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A COMBINED GENERALIZED KAUP–NEWELL SOLITON HIERARCHY AND ITS HEREDITARY RECURSION OPERATOR AND BI-HAMILTONIAN STRUCTURE

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On the basis of a specific matrix Lie algebra, we propose a Kaup–Newell-type matrix eigenvalue problem with four potentials and compute an associated soliton hierarchy within the zero-curvature formulation. A hereditary recursion operator and a bi-Hamiltonian structure are presented to show the Liouville integrability of the resulting soliton hierarchy. An illustrative example is a novel model consisting of combined derivative nonlinear Schrödinger equations with two arbitrary constants.

Keywords: matrix eigenvalue problem, zero-curvature equation, integrable hierarchy, derivate nonlinear Schrödinger equations

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1. Introduction

Integrable models are associated with matrix eigenvalue problems [1], [2], called Lax pairs [3], and they possess hereditary recursion operators, which generate symmetries from symmetries, and bi-Hamiltonian structures, which connect symmetries with conserved quantities [4]. Matrix eigenvalue problems are also used to establish inverse scattering transforms, which solve initial value problems [1]. Integrable models have various applications in physical sciences and engineering, such as water waves, nonlinear optics, and quantum mechanics [2].

There are well-known examples of integrable models, including the Ablowitz–Kaup–Newell–Segur integrable models [5] and their integrable couplings [6]. Matrix Lie algebras lay a strong foundation for integrable models within the zero-curvature formulation [6]–[8]. It is always intriguing to see what kind of matrix eigenvalue problems can yield integrable models. In this paper, we propose a novel Kaup–Newell-type 4×4 matrix eigenvalue problem and construct an associated soliton hierarchy, starting from a specific matrix Lie algebra.

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The zero-curvature formulation is powerful in constructing integrable models (see [8], [9] for details). As usual, we let $u = (u_1, \ldots, u_q)^{\mathrm{T}}$ denote a column potential vector and λ denote the spectral parameter. Let $\tilde{g}$ be a given loop matrix algebra $\tilde{g}$ with the loop parameter λ. A matrix F_0 in $\tilde{g}$ is said to be pseudo-regular if it satisfies

$$\operatorname{Im} \operatorname{ad}_{F_0} \oplus \operatorname{Ker} \operatorname{ad}_{F_0} = \tilde{g}, \qquad [\operatorname{Ker} \operatorname{ad}_{F_0}, \operatorname{Ker} \operatorname{ad}_{F_0}] = 0, \tag{1.1}$$

where ad_{F_0} denotes the adjoint action of F_0 on $\tilde{g}$. We take one pseudo-regular matrix F_0 and q linear independent matrices $F_1, \ldots, F_q$ in $\tilde{g}$ to introduce a spatial spectral matrix

$$\mathcal{M} = \mathcal{M}(u, \lambda) = F_0(\lambda) + u_1 F_1(\lambda) + \cdots + u_q F_q(\lambda). \tag{1.2}$$

Then we compute a Laurent series solution

$$Z = \sum_{n \geqslant 0} \lambda^{-n} Z^{[n]},$$

of the stationary zero-curvature equation

$$Z_x = [\mathcal{M}, Z] \tag{1.3}$$

in the underlying loop algebra $\tilde{g}$. The pseudoregularity guarantees the existence of such Laurent series solutions.

The second step is to find an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (\lambda^m Z)_+ + \Delta_m, \qquad (\lambda^m Z)_+ = \sum_{n=0}^{m} \lambda^{m-n} Z^{[n]}, \qquad m \geqslant 0, \quad \Delta_m \in \tilde{g}, \tag{1.4}$$

that provide the other parts of Lax pairs, such that the zero curvature equations

$$\mathcal{M}_{t_m} - \mathcal{N}^{[m]}_x + [\mathcal{M}, \mathcal{N}^{[m]}] = 0, \qquad m \geqslant 0, \tag{1.5}$$

produce a soliton hierarchy

$$u_{t_m} = X^{[m]} = X^{[m]}(u), \qquad m \geqslant 0. \tag{1.6}$$

The zero-curvature equations in (1.5) actually represent the solvability conditions of the spatial and temporal matrix eigenvalue problems:

$$\varphi_x = \mathcal{M} \varphi, \qquad \varphi_{t_m} = \mathcal{N}^{[m]} \varphi, \qquad m \geqslant 0. \tag{1.7}$$

To determine the modification terms Δ_m, $m \geqslant 0$, we often need the trial and error strategy.

