Theoretical and Mathematical Physics, **216**(2): 1180–1188 (2023)

FOUR-COMPONENT INTEGRABLE HIERARCHIES OF HAMILTONIAN EQUATIONS WITH $(m+n+2)$TH-ORDER LAX PAIRS

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A class of higher-order matrix spectral problems is formulated and the associated integrable hierarchies are generated via the zero-curvature formulation. The trace identity is used to furnish Hamiltonian structures and thus explore the Liouville integrability of the obtained hierarchies. Illuminating examples are given in terms of coupled nonlinear Schrödinger equations and coupled modified Korteweg–de Vries equations with four components.

Keywords: Lax pair, zero-curvature equation, integrable hierarchy, Hamiltonian structure, NLS equations, mKdV equations

DOI: 10.1134/S0040577923080093

1. Introduction

Integrable Hamiltonian equations of infinite dimensions are a class of partial differential equations (PDEs) that possess infinitely many conserved functionals commuting with respect to the associated Poisson bracket [1]. Such Hamiltonian equations often have a rich array of analytic and geometric structures, the study of which can reveal new and unexpected connections to other areas of mathematical physics. The most famous example is the Korteweg–de Vries equation.

It is known that constructing integrable Hamiltonian equations is a challenging task, requiring a combination of physical intuition, mathematical insight, and technical expertise. A common approach in soliton theory is the zero-curvature formulation. One first formulates Lax pairs of matrix spectral problems and then generate integrable Hamiltonian PDEs via zero-curvature equations [2], [3]. Recursion structures behind matrix spectral problems guarantee the existence of integrable hierarchies of Hamiltonian equations, which commute with respect to the commutator of vector fields over the corresponding jet space.

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The work was supported in part by the NSFC grants 12271488, 11975145, 11972291, and 51771083, the Ministry of Science and Technology of China (G2021016032L), and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020).

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 216, No. 2, pp. 315–325, August, 2023. Received March 3, 2023. Revised April 30, 2023. Accepted May 3, 2023.

We consider PDEs with a vector potential or a dependent variable, $u = (u_1, \dots, u_q)^{\mathrm{T}}$. Let λ denote the spectral parameter in matrix spectral problems. The starting point is a loop algebra $\tilde{g}$ of matrices with the loop parameter λ. We take linearly independent elements $e_1, \dots, e_q$ and a pseudoregular element e_0, i.e., an element satisfying

$$\operatorname{Ker} \operatorname{ad}_{e_0} \oplus \operatorname{Im} \operatorname{ad}_{e_0} = \tilde{g}, \qquad [\operatorname{Ker} \operatorname{ad}_{e_0}, \operatorname{Ker} \operatorname{ad}_{e_0}] = 0. \tag{1.1}$$

We then specify a spectral matrix as

$$U = U(u, \lambda) = e_0(\lambda) + u_1 e_1(\lambda) + \dots + u_q e_q(\lambda). \tag{1.2}$$

The properties of the pseudoregular element e_0 ensure that there exists a Laurent-series solution $Z = \sum_{s \geqslant 0} \lambda^{-s} Z^{[s]}$ of the stationary zero-curvature equation

$$Z_x = i[U, Z]. \tag{1.3}$$

Now, after introducing

$$V^{[r]} = V^{[r]}(u, \lambda) = (\lambda^r Z)_+ + \Delta_r = \sum_{s=0}^{r} \lambda^s Z^{[r-s]} + \Delta_r, \qquad r \geqslant 0, \tag{1.4}$$

an integrable hierarchy of Hamiltonian equations can be represented as a hierarchy of zero-curvature equations

$$U_{t_r} - V_x^{[r]} + i[U, V^{[r]}] = 0, \qquad r \geqslant 0, \tag{1.5}$$

which are the compatibility conditions for the spatial and temporal matrix spectral problems

$$-i\phi_x = U\phi, \qquad -i\phi_{t_r} = V^{[r]}\phi, \quad r \geqslant 0, \tag{1.6}$$

with ϕ being an eigenfunction. Their Hamiltonian structures and corresponding Liouville integrability are typically shown by applying the trace identity [4], [5],

$$\frac{\delta}{\delta u} \int \operatorname{tr}\left(Z \frac{\partial U}{\partial \lambda}\right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \operatorname{tr}\left(Z \frac{\partial U}{\partial u}\right), \tag{1.7}$$

where $\delta/\delta u$ is the variational derivative with respect to u and the constant γ is determined by

$$\gamma = -\frac{\lambda}{2} \frac{\partial}{\partial \lambda} \ln |\operatorname{tr}(Z^2)|. \tag{1.8}$$

