u,\lambda)$ and $V = V(u,\lambda)$ are two given square matrices, and ϕ is a matrix eigenfunction. Nonlinear integrable equations are generated from the zero-curvature equation

$$U_t - V_x + i[U,V] = 0. \tag{3.2}$$

3.1. AKNS NLS and mKdV equations. We take arbitrary constants α_1, α_2, β_1, and β_2 and define

$$\alpha = \alpha_1 - \alpha_2, \qquad \beta = \beta_1 - \beta_2. \tag{3.3}$$

To generalize the AKNS integrable equations [7], we consider the matrix potentials

$$u = u(p,q), \quad p = p(x,t) = (p_{jk})_{1\leqslant j\leqslant m, 1\leqslant k\leqslant n}, \quad q = q(x,t) = (q_{kj})_{1\leqslant k\leqslant n, 1\leqslant j\leqslant m}. \tag{3.4}$$

The Lax pair takes the form [21]

$$U = \lambda\Lambda + P(u), \qquad V^{[r]} = \lambda^r\Omega + Q^{[r]}(u,\lambda), \tag{3.5}$$

where $r \in \mathbb{N}$, $\Lambda = \operatorname{diag}(\alpha_1 I_m, \alpha_2 I_n)$, $\Omega = \operatorname{diag}(\beta_1 I_m, \beta_2 I_n)$, P and $Q^{[r]}$ are traceless, and $\deg_\lambda Q^{[r]} \leqslant r-1$. The matrix PT-symmetric NLS equations correspond to the potential matrix

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \tag{3.6}$$

and the matrix

$$Q^{[2]} = \frac{\beta}{\alpha}\lambda \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} - \frac{\beta}{\alpha^2}\begin{bmatrix} pq & ip_x \\ -iq_x & -qp \end{bmatrix}. \tag{3.7}$$

The matrix NLS equations are derived from the zero-curvature equation

$$U_t - V_x^{[2]} + i[U, V^{[2]}] = 0 \tag{3.8}$$

and are given by

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2pqp), \qquad q_t = \frac{\beta}{\alpha^2} i(q_{xx} + 2qpq). \tag{3.9}$$

This includes the case $m = 1$ and $n = 2$, which was studied in [22]. Similarly, the matrix PT-symmetric mKdV equations arise from the zero-curvature equation

$$U_t - V_x^{[3]} + i[U, V^{[3]}] = 0, \tag{3.10}$$

where $Q^{[3]}$ is defined by

$$Q^{[3]} = \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P,P_x] + P_{xx} + 2P^3), \tag{3.11}$$

with $I_{m,n} = \operatorname{diag}(I_m, -I_n)$. The matrix mKdV equations are given by

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp), \qquad q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x). \tag{3.12}$$

3.2. Other spectral matrices. Other examples of successful spectral matrices associated with $sl(2,\mathbb{R})$ are

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, \quad U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, \quad U = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix}, \quad U = \begin{bmatrix} \lambda r & \lambda p \\ \lambda q & -\lambda r \end{bmatrix}, \tag{3.13}$$

where $pq + r^2 = 1$. They lead to the respective AKNS, Kaup–Newell, Wadati–Konno–Ichikawa, and Heisenberg soliton hierarchies

Generalized spectral matrices associated with $sl(2,\mathbb{R})$ include (see, e.g., [23]–[27])

$$\begin{gathered} U = \begin{bmatrix} \lambda + \alpha pq & p \\ q & -\lambda - \alpha pq \end{bmatrix}, \qquad U = \begin{bmatrix} \lambda^2 + \alpha pq & \lambda p \\ \lambda q & -\lambda^2 - \alpha pq \end{bmatrix}, \\ U = \begin{bmatrix} \lambda + \alpha\sqrt{pq+1} & \lambda p \\ \lambda q & -\lambda - \alpha\sqrt{pq+1} \end{bmatrix}, \end{gathered} \tag{3.14}$$

where α is a constant. They give rise to generalized soliton hierarchies.

Examples of successful spectral matrices associated with $so(3,\mathbb{R})$ (see, e.g., [28]–[31]) include

$$\begin{gathered} U = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix}, \\ U = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & -\lambda q & -\lambda r \\ \lambda q & 0 & -\lambda p \\ \lambda r & \lambda p & 0 \end{bmatrix}, \end{gathered} \tag{3.15}$$

where $p^2 + q^2 + r^2 = 1$. They yield the respective AKNS, Kaup–Newell, Wadati–Konno–Ichikawa, and Heisenberg-type soliton hierarchies.