The third step is to furnish a bi-Hamiltonian structure for the resulting soliton hierarchy (1.6), by determining a hereditary recursion operator and applying the so-called trace identity,

$$\frac{\delta}{\delta u} \int \operatorname{tr}\Big(Z \frac{\partial \mathcal{M}}{\partial \lambda} \Big)\, dx = \lambda^{-\kappa} \frac{\partial}{\partial \lambda} \lambda^{\kappa} \operatorname{tr}\Big(Z \frac{\partial \mathcal{M}}{\partial u} \Big), \tag{1.8}$$

where $\delta/\delta u$ is the variational derivative with respect to u, and κ is a constant independent of the spectral parameter λ. It then follows that every member of the soliton hierarchy has a bi-Hamiltonian structure and thus Liouville integrability (see, e.g., [8]–[10]).

Various hierarchies of Liouville integrable models exist in the literature [5]–[20]. Among the one-component integrable hierarchies are the Korteweg–de Vries hierarchy, the nonlinear Schrödinger hierarchy, and the modified Korteweg–de Vries hierarchy [1], [2]. The case of two components is also very popular, and the well-known examples include the Ablowitz–Kaup–Newell–Segur integrable hierarchy [5], the Heisenberg integrable hierarchy [21], the Kaup–Newell integrable hierarchy [22], and the Wadati–Konno–Ichikawa integrable hierarchy [23]. All those soliton hierarchies are associated with 2×2 spectral matrices. The case of higher-order spectral matrices leads to a higher level of difficulty.

Our aim in this paper is to propose a specific Kaup–Newell-type 4×4 spectral matrix and construct an associated hierarchy of four-component Liouville integrable models within the zero-curvature formulation, based on a special matrix Lie algebra. A hereditary recursion operator and a bi-Hamiltonian structure are determined to show the Liouville integrability for the resulting soliton hierarchy. An illustrative example is presented, which consists of combined generalized integrable derivative nonlinear Schrödinger equations. A summary and concluding remarks are given in the final section (Sec. 4).

2. A soliton hierarchy with four potentials

We take an arbitrary real number δ and a square matrix T of order $r \in \mathbb{N}$, such that

$$T^2 = I_r, \tag{2.1}$$

where I_r stands for the identity matrix of order r. We define a set $\tilde{g}$ of block matrices

$$\tilde{g} = \left\{ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2r\times 2r} \,\middle|\, A_4 = TA_1T^{-1},\ A_3 = \delta T A_2 T^{-1} \right\}. \tag{2.2}$$

Obviously, this forms a matrix Lie algebra, with the matrix commutator $[A, B] = AB - BA$ being the Lie bracket. In what follows, we use this matrix Lie algebra in the case $r = 2$, $\delta = 1$ and

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{2.3}$$

to formulate a specific spectral matrix.

Let $u = u(x,t) = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$ be a column vector with four potentials, and α_1 and α_2 be two arbitrary real numbers; we assume that

$$\alpha = \alpha_1 - \alpha_2 \neq 0. \tag{2.4}$$

Motivated by recent diverse studies on matrix eigenvalue problems involving four potentials (see, e.g., [24]–[27] and [28], [29] for examples of respective arbitrary-order and fourth-order matrix eigenvalue problems), we introduce and study a matrix eigenvalue problem of the form

$$\varphi_x = \mathcal{M}\varphi = \mathcal{M}(u,\lambda)\varphi, \qquad \mathcal{M} = \begin{bmatrix} 0 & \lambda u_1 & \lambda u_2 & \alpha_1\lambda^2 \\ \lambda u_3 & 0 & \alpha_2\lambda^2 & \lambda u_4 \\ \lambda u_4 & \alpha_2\lambda^2 & 0 & \lambda u_3 \\ \alpha_1\lambda^2 & \lambda u_2 & \lambda u_1 & 0 \end{bmatrix}, \tag{2.5}$$

where λ is again the spectral parameter. This spectral matrix $\mathcal{M}$ is built from the above matrix Lie algebra $\tilde{g}$, and it is a kind of 4×4 matrix generalization of the Kaup–Newell eigenvalue problem [22]. It is not an easy job to determine a Lax pair of 4×4 matrix eigenvalue problems that produces a hierarchy of integrable models. Interestingly, the above eigenvalue problem generates an associated integrable hierarchy possessing a hereditary recursion operator and a bi-Hamiltonian structure. All equations in the hierarchy exhibit specific combined structures.