Many integrable hierarchies of Hamiltonian equations are presented in the zero-curvature formulation, based on the special linear algebras (see, e.g., [2], [6]–[13]) and the special orthogonal algebras (see, e.g., [14]–[17]). The combination of Hamiltonian structures with recursion structures yields bi-Hamiltonian structures, which exhibit the Liouville integrability of the Hamiltonian equations [18]. Integrable hierarchies with two scalar potentials include the Ablowitz–Kaup–Newell–Segur hierarchy [2], the Kaup–Newell hierarchy [19], the Wadati–Konno–Ichikawa hierarchy [20], and the Heisenberg hierarchy [21], which are associated with the four spectral matrices

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, \quad U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, \quad U = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix}, \quad U = \begin{bmatrix} \lambda r & \lambda p \\ \lambda q & -\lambda r \end{bmatrix},$$

where $pq + r^2 = 1$, and p and q are two scalar potentials. Similar integrable hierarchies are generated from the four counterparts of spectral matrices associated with $so(3, \mathbb{R})$,

$$U = \begin{bmatrix} 0 & q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -\lambda q & -\lambda r \\ \lambda q & 0 & -\lambda p \\ \lambda r & \lambda p & 0 \end{bmatrix},$$

where $p^2 + q^2 + r^2 = 1$ (see, e.g., [15]).

The aim of this paper is to formulate a class of higher-order matrix spectral problems with four components and compute the associated integrable hierarchies within the zero-curvature formulation. Hamiltonian structures of the resulting hierarchies are established by the trace identity. Two illustrative examples are coupled integrable nonlinear Schrödinger equations and coupled integrable modified Korteweg–de Vries equations. The last section is devoted to the concluding remarks.

2. Higher-order Lax pairs and integrable hierarchies

Let m and n be two arbitrary natural numbers and $\delta = \pm 1$. Within the zero-curvature formulation, we introduce an $(m + n + 2)$th-order matrix spectral problem

$$-i\phi_x = U\phi, \qquad U = U(u, \lambda) = \left[\begin{array}{c|cc|c} \lambda & \mathbf{p}_1 & \mathbf{p}_2 & 0 \\ \hline \mathbf{q}_1 & & & \delta \mathbf{p}_1^{\mathrm{T}} \\ & & 0 & \\ \mathbf{q}_2 & & & \mathbf{p}_2^{\mathrm{T}} \\ \hline 0 & \delta \mathbf{q}_1^{\mathrm{T}} & \mathbf{q}_2^{\mathrm{T}} & -\lambda \end{array}\right]_{(m+n+2)\times(m+n+2)}, \tag{2.1}$$

where

$$\mathbf{p}_1 = (\underbrace{p_1, \ldots, p_1}_{m}), \quad \mathbf{p}_2 = (\underbrace{p_2, \ldots, p_2}_{n}), \quad \mathbf{q}_1 = (\underbrace{q_1, \ldots, q_1}_{m})^{\mathrm{T}}, \quad \mathbf{q}_2 = (\underbrace{q_2, \ldots, q_2}_{n})^{\mathrm{T}},$$

and the potential vector u is given by $u = (p_1, p_2, q_1, q_2)^{\mathrm{T}}$. This spectral problem is different from the matrix Ablowitz–Kaup–Newell–Segur spectral problem (see, e.g., [2]).