To derive four-component generalizations, we proposed a novel 4×4 spectral matrix [32]

$$U = U(u,\lambda) = \begin{bmatrix} \alpha_1\lambda & \delta_1 u_1 & u_2 & 0 \\ \delta_1 u_3 & \alpha_2\lambda & 0 & u_4 \\ \delta_1\delta_2 u_4 & 0 & \alpha_2\lambda & -\delta_1 u_3 \\ 0 & \delta_1\delta_2 u_2 & -\delta_1 u_1 & \alpha_1\lambda \end{bmatrix}, \tag{3.16}$$

where $u = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$, and α_1, α_2, δ_1, and δ_2 are arbitrary constants, but $\alpha_1 \neq \alpha_2$. This spectral matrix generates a combined soliton hierarchy that includes the combined nonlinear Schrödinger equations:

$$\begin{aligned} u_{1,t} &= \frac{1}{\alpha^2}[\delta_1\beta u_{1,xx} + \delta_2\gamma u_{2,xx} - 2\delta_1(\delta_1\beta u_3 - \delta_2\gamma u_4)(\delta_1 u_1^2 - \delta_2 u_2^2) - \\ &\quad - 4\delta_1^2\delta_2(\gamma u_3 + \beta u_4)u_1 u_2], \\ u_{2,t} &= \frac{1}{\alpha^2}[-\delta_1\gamma u_{1,xx} + \delta_1\beta u_{2,xx} + 2\delta_1^2(\gamma u_3 + \beta u_4)(\delta_1 u_1^2 - \delta_2 u_2^2) - \\ &\quad - 4\delta_1^2(\delta_1\beta u_3 - \delta_2\gamma u_4)u_1 u_2], \\ u_{3,t} &= -\frac{1}{\alpha^2}[\delta_1\beta u_{3,xx} - \delta_2\gamma u_{4,xx} - 2\delta_1(\delta_1\beta u_1 + \delta_2\gamma u_2)(\delta_1 u_3^2 - \delta_2 u_4^2) + \\ &\quad + 4\delta_1^2\delta_2(\gamma u_1 - \beta u_2)u_3 u_4], \\ u_{4,t} &= -\frac{1}{\alpha^2}[\delta_1\gamma u_{3,xx} + \delta_1\beta u_{4,xx} - 2\delta_1^2(\gamma u_1 - \beta u_2)(\delta_1 u_3^2 - \delta_2 u_4^2) - \\ &\quad - 4\delta_1^2(\delta_1\beta u_1 + \delta_2\gamma u_2)u_3 u_4], \end{aligned} \tag{3.17}$$

where α is still defined to be $\alpha = \alpha_1 - \alpha_2$. More examples can be found in the literature (see, e.g., [33]).

4. Nonlocal integrable equations

We aim to discuss the classification of nonlocal integrable equations, derived either from single nonlocal group reductions or from pairs of group reductions containing nonlocal ones, in relation to the matrix AKNS spectral problem described above.

There are three cases for replacing the eigenvalue λ: $\lambda \to -\lambda^*, -\lambda, \lambda$, where λ^* is the complex conjugate of λ. Each of these cases generates nonlocal integrable NLS equations. Moreover, there are two cases for replacing λ: $\lambda \to -\lambda^*, \lambda$, each of which generates nonlocal integrable mKdV equations. The replacement $\lambda \to -\lambda$ yields only local reduced integrable mKdV equations [34], while the replacement $\lambda \to \lambda^*$ yields only local reduced NLS and mKdV integrable equations (see, e.g., [35]).

4.1. Single group reduction. By taking single group reduction, we can formulate nice types of the corresponding group reductions,

$$\begin{aligned} &U^\dagger(\tilde{x},\tilde{t},-\lambda^*) = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \\ &U^{\mathrm{T}}(\tilde{x},\tilde{t},-\lambda) = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \\ &U^{\mathrm{T}}(\tilde{x},\tilde{t},\lambda) = \Sigma U(x,t,\lambda)\Sigma^{-1}, \end{aligned} \tag{4.1}$$

where $(\tilde{x},\tilde{t}\,) = (-x,t), (x,-t), (-x,-t)$, and

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \qquad \Sigma_j^\dagger = \Sigma_j \ \text{ or } \ \Sigma_j^{\mathrm{T}} = \Sigma_j, \quad j=1,2, \tag{4.2}$$

where $W^\dagger$ (or W^{T}) denotes the Hermitian conjugate (or transpose) of a matrix W.