To construct an associated soliton hierarchy, we usually start with the corresponding stationary zero-curvature equation (1.3). Based on the presented matrix Lie algebra, we try to take

$$Z = \begin{bmatrix} a & b & e & f \\ c & -a & -f & g \\ g & -f & -a & c \\ f & e & b & a \end{bmatrix} = \sum_{n\geqslant 0} \lambda^{-n} Z^{[n]}. \tag{2.6}$$

The reason to take this form is that with the specific spectral matrix $\mathcal{M}$ in (2.5), an arbitrary matrix in $\tilde{g}$ leads to a commutator matrix of the above form (2.6). Consequently, the corresponding stationary zero-curvature equation (1.3) becomes

$$\begin{cases} a_x = \lambda c u_1 + \lambda g u_2 - \lambda b u_3 - \lambda e u_4, \\ b_x = \alpha\lambda^2 e - 2\lambda a u_1 - 2\lambda f u_2, \\ c_x = -\alpha\lambda^2 g + 2\lambda a u_3 + 2\lambda f u_4, \end{cases} \tag{2.7}$$

$$\begin{cases} e_x = \alpha\lambda^2 b - 2\lambda f u_1 - 2\lambda a u_2, \\ g_x = -\alpha\lambda^2 c + 2\lambda f u_3 + 2\lambda a u_4, \\ f_x = \lambda g u_1 + \lambda c u_2 - \lambda e u_3 - \lambda b u_4. \end{cases} \tag{2.8}$$

Given these equations, the basic objects of a solution Z can be assumed to be as follows:

$$\begin{aligned} a &= \sum_{n\geqslant 0} \lambda^{-2n} a^{[n]}, & b &= \sum_{n\geqslant 0} \lambda^{-2n-1} b^{[n]}, & c &= \sum_{n\geqslant 0} \lambda^{-2n-1} c^{[n]}, \\ e &= \sum_{n\geqslant 0} \lambda^{-2n-1} e^{[n]}, & f &= \sum_{n\geqslant 0} \lambda^{-2n} f^{[n]}, & g &= \sum_{n\geqslant 0} \lambda^{-2n-1} g^{[n]}. \end{aligned} \tag{2.9}$$

To determine a solution Z recursively, the following two equations

$$\begin{aligned} -\alpha\lambda f_x &= u_3 b_x + u_1 c_x + u_4 e_x + u_2 g_x, \\ -\alpha\lambda a_x &= u_4 b_x + u_2 c_x + u_3 e_x + u_1 g_x \end{aligned} \tag{2.10}$$

are crucial, which can be verified directly. At this moment, we can see that Eqs. (2.7) and (2.8) lead to the two initial equations

$$\begin{aligned} a_x^{[0]} &= u_1 c^{[0]} + u_2 g^{[0]} - u_3 b^{[0]} - u_4 e^{[0]}, \\ f_x^{[0]} &= u_1 g^{[0]} + u_2 c^{[0]} - u_3 e^{[0]} - u_4 b^{[0]} \end{aligned} \tag{2.11}$$

and the recursion relations for the Laurent series solution:

$$\begin{cases} a_x^{[n+1]} = -\frac{1}{\alpha}(u_4 b_x^{[n]} + u_2 c_x^{[n]} + u_3 e_x^{[n]} + u_1 g_x^{[n]}), \\ f_x^{[n+1]} = -\frac{1}{\alpha}(u_3 b_x^{[n]} + u_1 c_x^{[n]} + u_4 e_x^{[n]} + u_2 g_x^{[n]}), \end{cases} \tag{2.12}$$

$$\begin{cases} b^{[n+1]} = \frac{1}{\alpha}(e_x^{[n]} + 2u_1 f^{[n+1]} + 2u_2 a^{[n+1]}), \\ c^{[n+1]} = \frac{1}{\alpha}(-g_x^{[n]} + 2u_3 f^{[n+1]} + 2u_4 a^{[n+1]}), \end{cases} \tag{2.13}$$

$$\begin{cases} e^{[n+1]} = \frac{1}{\alpha}(b_x^{[n]} + 2u_1 a^{[n+1]} + 2u_2 f^{[n+1]}), \\ g^{[n+1]} = \frac{1}{\alpha}(-c_x^{[n]} + 2u_3 a^{[n+1]} + 2u_4 f^{[n+1]}), \end{cases} \tag{2.14}$$