As usual, we seek a Laurent-series solution of the stationary zero-curvature equation (1.3), and based on machine learning, we can take the solution Z in the form

$$Z = \left[\begin{array}{c|cc|c} a & \mathbf{b}_1 & \mathbf{b}_2 & 0 \\ \hline \mathbf{c}_1 & 0 & \mathbf{d} & \delta \mathbf{b}_1^{\mathrm{T}} \\ \mathbf{c}_2 & -\delta \mathbf{d}^{\mathrm{T}} & 0 & \mathbf{b}_2^{\mathrm{T}} \\ \hline 0 & \delta \mathbf{c}_1^{\mathrm{T}} & \mathbf{c}_2^{\mathrm{T}} & -a \end{array}\right]_{(m+n+2)\times(m+n+2)} = \sum_{s \geqslant 0} \lambda^{-s} Z^{[s]}, \tag{2.2}$$

where

$$\mathbf{b}_1 = (\underbrace{b_1, \ldots, b_1}_{m}), \quad \mathbf{b}_2 = (\underbrace{b_2, \ldots, b_2}_{n}), \quad \mathbf{c}_1 = (\underbrace{c_1, \ldots, c_1}_{m})^{\mathrm{T}}, \quad \mathbf{c}_2 = (\underbrace{c_2, \ldots, c_2}_{n})^{\mathrm{T}},$$

$$\mathbf{d} = d \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m \times n},$$

and the Laurent expansions are

$$a=\sum_{s\geqslant 0}\lambda^{-s}a^{[s]},\quad b_j=\sum_{s\geqslant 0}\lambda^{-s}b_j^{[s]},\quad c_j=\sum_{s\geqslant 0}\lambda^{-s}c_j^{[s]},\quad d=\sum_{s\geqslant 0}\lambda^{-s}d^{[s]}, \tag{2.3}$$

with $j=1,2$.

Then, it is straightforward to see that the corresponding stationary zero-curvature equation (1.3) leads to

$$\begin{aligned}
&b_{1,x}=i(\lambda b_1-p_1a-\delta np_2d), \qquad b_{2,x}=i(\lambda b_2-p_2a+mp_1d),\\
&c_{1,x}=-i(\lambda c_1-q_1a+nq_2d), \qquad c_{2,x}=-i(\lambda c_2-q_2a-\delta mq_1d),\\
&d_x=i(q_1b_2-\delta q_2b_1+\delta p_1c_2-p_2c_1),\\
&a_x=i(mp_1c_1+np_2c_2-mq_1b_1-nq_2b_2)=-\lambda^{-1}(mq_1b_{1,x}+nq_2b_{2,x}+mp_1c_{1,x}+np_2c_{2,x}).
\end{aligned} \tag{2.4}$$

These equations equivalently generate the initial conditions

$$a_x^{[0]}=0,\qquad b_1^{[0]}=b_2^{[0]}=c_1^{[0]}=c_2^{[0]}=0,\qquad d_x^{[0]}=0 \tag{2.5}$$

and the recursion relation

$$\begin{aligned}
b_1^{[s+1]}&=-ib_{1,x}^{[s]}+p_1a^{[s]}+\delta np_2d^{[s]},\\
b_2^{[s+1]}&=-ib_{2,x}^{[s]}+p_2a^{[s]}-mp_1d^{[s]},\\
c_1^{[s+1]}&=ic_{1,x}^{[s]}+q_1a^{[s]}-nq_2d^{[s]},\\
c_2^{[s+1]}&=ic_{2,x}^{[s]}+q_2a^{[s]}+\delta mq_1d^{[s]},\\
d_x^{[s+1]}&=i(q_1b_2^{[s+1]}-\delta q_2b_1^{[s+1]}+\delta p_1c_2^{[s+1]}-p_2c_1^{[s+1]}),\\
a_x^{[s+1]}&=i(-mq_1b_1^{[s+1]}-nq_2b_2^{[s+1]}+mp_1c_1^{[s+1]}+np_2c_2^{[s+1]})=\\
&=-(mq_1b_{1,x}^{[s]}+nq_2b_{2,x}^{[s]}+mp_1c_{1,x}^{[s]}+np_2c_{2,x}^{[s]}),
\end{aligned} \tag{2.6}$$

where $s\geqslant 0$.