Based on the specific form of $Q^{[2]}$, we obtain three types of nonlocal integrable NLS reductions: reverse-space, reverse-time, and reverse-spacetime reductions,

$$\begin{aligned} &U^\dagger(-x,t,-\lambda^*) = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \\ &U^{\mathrm{T}}(x,-t,-\lambda) = -\Sigma U(x,t,\lambda)\Sigma^{-1}, \\ &U^{\mathrm{T}}(-x,-t,\lambda) = \Sigma U(x,t,\lambda)\Sigma^{-1}, \end{aligned} \tag{4.3}$$

which preserve the integrability conditions, ensuring the invariance of the zero-curvature equations. The corresponding potential reductions are given by

$$\begin{aligned} &q(x,t) = -\Sigma_2^{-1} p^\dagger(-x,t)\Sigma_1, \\ &q(x,t) = -\Sigma_2^{-1} p^{\mathrm{T}}(x,-t)\Sigma_1, \\ &q(x,t) = \Sigma_2^{-1} p^{\mathrm{T}}(-x,-t)\Sigma_1. \end{aligned} \tag{4.4}$$

They respectively generate the nonlocal reverse-space integrable NLS equations

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} - 2p\Sigma_2^{-1}p^\dagger(-x,t)\Sigma_1 p), \tag{4.5}$$

the nonlocal reverse-time integrable NLS equations

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} - 2p\Sigma_2^{-1}p^{\mathrm{T}}(x,-t)\Sigma_1 p) \tag{4.6}$$

and the nonlocal reverse-spacetime integrable NLS equations

$$p_t = -\frac{\beta}{\alpha^2} i(p_{xx} + 2p\Sigma_2^{-1}p^{\mathrm{T}}(-x,-t)\Sigma_1 p). \tag{4.7}$$

In (4.5), Σ_1 and Σ_2 are arbitrary invertible Hermitian matrices, while in (4.6) and (4.7), Σ_1 and Σ_2 are arbitrary invertible symmetric matrices. It is easy to see that all three types of nonlocal integrable NLS equations are PT-symmetric.

Based on the specific form of $Q^{[3]}$, we derive two types of complex and real reverse-spacetime integrable reductions,

$$U^{\dagger}(-x,-t,-\lambda^*) = -\Sigma U(x,t,\lambda)\Sigma^{-1} \tag{4.8}$$

and

$$U^{\mathrm{T}}(-x,-t,\lambda) = \Sigma U(x,t,\lambda)\Sigma^{-1}, \tag{4.9}$$

which maintain the integrability conditions and guarantee the invariance of the zero-curvature equations. The respective potential reductions are presented by

$$q(x,t) = -\Sigma_2^{-1}p^{\dagger}(-x,-t)\Sigma_1 \tag{4.10}$$

and

$$q(x,t) = \Sigma_2^{-1}p^{\mathrm{T}}(-x,-t)\Sigma_1. \tag{4.11}$$

The two reductions respectively lead to nonlocal complex reverse-spacetime integrable mKdV equations

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} - 3p\Sigma_2^{-1}p^{\dagger}(-x,-t)\Sigma_1 p_x - 3p_x\Sigma_2^{-1}p^{\dagger}(-x,-t)\Sigma_1 p) \tag{4.12}$$

and nonlocal real reverse-spacetime integrable mKdV equations

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3p\Sigma_2^{-1}p^{\mathrm{T}}(-x,-t)\Sigma_1 p_x + 3p_x\Sigma_2^{-1}p^{\mathrm{T}}(-x,-t)\Sigma_1 p). \tag{4.13}$$

In (4.12), Σ_1 and Σ_2 are arbitrary invertible Hermitian matrices, while in (4.13), Σ_1 and Σ_2 are arbitrary invertible symmetric matrices. It is also easy to see that both types of nonlocal integrable mKdV equations are PT-symmetric.