where $n \geqslant 0$. To achieve the uniqueness of the Laurent series solution, we fix the initial data by solving (2.11),

$$\begin{aligned} b^{[0]} &= \beta u_1 + \gamma u_2, \qquad & c^{[0]} &= \beta u_3 + \gamma u_4, \\ e^{[0]} &= \beta u_2 + \gamma u_1, \qquad & g^{[0]} &= \beta u_4 + \gamma u_3, \\ a^{[0]} &= \text{const}, \qquad & f^{[0]} &= \text{const} \end{aligned} \tag{2.15}$$

with two arbitrary constants β and γ, and select the constants of integration in (2.12) to be zero:

$$a^{[n]}|_{u=0} = 0, \qquad f^{[n]}|_{u=0} = 0, \qquad n \geqslant 1. \tag{2.16}$$

The initial values for $a^{[0]}$ and $f^{[0]}$ do not affect all other coefficients in the Laurent series solution, but the two constants β and γ bring the diversity of the associated integrable models, exhibiting a combined structure in the resulting models. We can now compute that

$$\begin{cases} a^{[1]} = -\frac{1}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2], \\ f^{[1]} = -\frac{1}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2], \end{cases}$$

$$\begin{cases} b^{[1]} = \frac{1}{\alpha}\big\{\gamma u_{1,x} + \beta u_{2,x} - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1 - \\ \qquad -\frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_2\big\}, \\ c^{[1]} = \frac{1}{\alpha}\big\{-\gamma u_{3,x} - \beta u_{4,x} - \frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3 - \\ \qquad -\frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_4\big\}, \end{cases}$$

$$\begin{cases} e^{[1]} = \frac{1}{\alpha}\big\{\beta u_{1,x} + \gamma u_{2,x} - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_1 - \\ \qquad -\frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\big\}, \\ g^{[1]} = \frac{1}{\alpha}\big\{-\beta u_{3,x} - \gamma u_{4,x} - \frac{2}{\alpha}[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_3 - \\ \qquad -\frac{2}{\alpha}[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\big\}. \end{cases}$$

Based on a careful inspection of the corresponding zero-curvature equation, we can introduce the temporal matrix eigenvalue problems

$$\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(u,\lambda)\varphi, \qquad \mathcal{N}^{[m]} = \lambda(\lambda^{2m+1}Z)_+, \qquad m \geqslant 0 \tag{2.17}$$

where the subscript + denotes the polynomial part of λ as in (1.4), to generate associated integrable models. The solvability conditions of the spatial and temporal matrix eigenvalue problems in (2.5) and (2.17) are exactly the zero-curvature equations in (1.5). All those zero-curvature equations yield a soliton hierarchy with four potentials:

$$u_{t_m} = X^{[m]} = X^{[m]}(u) = (b_x^{[m]}, e_x^{[m]}, c_x^{[m]}, g_x^{[m]})^{\mathrm{T}}, \qquad m \geqslant 0. \tag{2.18}$$

More precisely, we obtain a hierarchy of models with four equations

$$u_{1,t_m} = b_x^{[m]}, \qquad u_{2,t_m} = e_x^{[m]}, \qquad u_{3,t_m} = c_x^{[m]}, \qquad u_{4,t_m} = g_x^{[m]}, \qquad m \geqslant 0. \tag{2.19}$$

The first nonlinear example in this hierarchy is the model of generalized integrable derivative nonlinear Schrödinger equations

$$\begin{cases} u_{1,t_1} = \frac{1}{\alpha}(\gamma u_{1,xx} + \beta u_{2,xx}) - \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_1\}_x - \\ \qquad -\frac{2}{\alpha^2}\{(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_2\}_x, \\ u_{2,t_1} = \frac{1}{\alpha}(\beta u_{1,xx} + \gamma u_{2,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_1\}_x - \\ \qquad -\frac{2}{\alpha^2}\{(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_2\}_x, \\ u_{3,t_1} = -\frac{1}{\alpha}(\gamma u_{3,xx} + \beta u_{4,xx}) - \frac{2}{\alpha^2}\{[(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_3\}_x - \\ \qquad -\frac{2}{\alpha^2}\{(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_4\}_x, \\ u_{4,t_2} = -\frac{1}{\alpha}(\beta u_{3,xx} + \gamma u_{4,xx}) - \frac{2}{\alpha^2}\{[(\gamma u_3 + \beta u_4)u_1 + (\beta u_3 + \gamma u_4)u_2]u_3\}_x - \\ \qquad -\frac{2}{\alpha^2}\{(\beta u_3 + \gamma u_4)u_1 + (\gamma u_3 + \beta u_4)u_2]u_4\}_x. \end{cases} \tag{2.20}$$

This system provides a combined integrable model with four components, which broadens the category of coupled integrable models of nonlinear Schrödinger type equations (see, e.g., [30]–[32]). One point worth mentioning is that each equation in (2.20) contains a linear combination of two derivative terms of the second order, and we therefore say that such a model is a combined model.