In what follows, we take the initial values and choose the integration constants to be zero,

$$a^{[0]}=1,\quad d^{[0]}=0,\qquad a^{[s]}|_{u=0}=0,\quad d^{[s]}|_{u=0}=0,\quad s\geqslant 1, \tag{2.7}$$

to uniquely determine the solution Z. We can then find the first four sets of $a^{[s]}$, $b_1^{[s]}$, $b_2^{[s]}$, $c_1^{[s]}$, $c_2^{[s]}$ and $d^{[s]}$:

$$a^{[1]}=0,\qquad b_1^{[1]}=p_1,\quad b_2^{[1]}=p_2,\quad c_1^{[1]}=q_1,\quad c_2^{[1]}=q_2,\quad d^{[1]}=0;$$

$$\begin{aligned}
&a^{[2]}=-mp_1q_1-np_2q_2,\qquad b_1^{[2]}=-ip_{1,x},\quad b_2^{[2]}=-ip_{2,x},\\
&c_1^{[2]}=iq_{1,x},\quad c_2^{[2]}=iq_{2,x},\quad d^{[2]}=-\delta p_1q_2+p_2q_1;
\end{aligned}$$

$$\begin{aligned}
&a^{[3]}=-i(mp_1q_{1,x}-mp_{1,x}q_1+np_2q_{2,x}-np_{2,x}q_2),\\
&b_1^{[3]}=-p_{1,xx}-mp_1^2q_1-2np_1p_2q_2+\delta np_2^2q_1,\\
&b_2^{[3]}=-p_{2,xx}+\delta mp_1^2q_2-2mp_1p_2q_1-np_2^2q_2,\\
&c_1^{[3]}=-q_{1,xx}-mp_1q_1^2+\delta np_1q_2^2-2np_2q_1q_2,\\
&c_2^{[3]}=-q_{2,xx}-2mp_1q_1q_2+\delta mp_2q_1^2-np_2q_2^2,
\end{aligned}$$

$$
\begin{aligned}
d^{[3]} &= -i(\delta p_1 q_{2,x} - p_2 q_{1,x} - \delta p_{1,x} q_2 + p_{2,x} q_1);\\
a^{[4]} &= \frac{3}{2} m^2 p_1^2 q_1^2 - \frac{3}{2}\delta mn p_1^2 q_2^2 + 6mn p_1 p_2 q_1 q_2 - \frac{3}{2}\delta mn p_2^2 q_1^2 + \frac{3}{2} n^2 p_2^2 q_2^2 +\\
&\quad + m p_1 q_{1,xx} + m p_{1,x,x} q_1 + n p_2 q_{2,xx} + n p_{2,xx} q_2 - m p_{1,x} q_{1,x} - n p_{2,x} q_{2,x},\\
b_1^{[4]} &= i(p_{1,xxx} + 3m p_1 p_{1,x} q_1 + 3n p_1 p_{2,x} q_2 - 3\delta n p_2 p_{2,x} q_1 + 3n p_{1,x} p_2 q_2),\\
b_2^{[4]} &= i(p_{2,xxx} + 3m p_1 p_{2,x} q_1 - 3\delta m p_1 p_{1,x} q_2 + 3m p_{1,x} p_2 q_1 + 3n p_2 p_{2,x} q_2),\\
c_1^{[4]} &= -i(q_{1,xxx} + 3m p_1 q_1 q_{1,x} - 3\delta n p_1 q_2 q_{2,x} + 3n p_2 q_1 q_{2,x} + 3n p_2 q_{1,x} q_2),\\
c_2^{[4]} &= -i(q_{2,xxx} + 3m p_1 q_1 q_{2,x} + 3m p_1 q_{1,x} q_2 - 3\delta m p_2 q_1 q_{1,x} + 3n p_2 q_2 q_{2,x}),\\
d^{[4]} &= 3(m p_1 q_1 + n p_2 q_2)(\delta p_1 q_2 - p_2 q_1) + \delta p_{1,xx} q_2 - p_{2,xx} q_1 -\\
&\quad - p_2 q_{1,xx} + \delta p_1 q_{2,xx} - \delta p_{1,x} q_{2,x} + p_{2,x} q_{1,x}.
\end{aligned}
$$