4.2. Pairs of group reductions. For the NLS equations, we note that we have one type of local reduction in the case of replacing λ: $\lambda \to \lambda^*$, and three types of nonlocal reductions in the cases of replacing λ: $\lambda \to -\lambda, -\lambda^*, \lambda$. Clearly, we can formulate six types of nonlocal reductions: three pairs of local and nonlocal reductions and three pairs of nonlocal and nonlocal reductions. These are classified as follows:

$$\text{types } (\lambda^*,-\lambda),\ (\lambda^*,-\lambda^*),\ (\lambda^*,\lambda),\ (-\lambda,-\lambda^*),\ (-\lambda,\lambda) \text{ and } (-\lambda^*,\lambda). \tag{4.14}$$

For the mKdV equations, we note that there are two types of local reductions when replacing λ: $\lambda \to \lambda^*, -\lambda$, and two types of nonlocal reductions when replacing λ: $\lambda \to -\lambda^*, \lambda$. It follows that we can formulate one type of local reduction $(\lambda^*,-\lambda)$ and five types of nonlocal reductions:

$$\text{Types } (\lambda^*,-\lambda^*),\ (\lambda^*,\lambda),\ (-\lambda,-\lambda^*),\ (-\lambda,\lambda) \text{ and } (-\lambda^*,\lambda). \tag{4.15}$$

The pair of local reductions $(\lambda^*,-\lambda)$ gives rise to Sasa–Satsuma-type integrable mKdV equations [21].

We specify pairs of group reductions for the spectral matrix

$$\begin{aligned} U^{\dagger}(\tilde{x},\tilde{t},\pm\lambda^*) \quad &\text{or} \quad U^{\mathrm{T}}(\tilde{x},\tilde{t},\pm\lambda) = \pm\Sigma U(x,t,\lambda)\Sigma^{-1}, \\ U^{\dagger}(\tilde{x},\tilde{t},\pm\lambda^*) \quad &\text{or} \quad U^{\mathrm{T}}(\tilde{x},\tilde{t},\pm\lambda) = \pm\Delta U(x,t,\lambda)\Delta^{-1}, \end{aligned} \tag{4.16}$$

where Σ and Δ are two Hermitian, or symmetric, matrices, defined by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}. \tag{4.17}$$

In what follows, we consider two specific cases where $m = 2$ and $n = 1$:

$$\Sigma_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \qquad \Sigma_2^{-1} = 1, \qquad \Delta_1 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \qquad \Delta_2^{-1} = 1, \tag{4.18}$$

and

$$\Sigma_1 = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \qquad \Sigma_2^{-1} = 1, \qquad \Delta_1 = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \qquad \Delta_2^{-1} = 1, \tag{4.19}$$

where σ and δ are real and satisfy $\sigma^2 = \delta^2 = 1$. These two group reductions generate scalar nonlocal integrable NLS and mKdV equations, which are listed as follows (see also [15]–[18]).

Six classes of nonlocal integrable NLS equations (where $p_1 = p_{11}$ is assumed):

Class 1: Type $(\lambda^*, -\lambda)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} + 2\sigma(p_1 p_1^* + p_1(x,-t)p_1^*(x,-t))p_1], \\ p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\delta(p_1 p_1(x,-t) + p_1^* p_1^*(x,-t))p_1]. \end{aligned} \tag{4.20}$$

Class 2: Type $(\lambda^*, -\lambda^*)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} + 2\sigma(p_1 p_1^* + p_1(-x,t)p_1^*(-x,t))p_1], \\ p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\delta(p_1 p_1^*(-x,t) + p_1^* p_1(-x,t))p_1]. \end{aligned} \tag{4.21}$$

Class 3: Type (λ^*, λ) equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} + 2\sigma(p_1 p_1^* + p_1(-x,-t)p_1^*(-x,-t))p_1], \\ p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} + 2\delta(p_1 p_1(-x,-t) + p_1^* p_1^*(-x,-t))p_1]. \end{aligned} \tag{4.22}$$

Class 4: Type $(-\lambda, -\lambda^*)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\sigma(p_1 p_1^*(-x,t) + p_1(x,-t)p_1^*(-x,-t))p_1], \\ p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\delta(p_1 p_1(x,-t) + p_1^*(-x,t)p_1^*(-x,-t))p_1]. \end{aligned} \tag{4.23}$$