Two special subcases, $\beta = 0$ and $\gamma = 0$, in the resulting soliton hierarchy provide uncombined integrable models.

If we take $\alpha = \beta = 1$ and $\gamma = 0$ in model (2.20), we obtain a coupled integrable derivative nonlinear Schrödinger model

$$\begin{cases} u_{1,t_1} = u_{2,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 + u_2u_3)u_2]_x, \\ u_{2,t_1} = u_{1,xx} - 2[(u_1u_4 + u_2u_3)u_1 + (u_1u_3 + u_2u_4)u_2]_x, \\ u_{3,t_1} = -u_{4,xx} - 2[(u_1u_3 + u_2u_4)u_3 + (u_1u_4 + u_2u_3)u_4]_x, \\ u_{4,t_1} = -u_{3,xx} - 2[(u_1u_4 + u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x. \end{cases} \tag{2.21}$$

If we take $\alpha = \gamma = 1$ and $\beta = 0$ in model (2.20), we obtain another coupled integrable derivative nonlinear Schrödinger model,

$$\begin{cases} u_{1,t_1} = u_{1,xx} - 2[(u_1u_4 + u_2u_3)u_1 + (u_1u_3 + u_2u_4)u_2]_x, \\ u_{2,t_1} = u_{2,xx} - 2[(u_1u_3 + u_2u_4)u_1 + (u_1u_4 + u_2u_3)u_2]_x, \\ u_{3,t_1} = -u_{3,xx} - 2[(u_1u_4 + u_2u_3)u_3 + (u_1u_3 + u_2u_4)u_4]_x, \\ u_{4,t_1} = -u_{4,xx} - 2[(u_1u_3 + u_2u_4)u_3 + (u_1u_4 + u_2u_3)u_4]_x. \end{cases} \tag{2.22}$$

There is an interesting phenomenon that the obtained two models just exchange the first component with the second component and the third component with the fourth component in the vector fields in the right-hand sides. Surprisingly, those two uncombined models still commute with each other.

3. Recursion operator and the bi-Hamiltonian structure

To propose a bi-Hamiltonian structure to show the Liouville integrability of soliton hierarchy (2.19), we can use trace identity (1.8) for the spatial matrix eigenvalue problem in (2.5). From the Laurent series solution Z in (2.6), we readily obtain

$$\begin{aligned} \operatorname{tr}\left(Z\frac{\partial \mathcal{M}}{\partial \lambda}\right) &= 2(2\alpha\lambda f + bu_3 + cu_1 + eu_4 + gu_2), \\ \operatorname{tr}\left(Z\frac{\partial \mathcal{M}}{\partial u}\right) &= 2(\lambda c, \lambda g, \lambda b, \lambda e)^{\mathrm{T}}, \end{aligned} \tag{3.1}$$

and as a consequence of these computations, the trace identity generates

$$\frac{\delta}{\delta u}\int \lambda^{-2n-1}(2\alpha f^{[n+1]}+u_3b^{[n]}+u_4e^{[n]}+u_1c^{[n]}+u_2g^{[n]})\,dx = \\ = \lambda^{-\kappa}\frac{\partial}{\partial\lambda}\lambda^{\kappa-2n}(c^{[n]},g^{[n]},b^{[n]},e^{[n]})^{\mathrm{T}}, \qquad n\geqslant 0. \tag{3.2}$$