Now, we introduce the temporal matrix spectral problems

$$
-i\phi_{t_r} = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi, \qquad V^{[r]} = (\lambda^r Z)_+ = \sum_{s=0}^{r} \lambda^s Z^{[r-s]}, \quad r \geqslant 0, \tag{2.8}
$$

which are the other parts of Lax pairs of matrix spectral problems in the zero-curvature formulation. The compatibility conditions for the spatial and temporal matrix spectral problems, Eqs. (2.1) and (2.8), are the zero-curvature equations (1.5). These equations yield a four-component integrable hierarchy

$$
u_{t_r} = K^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]})^{\mathrm{T}}, \qquad r \geqslant 0, \tag{2.9}
$$

or, more precisely,

$$
p_{1,t_r} = ib_1^{[r+1]}, \quad p_{2,t_r} = ib_2^{[r+1]}, \quad q_{1,t_r} = -ic_1^{[r+1]}, \quad q_{2,t_r} = -ic_2^{[r+1]}, \qquad r \geqslant 0. \tag{2.10}
$$

The first two nonlinear examples in the above integrable hierarchy are the coupled nonlinear Schrödinger equations

$$
\begin{aligned}
ip_{1,t_2} &= p_{1,xx} + m p_1^2 q_1 + 2n p_1 p_2 q_2 - \delta n p_2^2 q_1,\\
ip_{2,t_2} &= p_{2,xx} - \delta m p_1^2 q_2 + 2m p_1 p_2 q_1 + n p_2^2 q_2,\\
iq_{1,t_2} &= -q_{1,xx} - m p_1 q_1^2 + \delta n p_1 q_2^2 - 2n p_2 q_1 q_2,\\
iq_{2,t_2} &= -q_{2,xx} - 2m p_1 q_1 q_2 + \delta m p_2 q_1^2 - n p_2 q_2^2
\end{aligned} \tag{2.11}
$$

and the coupled modified Korteweg–de Vries equations

$$
\begin{aligned}
p_{1,t_3} &= p_{1,xxx} + 3m p_1 p_{1,x} q_1 + 3n p_1 p_{2,x} q_2 - 3\delta n p_2 p_{2,x} q_1 + 3n p_{1,x} p_2 q_2,\\
p_{2,t_3} &= p_{2,xxx} + 3m p_1 p_{2,x} q_1 - 3\delta m p_1 p_{1,x} q_2 + 3m p_{1,x} p_2 q_1 + 3n p_2 p_{2,x} q_2,\\
q_{1,t_3} &= q_{1,xxx} + 3m p_1 q_1 q_{1,x} - 3\delta n p_1 q_2 q_{2,x} + 3n p_2 q_1 q_{2,x} + 3n p_2 q_{1,x} q_2,\\
q_{2,t_3} &= q_{2,xxx} + 3m p_1 q_1 q_{2,x} + 3m p_1 q_{1,x} q_2 - 3\delta m p_2 q_1 q_{1,x} + 3n p_2 q_2 q_{2,x},
\end{aligned} \tag{2.12}
$$

where m and n are two arbitrary natural numbers and $\delta = \pm 1$.

They provide two examples of coupled integrable nonlinear Schrödinger equations and coupled integrable modified Korteweg–de Vries equations.

3. Hamiltonian structures

To obtain Hamiltonian structures for integrable hierarchy (2.9), we apply trace identity (1.7) to the matrix spectral problem in (2.1). Noting that the solution Z is given by (2.2), we can directly compute that

$$\operatorname{tr}\left(Z\frac{\partial U}{\partial \lambda}\right) = 2a, \qquad \operatorname{tr}\left(Z\frac{\partial U}{\partial u}\right) = (2mc_1, 2nc_2, 2mb_1, 2nb_2)^{\mathrm{T}}.$$

It follows that using trace identity (1.7) leads to

$$\frac{\delta}{\delta u}\int \lambda^{-s-1}a^{[s+1]}\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma-s}(mc_1^{[s]}, nc_2^{[s]}, mb_1^{[s]}, nb_2^{[s]})^{\mathrm{T}}, \qquad s \geqslant 0.$$