Class 5: Type $(-\lambda, \lambda)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\sigma(p_1 p_1(x,-t) + p_1(-x,t)p_1(-x,-t))p_1], \\ p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} + 2\delta(p_1 p_1(-x,t) + p_1(x,-t)p_1(-x,-t))p_1]. \end{aligned} \tag{4.24}$$

Class 6: Type $(-\lambda^*, \lambda)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} - 2\sigma(p_1 p_1^*(-x,t) + p_1(-x,-t)p_1^*(x,-t))p_1], \\ p_{1,t} &= -\frac{\beta}{\alpha^2} i[p_{1,xx} + 2\delta(p_1 p_1(-x,-t) + p_1^*(-x,t)p_1^*(x,-t))p_1]. \end{aligned} \tag{4.25}$$

Five classes of nonlocal integrable mKdV equations (where $p_1 = p_{11}$ is again assumed):
Class 1: Type$(\lambda^*, -\lambda^*)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\sigma|p_1|^2 p_{1,x} + 3\sigma p_1^*(-x,-t)(p_1 p_1(-x,-t))_x], \\ p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\delta p_1 p_1^*(-x,-t) p_{1,x} - 3\delta p_1^*(p_1 p_1(-x,-t))_x]. \end{aligned} \tag{4.26}$$

Class 2: Type (λ^*, λ) equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\sigma|p_1|^2 p_{1,x} + 3\sigma p_1(-x,-t)(p_1 p_1^*(-x,-t))_x], \\ p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\delta p_1 p_1(-x,-t) p_{1,x} + 3\delta p_1^*(p_1 p_1^*(-x,-t))_x]. \end{aligned} \tag{4.27}$$

Class 3: Type $(-\lambda, -\lambda^*)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1^2 p_{1,x} - 3\sigma p_1^*(-x,-t)(p_1 p_1^*(-x,-t))_x], \\ p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\delta p_1 p_1^*(-x,-t) p_{1,x} - 3\delta p_1(p_1 p_1^*(-x,-t))_x]. \end{aligned} \tag{4.28}$$

Class 4: Type $(-\lambda, \lambda)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1^2 p_{1,x} - 3\sigma p_1(-x,-t)(p_1 p_1(-x,-t))_x], \\ p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\delta p_1 p_1(-x,-t) p_{1,x} + 3\delta(p_1 p_1(-x,-t))_x p_1]. \end{aligned} \tag{4.29}$$

Class 5: Type $(-\lambda^*, \lambda)$ equations

$$\begin{aligned} p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} - 6\sigma p_1 p_1^*(-x,-t) p_{1,x} - 3\sigma p_1(-x,-t)(|p_1|^2)_x], \\ p_{1,t} &= -\frac{\beta}{\alpha^3}[p_{1,xxx} + 6\delta p_1 p_1(-x,-t) p_{1,x} + 3\delta p_1^*(-x,-t)(|p_1|^2)_x]. \end{aligned} \tag{4.30}$$

5. Soliton solutions

We construct soliton solutions using the Darboux transformation (DT) method [36]–[38], which is equivalent to the reflectionless Riemann–Hilbert problem. The resulting Darboux matrices encompass the most general situation, including the so-called generalized Darboux matrices (see, e.g., [39], [40]).

5.1. General framework of binary DTs. We assume that a binary Darboux transformation is given by

$$\phi' = T^+\phi = T^+(u,\lambda)\phi, \qquad \tilde{\phi}' = \tilde{\phi}T^- = \tilde{\phi}T^-(u,\lambda), \qquad u' = f(u), \tag{5.1}$$

such that

$$-i\phi'_x = U'\phi', \quad -i\phi'_t = V'\phi' \quad \text{and} \quad i\tilde{\phi}'_x = \tilde{\phi}'U', \quad i\tilde{\phi}'_t = \tilde{\phi}'V', \tag{5.2}$$

where

$$U' = U(u',\lambda), \qquad V' = V(u',\lambda). \tag{5.3}$$

The conditions for T^+ and T^- are

$$-iT^+_x T^- + T^+UT^- = U', \qquad -iT^+_t T^- + T^+VT^- = V'. \tag{5.4}$$

5.2. Darboux matrices. To construct Darboux matrices, we choose two sets of arbitrary numbers $\{\lambda_k, \hat{\lambda}_k \in \mathbb{C}\}_{k=1}^N$, where $N \in \mathbb{N}$. Then we define the Darboux matrices