Checking with $n=1$ leads to $\kappa=0$, and we therefore arrive at

$$\frac{\delta}{\delta u}\mathcal{H}^{[n]}=(c^{[n]},g^{[n]},b^{[n]},e^{[n]})^{\mathrm{T}}, \qquad n\geqslant 0, \tag{3.3}$$

where the Hamiltonian functionals are given by

$$\begin{aligned}\mathcal{H}^{[0]} &= \int \frac{1}{2}[u_1(\beta u_3+\gamma u_4)+u_2(\beta u_4+\gamma u_3)+ \\ &\quad + u_3(\beta u_1+\gamma u_2)+u_4(\beta u_2+\gamma u_1)]\,dx, \\ \mathcal{H}^{[n]} &= -\int \frac{1}{2n}(2\alpha f^{[n+1]}+u_3b^{[n]}+u_1c^{[n]}+u_4e^{[n]}+u_2g^{[n]})\,dx, \qquad n\geqslant 1.\end{aligned} \tag{3.4}$$

The above Hamiltonian functionals $\mathcal{H}^{[0]}$ have been computed directly. The results in (3.3) enable us to produce a Hamiltonian structure for soliton hierarchy (2.19),

$$u_{t_m}=X^{[m]}=J_1\frac{\delta\mathcal{H}^{[m]}}{\delta u}, \qquad m\geqslant 0, \tag{3.5}$$

where the Hamiltonian operator J_1 is defined by

$$J_1=\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \partial & 0 \\ & & 0 & \partial \\ \hline \partial & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \partial & & \end{array}\right], \tag{3.6}$$

and the functionals $\mathcal{H}^{[m]}$ are determined by (3.4). With this Hamiltonian structure, we have an interrelation $Y=J_1\frac{\delta\mathcal{H}}{\delta u}$ between a symmetry Y and a conserved functional $\mathcal{H}$ of each model in the hierarchy.

The characteristic Abelian algebra of vector fields $X^{[n]}$

$$[\![X^{[n_1]},X^{[n_2]}]\!]=X^{[n_1]\prime}(u)[X^{[n_2]}]-X^{[n_2]\prime}(u)[X^{[n_1]}]=0, \qquad n_1,n_2\geqslant 0, \tag{3.7}$$

can be derived from the algebra of Lax operators

$$[\![\mathcal{N}^{[n_1]},\mathcal{N}^{[n_2]}]\!]=\mathcal{N}^{[n_1]\prime}(u)[X^{[n_2]}]-\mathcal{N}^{[n_2]\prime}(u)[X^{[n_1]}]+[\mathcal{N}^{[n_1]},\mathcal{N}^{[n_2]}]=0 \tag{3.8}$$

for $n_1,n_2\geqslant 0$. This is a direct result from the relation between the isospectral zero-curvature equations (see [33] for details).

On the other hand, based on the recursion relation $X^{[m+1]} = \Phi X^{[m]}$, we can derive a hereditary recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ [4] for soliton hierarchy (2.19). The operator Φ is given by

$$\begin{cases} \Phi_{11} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_3 + \partial u_2\partial^{-1}u_4), \\ \Phi_{12} = \frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_4 + \partial u_2\partial^{-1}u_3), \\ \Phi_{13} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_1 + \partial u_2\partial^{-1}u_2), \\ \Phi_{14} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_2 + \partial u_2\partial^{-1}u_1); \end{cases} \tag{3.9}$$

$$\begin{cases} \Phi_{21} = \frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_4 + \partial u_2\partial^{-1}u_3), \\ \Phi_{22} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_3 + \partial u_2\partial^{-1}u_4), \\ \Phi_{23} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_2 + \partial u_2\partial^{-1}u_1), \\ \Phi_{24} = -\frac{2}{\alpha^2}(\partial u_1\partial^{-1}u_1 + \partial u_2\partial^{-1}u_2); \end{cases} \tag{3.10}$$

$$\begin{cases} \Phi_{31} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3 + \partial u_4\partial^{-1}u_4), \\ \Phi_{32} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4 + \partial u_4\partial^{-1}u_3), \\ \Phi_{33} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1 + \partial u_4\partial^{-1}u_2), \\ \Phi_{34} = -\frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2 + \partial u_4\partial^{-1}u_1); \end{cases} \tag{3.11}$$

$$\begin{cases} \Phi_{41} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_4 + \partial u_4\partial^{-1}u_3), \\ \Phi_{42} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_3 + \partial u_4\partial^{-1}u_4), \\ \Phi_{43} = -\frac{1}{\alpha}\partial_x - \frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_2 + \partial u_4\partial^{-1}u_1), \\ \Phi_{44} = -\frac{2}{\alpha^2}(\partial u_3\partial^{-1}u_1 + \partial u_4\partial^{-1}u_2). \end{cases} \tag{3.12}$$