Considering the case with $s = 2$, we obtain $\gamma = 0$; we then have the variational identities

$$\frac{\delta \mathcal{H}^{[s]}}{\delta u} = (mc_1^{[s+1]}, nc_2^{[s+1]}, mb_1^{[s+1]}, nb_2^{[s+1]})^{\mathrm{T}}, \qquad s \geqslant 0, \tag{3.1}$$

where the Hamiltonian functionals are

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+2]}}{s+1}\,dx, \qquad s \geqslant 0, \tag{3.2}$$

the first three of which are given by

$$\begin{aligned}
\mathcal{H}^{[0]} &= \int (mp_1q_1 + np_2q_2)\,dx,\\
\mathcal{H}^{[1]} &= \int \frac{i}{2}(mp_1q_{1,x} - mp_{1,x}q_1 + np_2q_{2,x} - np_{2,x}q_2)\,dx,\\
\mathcal{H}^{[2]} &= \int \Big[\frac{1}{2}(-m^2p_1^2q_1^2 + \delta mnp_1^2q_2^2 - 4mnp_1p_2q_1q_2 + \delta mnp_2^2q_1^2 - n^2p_2^2q_2^2) - \\
&\quad - \frac{1}{3}(mp_1q_{1,xx} + mp_{1,x,x}q_1 + np_2q_{2,xx} + np_{2,xx}q_2 - mp_{1,x}q_{1,x} - np_{2,x}q_{2,x})\Big]\,dx.
\end{aligned} \tag{3.3}$$

From these identities, we can easily obtain the Hamiltonian structures for the associated integrable equations,

$$u_{t_r} = K^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]})^{\mathrm{T}} = J\frac{\delta \mathcal{H}^{[r]}}{\delta u}, \qquad r \geqslant 0, \tag{3.4}$$

where

$$J = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & i/m & 0 \\ & & 0 & i/n \\ \hline -i/m & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -i/n & & \end{array}\right]. \tag{3.5}$$

The associated Hamiltonian structures show a connection $S = J\frac{\delta \mathcal{H}}{\delta u}$ between a conserved functional $\mathcal{H}$ and a symmetry S, which can be used to show the Liouville integrability of hierarchy (2.9).

A basic feature of integrability is the commutativity of the vector fields $K^{[r]}$:

$$[\![K^{[s_1]}, K^{[s_2]}]\!] = K^{[s_1]\prime}(u)[K^{[s_2]}] - K^{[s_2]\prime}(u)[K^{[s_1]}] = 0, \qquad s_1, s_2 \geqslant 0. \tag{3.6}$$

It is guaranteed by the Lax operator algebra

$$[\![V^{[s_1]}, V^{[s_2]}]\!] = V^{[s_1]\prime}(u)[K^{[s_2]}] - V^{[s_2]\prime}(u)[K^{[s_1]}] + [V^{[s_1]}, V^{[s_2]}] = 0, \qquad s_1, s_2 \geqslant 0, \tag{3.7}$$

which is a consequence of the isospectral zero-curvature equations (see [22] for details).

In addition, from the recursion relation $K^{[r+1]} = \Phi K^{[r]}$, we find the entries of the recursion operator $\Phi = (\Phi_{jk})_{4\times 4}$ to be

$$\begin{aligned}
&\Phi_{11} = i(-\partial_x - mp_1\partial^{-1}q_1 - np_2\partial^{-1}q_2), && \Phi_{12} = i(-np_1\partial^{-1}q_2 + \delta np_2\partial^{-1}q_1),\\
&\Phi_{13} = i(-mp_1\partial^{-1}p_1 + \delta np_2\partial^{-1}q_2), && \Phi_{14} = i(-np_1\partial^{-1}p_2 - np_2\partial^{-1}p_1),\\
&\Phi_{21} = i(-mp_2\partial^{-1}q_1 + \delta mp_1\partial^{-1}q_2), && \Phi_{22} = i(-\partial_x - np_2\partial^{-1}q_2 - mp_1\partial^{-1}q_1),\\
&\Phi_{23} = i(-mp_2\partial^{-1}p_1 - mp_1\partial^{-1}p_2), && \Phi_{24} = i(np_2\partial^{-1}p_2 + \delta mp_1\partial^{-1}p_1),\\
&\Phi_{31} = i(mq_1\partial^{-1}q_1 - \delta nq_2\partial^{-1}q_2), && \Phi_{32} = i(nq_1\partial^{-1}q_2 + nq_2\partial^{-1}q_1),\\
&\Phi_{33} = i(\partial_x + mq_1\partial^{-1}p_1 + nq_2\partial^{-1}p_2), && \Phi_{34} = i(nq_1\partial^{-1}p_2 - \delta nq_2\partial^{-1}p_1),\\
&\Phi_{41} = i(mq_2\partial^{-1}q_1 + mq_1\partial^{-1}q_2), && \Phi_{42} = i(nq_2\partial^{-1}q_2 - \delta mq_1\partial^{-1}q_1),\\
&\Phi_{43} = i(mq_2\partial^{-1}p_1 - \delta mq_1\partial^{-1}p_2), && \Phi_{44} = i(\partial_x + nq_2\partial^{-1}p_2 + mq_1\partial^{-1}p_1).
\end{aligned}$$