$$T^+ = I_{m+n} - \sum_{j,l=1}^{N} \frac{v_j (M^{-1})_{jl} \hat{v}_l}{\lambda - \hat{\lambda}_l}, \qquad T^- = I_{m+n} + \sum_{j,l=1}^{N} \frac{v_j (M^{-1})_{jl} \hat{v}_l}{\lambda - \lambda_j}, \tag{5.5}$$

where the eigenvectors and adjoint eigenvectors are defined by

$$\begin{aligned} -iv_{k,x} &= U(u, \lambda_k) v_k, \qquad & -iv_{k,t} &= V^{[r]}(u, \lambda_k) v_k, \\ i\hat{v}_{k,x} &= \hat{v}_k U(u, \hat{\lambda}_k), \qquad & i\hat{v}_{k,t} &= \hat{v}_k V^{[r]}(u, \hat{\lambda}_k), \end{aligned} \qquad 1 \leqslant k \leqslant N, \tag{5.6}$$

with $r = 2, 3$ corresponding to the respective NLS and mKdV equations.

5.3. *M*-matrices. To satisfy the corresponding spectral problems, we introduce the square matrix M as

$$M = (m_{jl})_{N \times N}, \qquad m_{jl} = \begin{cases} \dfrac{\hat{v}_j v_l}{\lambda_l - \hat{\lambda}_j}, & \text{if } \lambda_l \neq \hat{\lambda}_j, \\ m^c_{jl}(x,t), & \text{if } \lambda_l = \hat{\lambda}_j, \end{cases} \qquad 1 \leqslant j, l \leqslant N, \tag{5.7}$$

where we require the orthogonality condition

$$\hat{v}_j v_l = 0, \tag{5.8}$$

and the two ODE evolution properties

$$m^c_{jl,x} = i\hat{v}_j \frac{U(\lambda_l) - U(\hat{\lambda}_j)}{\lambda_l - \hat{\lambda}_j} v_l, \qquad m^c_{jl,t} = i\hat{v}_j \frac{V^{[r]}(\lambda_l) - V^{[r]}(\hat{\lambda}_j)}{\lambda_l - \hat{\lambda}_j} v_l \tag{5.9}$$

if $\lambda_l = \hat{\lambda}_j$, $1 \leqslant j, l \leqslant N$, where $r = 2, 3$. It is straightforward to see that T^+ and $(T^-)^{-1}$ are inverse to each other. These conditions are both necessary and sufficient to guarantee the correctness of the binary Darboux transformations.

We note that the case $\lambda_l = \hat{\lambda}_j$, $1 \leqslant j, l \leqslant N$, can yield the so-called generalized Darboux matrices. Furthermore, an iterated sequence of these Darboux matrices in the standard case (i.e., without the aforementioned condition) can be linked to the study of the algebraic properties of n-simplex maps, such as the local Yang–Baxter equation [41]. A decomposition of these Darboux matrices has also been explored in the literature (see, e.g., [42]).

5.4. Solitons by asymptotic expansions. We now expand T^+ at $\lambda = \infty$ as

$$T^+(x, \lambda) = I_{m+n} + \frac{1}{\lambda} T_1^+(x) + O\left(\frac{1}{\lambda^2}\right), \qquad \lambda \to \infty, \tag{5.10}$$

to obtain the potential matrix

$$P = -[\Lambda, T_1^+] = \lim_{\lambda \to \infty} \lambda [T^+(\lambda), \Lambda]. \tag{5.11}$$

In other words, we obtain the so-called soliton solutions as

$$p_{jk} = -\alpha (T_1^+)_{j,k+m}, \qquad 1 \leqslant j \leqslant m, \ 1 \leqslant k \leqslant n, \tag{5.12}$$

where $T_1^+ = ((T_1^+)_{jk})_{(m+n)\times(m+n)}$. The reduction properties for the new potential matrices are inherited from the original group reduction properties.

6. Conclusions, future work, and open questions

In this paper, we have discussed a method of group reductions for matrix spectral problems to generate nonlocal integrable equations within the Lax pair formulation. A comprehensive classification of nonlocal NLS and mKdV integrable equations generated both by single group reduction and by pairs of group reductions from the AKNS matrix spectral problems has been provided. Moreover, we have presented a general framework for binary Darboux transformations to derive soliton solutions for the resulting nonlocal reduced integrable equations, alongside novel solution phenomena in the nonlocal setting.