The hereditariness of Φ means [34] that it satisfies

$$L_{\Phi X}\Phi = \Phi L_X \Phi \tag{3.13}$$

for an arbitrary vector field X, where the Lie derivative $L_X\Phi$ is defined by

$$(L_X\Phi)Y = \Phi[\![X, Y]\!] - [\![X, \Phi Y]\!].$$

An operator $\Psi = \Psi(x, t, u, u_x, \dots)$ is a recursion operator of a given evolution equation $u_t = X(u)$ iff Ψ satisfies

$$\frac{\partial \Psi}{\partial t} + L_X\Psi = 0. \tag{3.14}$$

We can readily verify that $L_{X^{[0]}}\Phi = 0$, whence

$$L_{X^{[m]}}\Phi = L_{\Phi X^{[m-1]}}\Phi = \Phi L_{X^{[m-1]}}\Phi = \cdots = \Phi^m L_{X^{[0]}}\Phi = 0, \qquad m \geqslant 1. \tag{3.15}$$

This implies that Φ is a recursion operator for each model in hierarchy (2.19). Several symbolic algorithms are also available in the literature for computing recursion operators of given nonlinear partial differential equations directly (see, e.g., [35]).

By direct analysis, we can show that J_1 and $J_2 = \Phi J_1$ constitute a Hamiltonian pair. Namely, an arbitrary linear combination J of J_1 and J_2 is again Hamiltonian, because it satisfies

$$\int (Y^{[1]})^{\mathrm{T}} J'(u)[JY^{[2]}]Y^{[3]}\,dx + \mathrm{cycle}(Y^{[1]}, Y^{[2]}, Y^{[3]}) = 0, \tag{3.16}$$

where the $Y^{[i]}$ are arbitrary vector fields. Accordingly, hierarchy (2.19) possesses a bi-Hamiltonian structure [36]:

$$u_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta u} = J_2 \frac{\delta \mathcal{H}^{[m-1]}}{\delta u}, \qquad m \geqslant 1. \tag{3.17}$$

It then follows that the associated Hamiltonian functionals commute with each other under the corresponding two Poisson brackets [8]:

$$\begin{aligned} \{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} &= \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta u} \right)^{\mathrm{T}} J_1 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u} \, dx = 0, \\ \{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} &= \int \left(\frac{\delta \mathcal{H}^{[n_1]}}{\delta u} \right)^{\mathrm{T}} J_2 \frac{\delta \mathcal{H}^{[n_2]}}{\delta u} \, dx = 0, \end{aligned} \qquad n_1, n_2 \geqslant 0. \tag{3.18}$$

The bi-Hamiltonian structure also implies the hereditary property of the recursion operator Φ [4].

To conclude, each model in hierarchy (2.19) is Liouville integrable and possesses infinitely many commuting symmetries $\{X^{[n]}\}_{n=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$. One particular illustrative integrable model is the system in (2.20), which adds to the existing category of nonlinear combined Liouville integrable Hamiltonian models with four components.

4. Conclusions

Based on a specific matrix Lie algebra, a Kaup–Newell type (4×4) matrix eigenvalue problem was proposed and a hierarchy of four-component integrable models was successfully generated within the zero-curvature formulation. The key is to find a particular Laurent series solution of the corresponding stationary zero-curvature equation. The resulting soliton hierarchy was shown to be bi-Hamiltonian and therefore Liouville integrable, by determining a hereditary recursion operator and a Hamiltonian structure.

It would be interesting to explore what kind of mathematical structures of solitons could exist for the resulting integrable models. Various powerful and effective approaches could be used, including the Riemann–Hilbert technique [37], the Zakharov–Shabat dressing method [38], the Darboux transformation [39]–[41], and the determinant approach [42]. Besides soliton solutions, lump, kink, breather, and rogue-wave solutions, particular interactions between them (see, e.g., [43]–[50]) are also of great interest. They can be derived from soliton solutions by conducting wave number reductions. Nonlocal reduced integrable models are another important area; they can be generated by taking nonlocal group reductions or similarity transformations of matrix eigenvalue problems (see, e.g., [51]–[53]). It is a novel topic to explore solitons of nonlocal integrable models that are significant in mathematics and physics.

Integrable models are of great interest, and they are built around connections with various branches of mathematics, such as algebraic geometry, Lie theory, and the theory of Hamiltonian equations. The interplay between mathematics and physics enriches both fields and often leads to discoveries of new mathematical structures.

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