Obviously, the operator ΦJ is skewsymmetric, and therefore the conserved functionals commute with respect to the corresponding Poisson bracket [4]:

$$\{\mathcal{H}^{[s_1]}, \mathcal{H}^{[s_2]}\}_J = \int \left(\frac{\delta \mathcal{H}^{[s_1]}}{\delta u}\right)^{\mathrm{T}} J \frac{\delta \mathcal{H}^{[s_2]}}{\delta u}\, dx = 0, \qquad s_1, s_2 \geqslant 0. \tag{3.8}$$

Finally, a combination of the Hamiltonian operator J with the recursion operator Φ [23] yields a bi-Hamiltonian structure [18] for hierarchy (2.9). To conclude, each equation in hierarchy (2.9) possesses infinitely many commuting symmetries $\{K^{[s]}\}_{s=0}^{\infty}$ and conserved functionals $\{\mathcal{H}^{[s]}\}_{s=0}^{\infty}$, and is therefore Liouville integrable, due to (3.6) and (3.8). In particular, Eqs. (2.11) and (2.12) present two simplest examples of nonlinear integrable Hamiltonian equations in the hierarchy.

4. Concluding remarks

A class of higher-order matrix spectral problems has been formulated and their associated integrable hierarchies of Hamiltonian equations have been generated within the zero-curvature formulation. A Laurent-series solution of the corresponding stationary zero-curvature equation is an essential ingredient of the construction. All equations in the resulting hierarchies have been shown to be Liouville integrable, with the Hamiltonian structures following from the trace identity.

We note that the matrix spectral problems in (2.1) are specific reductions of the matrix spectral problems in [24]–[26] and [17], which lead to integrable equations generalizing the Kulish–Sklyanin ones [27]. But how a successful reduction from a given matrix spectral problem can be found remains an open question. Any modified example of (2.1), where $\delta \mathbf{p}_1^{\mathrm{T}}$ and $\delta \mathbf{q}_1^{\mathrm{T}}$ are changed to $(\delta_1 p_1, \ldots, \delta_m p_1)^{\mathrm{T}}$ and $(\delta_1 q_1, \ldots, \delta_m q_1)$ with the δ_i being ± 1 but not the same (for example, $(\delta p_1, \ldots, \delta p_1, -\delta p_1)^{\mathrm{T}}$ and $(\delta q_1, \ldots, \delta q_1, -\delta q_1)$), does not work because it has no nonzero Laurent-series solution.

On the other hand, one could generalize the previous matrix spectral problems in (2.1) by adding a third pair of potentials p_3 and q_3. The task is then to derive a meaningful Laurent-series solution of the corresponding stationary zero-curvature equation. When the spectral matrix in a spectral problem is of higher order, it would be difficult to compute a required Laurent-series solution. In our example, such a Laurent-series solution was determined by some deep learning technique.

It is always interesting to explore solution structures of integrable equations by incorporating and integrating a wide variety of techniques in soliton theory. Those methods contain the Riemann–Hilbert technique [28], the Zakharov–Shabat dressing method [29], the Darboux transformation [30], [31], and the determinant approach [32], [33]. Reductions from the τ-function theory are particularly interesting. Special kinds of solutions such as lump wave and rogue wave solutions can often be generated by taking wave-number reductions of N-soliton solutions (see, e.g., [34]–[41]).

Nonlocal integrable equations could also be considered if nonlocal group reductions were used for the considered matrix spectral problems (see, e.g., [42]–[45] for novel kinds of nonlocal integrable NLS equations). However, comparatively little is known about nonlocal integrable equations, and further investigation is required.

Conflicts of interest. The author declares no conflicts of interest.

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