Regarding nonlocal differential equations (DEs), there are several directions that we are particularly interested in exploring. Below are two of these directions.

Other applications of group reductions. It is one of our future problems to explore how to apply the group reduction idea to other matrix spectral problems. For instance, we can consider many 4×4 matrix spectral problems that generate four-component Hamiltonian integrable equations. One of the involved spectral matrices is

$$U = U(u, \lambda) = \begin{bmatrix} \alpha_1 \lambda & u_1 & u_2 & 0 \\ u_3 & \alpha_2 \lambda & 0 & u_4 \\ u_4 & 0 & \alpha_2 \lambda & -u_3 \\ 0 & u_2 & -u_1 & \alpha_1 \lambda \end{bmatrix}, \tag{6.1}$$

where $u = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$, λ is the spectral parameter, and α_1 and α_2 are two distinct constants.

Solution structures of nonlocal linear DEs. Another very interesting problem is how to find solutions of nth-order nonlocal linear DEs. For example, how to systematically derive the solution formulas for the equation

$$x^{(n)}(t) = \lambda x'(t) + \mu x'(-t) + \nu x(t) + \delta x(-t) + f(t), \tag{6.2}$$

where n is a natural number, $x^{(n)}$ denotes the nth-order derivative of x, $\lambda, \mu, \nu, \delta$ are real constants, and f is a continuous function.

Additionally, there are open questions regarding the determination of the well-posedness of nonlocal DEs.

Existence and uniqueness of nonlocal DEs. What conditions must be satisfied for the existence of a unique solution of nonlocal DEs of the types

$$x'(t) = f(t, x(t), x(-t)) \tag{6.3}$$

and

$$x'(t) = f\left(t, x(t), x\left(\frac{1}{t}\right)\right), \tag{6.4}$$

and how do these properties depend on the structure of the nonlocal terms and on the initial and boundary conditions?

Stability of nonlocal DEs. Under what conditions can we guarantee the stability properties of nonlocal DEs? How are the stability properties affected by the nonlocal structure and the solution conditions? Specifically, how can we establish uniform or asymptotic stability for the solutions of nonlocal equations of the types described above?

Well-posedness for nonlocal integrable equations. How can we establish the existence, uniqueness, and stability (spectral, orbital, and asymptotic) for the nonlocal integrable equations we have presented? This includes the well-posedness of the Cauchy problems for the nonlocal integrable NLS equations of the following types.
Type $(-\lambda, -\lambda^*)$:

$$\begin{aligned} p_{1,t} &= -i[p_{1,xx} - 2\sigma(p_1 p_1^*(-x,t) + p_1(x,-t)p_1^*(-x,-t))p_1], \\ p_{1,t} &= -i[p_{1,xx} - 2\delta(p_1 p_1(x,-t) + p_1^*(-x,t)p_1^*(-x,-t))p_1]; \end{aligned} \tag{6.5}$$

type $(-\lambda, \lambda)$:

$$\begin{aligned} p_{1,t} &= -i[p_{1,xx} - 2\sigma(p_1 p_1(x,-t) + p_1(-x,t)p_1(-x,-t))p_1], \\ p_{1,t} &= -i[p_{1,xx} + 2\delta(p_1 p_1(-x,t) + p_1(x,-t)p_1(-x,-t))p_1]; \end{aligned} \tag{6.6}$$

type $(-\lambda^*, \lambda)$:

$$\begin{aligned} p_{1,t} &= -i[p_{1,xx} - 2\sigma(p_1 p_1^*(-x,t) + p_1(-x,-t)p_1^*(x,-t))p_1], \\ p_{1,t} &= -i[p_{1,xx} + 2\delta(p_1 p_1(-x,-t) + p_1^*(-x,t)p_1^*(x,-t))p_1]; \end{aligned} \tag{6.7}$$

where σ and δ are both taken to be ± 1. The solution can exhibit either rarefaction waves or compression waves, depending on the initial conditions and the governing equations. Any contribution to these equations would introduce new ideas and techniques for tackling nonlocal DEs and determining their solution properties.

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